A GENERALIZED ACQUISITION SCHEME FOR VECTOR CROSS-PRODUCT DIRECTION FINDING WITH SPATIALLY SPREAD VECTOR-SENSOR COMPONENTS

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ABSTRACT

In this paper we propose a generalized non-collocated electromagnetic (EM) vector-sensor configuration allowing the use of the vector cross-product direction finding scheme. The presented work extends the results and the philosophy of [1] to a more general array configuration. We provide a sufficient condition ensuring identifiability of source DOA parameters and propose a novel algorithm allowing the DOA estimation for inter-antenna spacing larger than $\lambda/2$. The effectiveness of the proposed approach is illustrated by numerical simulations.

Index Terms— DOA estimation, vector sensor array, non-collocated antennas

1. INTRODUCTION

An EM vector-sensor consists of three orthogonally oriented electrically short dipoles and three orthogonally oriented magnetically small loops. All these components are spatially collocated, have the same symmetry center and aim at measuring the six components of an electromagnetic incident wave field. The expression of the array manifold for such a collocated vector-sensor, derived by Nehorai and Paldi in [2], is :

$$\mathbf{a} = \begin{bmatrix} \mathbf{e} \\ \mathbf{h} \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\theta & -\sin\phi \\ \sin\phi\cos\theta & \cos\phi \\ -\sin\theta & 0 \\ -\sin\phi & -\cos\phi\cos\theta \\ \cos\phi & -\sin\phi\cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{bmatrix} \sin\gamma \, e^{j\eta} \\ \cos\gamma \end{bmatrix}, \quad (1)$$

where $\mathbf{e} = [e_x, e_y, e_z]^T$ and $\mathbf{h} = [h_x, h_y, h_z]^T$ contain the electric and magnetic field components, respectively, along the $\{X, Y, Z\}$ axes of the attached Cartesian coordinates system. In (1), $\theta \in [0, \pi]$ is the elevation angle measured from the positive z axis, $\phi \in [0, 2\pi)$ denotes the azimuth angle measured from the x axis, $\gamma \in [0, \pi/2]$ refers to the auxiliary polarization angle and $\eta \in [-\pi, \pi)$ symbolizes the polarization phase difference. Given $\mathbf{e}$ and $\mathbf{h}$, the normalized Poynting-vector for an incident source can be calculated as

$$\mathbf{p} \stackrel{\text{def}}{=} \frac{\mathbf{e} \times \mathbf{h}^*}{\|\mathbf{e}\| \cdot \|\mathbf{h}\|} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \sin\theta \, \cos\phi \\ \sin\theta \, \sin\phi \\ \cos\theta \end{bmatrix}, \quad (2)$$

where "$\times$" symbolizes the vector cross-product, "$\| \cdot \|$" denotes the Frobenius norm of a vector and $\{u, v, w\}$ are the *direction-cosines* of the incident source along the three Cartesian axis. Various Direction Of Arrival (DOA) estimation schemes based on the vector-sensor array manifold in (1) have been proposed in the last two decades, *e.g.* [3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

However, the mathematical model of equation (1) unrealistically assumes that the mutual coupling between the six components of the vector-sensor is negligible. In practice, the mutual coupling can be reduced by using costly electromagnetic isolation devices, but it can never be entirely suppressed. An interesting, much cheaper alternative to electromagnetic isolation consists in using non collocating component antennas. In addition to the drastic mutual coupling reduction, this scheme also extends the geometric aperture of the array, thus leading to more accurate DOA estimates. Nevertheless, the use of these spatially spread configurations raises the delicate problem of adapting the direction finding algorithms developed for the collocated vector-sensors to the non-collocating case. A DOA estimation algorithm, that adapts the Poynting-vector method for a non-collocating vector-sensor was proposed in [1]. This method only works for a couple of special geometric configurations of the antennas, for which the three dipoles and the three loops are located on two parallel straight lines and verify particular spacing constraints between the elements.

In this paper we propose a generalized geometric configuration of the non-collocating vector-sensor components, allowing to use a vector cross-product direction finding scheme. The array layout in [1] can then be seen as a special/complementary case of this antenna array arrangement. We also propose in this paper an novel algorithm for estimating source DOAs from the data recorded with the proposed non-collocating vector-sensor, based on the minimization of a non-convex, non-linear cost function. Direction finding algorithms for polarized sources based on cost function minimizations, have been already proposed in the literature, *e.g.* [8, 9]. In [9] the cost function is used to disambiguate a fine DOA estimate and its minimization is performed by a search over a discrete grid, with a step well-determined in advance. A MUSIC-like cost function is employed in [8], preceded by a beamforming step aiming at reducing the signal subspace dimension and thus the optimization complexity. Unlike the above mentioned methods, the solution for minimizing the cost function proposed in this paper is based on a particular linear transform of the parameters of interest, allowing to define a low-dimension search grid. The remainder of the paper is organized as follows: in section 2 we introduce the generalized non-collocating vector-sensor configuration and in section 3 we give a sufficient condition for the parameter identifiability and present a novel algorithm for estimating the direction-cosines for this new antenna configuration. In section section 4 numerical simulations are used to illustrate the performances of the proposed algorithm. The last section provides some concluding remarks and comments.

2. A GENERALIZED ARRAY CONFIGURATION WITH SPATIALLY SPREAD VECTOR-SENSOR COMPONENTS

Starting from a collocated vector-sensor located at point $\mathbf{c} = [c_x, c_y, c_z]^T$ in the coordinate system $(OXYZ)$, we consider the array obtained by a symmetric spatial displacement of each dipole/loop pair $\{E_i, H_i\}$ $(i \in \{x, y, z\})$ along a straight line passing through $\mathbf{c}$, as illustrated in fig. 1. Thus, the point $\mathbf{c}$ can be used as a common phase reference for all the non-collocated elements of the spread vector-sensor. The distances of the dipoles/loops relative to their phase center are $\Delta X, \Delta Y$ and ΔZ, as shown in fig. 1, and the length of their projections on the three axes are denoted by $\Delta X_i, \Delta Y_i$ and ΔZ_i, with $i \in \{x, y, z\}$.

2.1. The "non-collocating" Poynting-vector

The spatial displacement of the six antennas of the vector-sensor, introduce additional phase-shifts in the expression of the array manifold given by eq. (1). Thus, the array manifold of the spatially

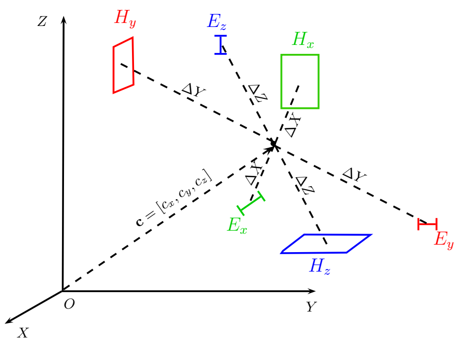

Fig. 1. The generalized spatially spread vector-sensor geometry. The rectangles represent the magnetic loops and the line segments the electric dipoles.

displaced vector-sensor is :

$$\tilde{\mathbf{a}} = \begin{bmatrix} e_x \; e^{-\frac{j2\pi}{\lambda}(c_x u + c_y v + c_z w)} \; e^{-\frac{j2\pi}{\lambda}(\Delta X_x u + \Delta X_y v + \Delta X_z w)} \\ e_y \; e^{-\frac{j2\pi}{\lambda}(c_x u + c_y v + c_z w)} \; e^{-\frac{j2\pi}{\lambda}(\Delta Y_x u + \Delta Y_y v + \Delta Y_z w)} \\ e_z \; e^{-\frac{j2\pi}{\lambda}(c_x u + c_y v + c_z w)} \; e^{-\frac{j2\pi}{\lambda}(\Delta Z_x u + \Delta Z_y v + \Delta Z_z w)} \\ h_x \; e^{-\frac{j2\pi}{\lambda}(c_x u + c_y v + c_z w)} \; e^{\frac{j2\pi}{\lambda}(\Delta X_x u + \Delta X_y v + \Delta X_z w)} \\ h_y \; e^{-\frac{j2\pi}{\lambda}(c_x u + c_y v + c_z w)} \; e^{\frac{j2\pi}{\lambda}(\Delta Y_x u + \Delta Y_y v + \Delta Y_z w)} \\ h_z \; e^{-\frac{j2\pi}{\lambda}(c_x u + c_y v + c_z w)} \; e^{\frac{j2\pi}{\lambda}(\Delta Z_x u + \Delta Z_y v + \Delta Z_z w)} \end{bmatrix}. \tag{3}$$

The expression of the"non-collocating" Poynting-vector can then be calculated similarly to eq. (2) and after some mathematical manipulations we get

$$\tilde{\mathbf{p}} = \begin{bmatrix} u \; e^{-\frac{j2\pi}{\lambda}((\Delta Y_x + \Delta Z_x)u + (\Delta Y_y + \Delta Z_y)v + (\Delta Y_z + \Delta Z_z)w)} \\ v \; e^{-\frac{j2\pi}{\lambda}((\Delta X_x + \Delta Z_x)u + (\Delta X_y + \Delta Z_y)v + (\Delta X_z + \Delta Z_z)w)} \\ w \; e^{-\frac{j2\pi}{\lambda}((\Delta X_x + \Delta Y_x)u + (\Delta X_y + \Delta Y_y)v + (\Delta Y_z + \Delta X_z)w)} \end{bmatrix}. \tag{4}$$

As expected, the expression of the Poynting-vector does not depend on the position of the array in the coordinate system, but only on the relative positions of the antennas. If we denote the coefficients of u, v and w in the exponent part of (4) by A_k, B_k and C_k ($k \in \{1, 2, 3\}$), respectively, $\tilde{\mathbf{p}}$ can be re-written as

$$\tilde{\mathbf{p}}(u, v, w) = \begin{bmatrix} u \; e^{-\frac{j2\pi}{\lambda}(A_1 u + B_1 v + C_1 w)} \\ v \; e^{-\frac{j2\pi}{\lambda}(A_2 u + B_2 v + C_2 w)} \\ w \; e^{-\frac{j2\pi}{\lambda}(A_3 u + B_3 v + C_3 w)} \end{bmatrix}. \tag{5}$$

In the next section we deal with the problem of estimating u, v and w from $\tilde{\mathbf{p}}$.

3. PARAMETERS ESTIMATION

Before presenting the algorithmic approach for estimating u, v and w, the identifiability of model (5) needs to be addressed.

3.1. Parameters identifiability

Compared to the "collocated" case, the entries of $\tilde{\mathbf{p}}$ are complex-valued, the parameters of interest being present both in the modulus and the exponent (phase). Therefore, estimating u, v and w from $\tilde{\mathbf{p}}$ may yield sign ambiguities *e.g.* there might exist cases where $\tilde{\mathbf{p}}(u, v, w) = \tilde{\mathbf{p}}(-u, v, w)$. This drawback can be avoided by considering jointly the relationships between the three elements of vector $\tilde{\mathbf{p}}$. We provide next a sufficient identifiability condition for the direction-cosines parameters in (5).

Proposition 1 *[Sufficient condition for the identifiability of model (5)] If the sign of u is known and the spacing between the six antennas is such that : $(i) A_1 \neq A_2 \neq A_3$, $(ii) B_1 = B_2 \neq B_3$, $(iii) C_1 = C_2 = C_3$, then, u, v and w can be uniquely estimated from $\tilde{\mathbf{p}}(u, v, w)$.*

Due to space limitations the proof of this proposition is omitted in the current version of the paper. This condition also holds if u is replaced by v or w, and/or if the roles of A_k, B_k, C_k ($k \in \{1, 2, 3\}$) are interchanged. From a geometric point of view, the conditions of *Proposition 1* imply $\Delta X_x \neq \Delta Y_x \neq \Delta Z_x$, $\Delta X_y = \Delta Y_y \neq \Delta Z_y$ and $\Delta X_z = \Delta Y_z = \Delta Z_z$, meaning that the dipoles and the loops are lying in two parallel planes, and any three of them are non-collinear. It is necessary to fix the sign of u to remove the ambiguity related to the lack of knowledge on the phase of the impinging source. A similar assumption is also made in [1].

3.2. Parameter estimation

In this paper we used the *Uni-Vector-Sensor ESPRIT* algorithm proposed in [4] for the numerical simulations presented in section 4. Nevertheless, any parameter estimation algorithm, capable of estimating a source steering vector could be used to this end. Once the steering vector is estimated, the corresponding Poynting vector $\hat{\tilde{\mathbf{p}}}$ can be easily computed by the vector cross-product in (2). Then, the estimation of the cosine-directions can be formulated as a minimization problem, where the objective function to minimize is given by

$$\mathcal{J}(u, v, w) = \left\| \hat{\tilde{\mathbf{p}}} - \tilde{\mathbf{p}}(u, v, w) \right\|^2. \tag{6}$$

It is worth noting that the formulation of the estimation problem given by eq. (6) is different from the one in [1], where "coarse" estimates of u, v and w computed from the moduli of the elements of $\hat{\tilde{\mathbf{p}}}$ are used to disambiguate the more accurate phase estimates. Moreover, the algorithm presented in [1] handles only a particular configuration of the array configuration where the three loops and the three dipoles are parallel to one of the coordinate system axes. Anyway, minimizing $\mathcal{J}(u, v, w)$ yields a non-convex and highly non-linear optimization problem. A straightforward minimization procedure is the use of a local optimization algorithm, in which case it is crucial to find a good initialization point. A solution is the use of the moduli of the entries of $\hat{\tilde{\mathbf{p}}}$ as initialization. If the signs of u, v and w are unknown, the eight possible sign combinations must be tested as initialization, and the one yielding the lowest value of $\mathcal{J}(u, v, w)$ is kept as solution. However, even for moderate signal to noise ratios, as the distance between sensors increases, the number of local minima of the objective function also increases, and this method fails almost systematically, as shown in the next section.

We propose next a novel minimization method that overcomes the problems described above, avoiding at the same time an exhaustive grid search for the parameters. The idea behind the proposed method is the use of a change of variables in (6) allowing to determine the periodicities in the phase of the entries of $\tilde{\mathbf{p}}$. By doing so, the minimization problem comes down to a discrete search over a low dimensional grid, guaranteeing to find the global minimizer of (6). To some extent, the proposed optimization procedure can be seen as an extension of the algorithm of [1] to the generalized array configuration introduced in section 2.

Let us define the vectors $\mathbf{a}_i = [A_i \ B_i \ C_i]^T$, with $i \in \{1, 2, 3\}$ and let $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]^T$. The identifiability conditions in *Proposition 1* guarantee that $\mathbf{A}$ is a non-singular 3×3 real matrix. If we denote $\boldsymbol{\omega} = [u, v, w]^T$, the objective function (6) can be written as :

$$\mathcal{J}(\boldsymbol{\omega}) = \left\| \hat{\tilde{\mathbf{p}}} - \boldsymbol{\omega} \circledast \begin{bmatrix} e^{-j\frac{2\pi}{\lambda}\mathbf{a}_1^T\boldsymbol{\omega}} \\ e^{-j\frac{2\pi}{\lambda}\mathbf{a}_2^T\boldsymbol{\omega}} \\ e^{-j\frac{2\pi}{\lambda}\mathbf{a}_3^T\boldsymbol{\omega}} \end{bmatrix} \right\|^2, \quad (7)$$

where "$\circledast$" symbolizes the Hadamard (element-wise) product of two vectors. Consider a new vector $\overline{\boldsymbol{\omega}} = [\overline{u}, \overline{v}, \overline{w}]^T$ such that $\overline{\boldsymbol{\omega}} = \mathbf{A}\boldsymbol{\omega}$ and define a new objective function

$$\mathcal{I}(\overline{\boldsymbol{\omega}}) = \mathcal{J}(\mathbf{A}^{-1}\overline{\boldsymbol{\omega}}) = \left\| \hat{\tilde{\mathbf{p}}} - (\mathbf{A}^{-1}\overline{\boldsymbol{\omega}}) \circledast \begin{bmatrix} e^{-j\frac{2\pi}{\lambda}\overline{u}} \\ e^{-j\frac{2\pi}{\lambda}\overline{v}} \\ e^{-j\frac{2\pi}{\lambda}\overline{w}} \end{bmatrix} \right\|^2. \quad (8)$$

As $\mathbf{A}$ is nonsingular, if $\overline{\boldsymbol{\omega}}^* = [\overline{u}^*, \overline{v}^*, \overline{w}^*]^T$ is a minimizer of $\mathcal{I}(\overline{\boldsymbol{\omega}})$, then $\boldsymbol{\omega}^* = \mathbf{A}^{-1}\overline{\boldsymbol{\omega}}^*$ is a minimizer of $\mathcal{J}(\boldsymbol{\omega})$. Let us suppose that u, v, w are non-negative, and let us consider only the exponential part of (8) :

$$\mathcal{I}_{exp}(\overline{\boldsymbol{\omega}}) = \left\| \hat{\tilde{\mathbf{p}}}./|\hat{\tilde{\mathbf{p}}}| - \begin{bmatrix} e^{-j\frac{2\pi}{\lambda}\overline{u}} \\ e^{-j\frac{2\pi}{\lambda}\overline{v}} \\ e^{-j\frac{2\pi}{\lambda}\overline{w}} \end{bmatrix} \right\|^2, \quad (9)$$

where "./" denotes the element-wise division and "|.|" denotes the element-wise modulus operation. If $[\overline{u}^\star, \overline{v}^\star, \overline{w}^\star]$ is a minimizer of the cost function (9), then any $[\overline{u}^\star + k_1\lambda, \ \overline{v}^\star + k_2\lambda, \ \overline{w}^\star + k_3\lambda]$, with $k_1, k_2, k_3 \in \mathbb{Z}$ (the set of integers), is also a minimizer of (9). The point is that the global minimizer of $\mathcal{I}(\overline{\boldsymbol{\omega}})$ is one of the minimizers of $\mathcal{I}_{exp}$. Thus, the proposed method for the minimization of the objective function (7) can be summarized as follows:

1. Perform the change of variables $\overline{\boldsymbol{\omega}} = \mathbf{A}\boldsymbol{\omega}$.
2. Find a minimizer $[\overline{u}^\star, \overline{v}^\star, \overline{w}^\star]$ of $\mathcal{I}_{exp}(\overline{\boldsymbol{\omega}})$. Any local optimization algorithm can be used to this end.
3. Find the global minimizer of $\mathcal{I}(\overline{\boldsymbol{\omega}})$ as
$\overline{\boldsymbol{\omega}}^* = \underset{k_1,k_2,k_3 \in \mathbb{Z}}{\text{argmin}} \ \mathcal{I}(\overline{u}^\star + k_1\lambda, \ \overline{v}^\star + k_2\lambda, \ \overline{w}^\star + k_3\lambda)$.
The search domain for $\overline{u}, \overline{v}, \overline{w}$ is bounded by $\overline{\boldsymbol{\omega}}_{\min} = \mathbf{A}\boldsymbol{\omega}_{\min}$ and $\overline{\boldsymbol{\omega}}_{\max} = \mathbf{A}\boldsymbol{\omega}_{\max}$, with $\boldsymbol{\omega}_{\max} = -\boldsymbol{\omega}_{\min} = [1 \ 1 \ 1]^T$ (see eq. (2)).
4. If the signs of the direction-cosines are *a priori* unknown, all the eight possible sign combinations must be tested in (9)[1]; the one yielding the lowest cost $\mathcal{I}(\overline{\boldsymbol{\omega}})$ is kept as the final solution.
5. Compute the minimizer of the original cost function $\mathcal{J}(\boldsymbol{\omega})$ as $\boldsymbol{\omega}^* = \mathbf{A}^{-1}\overline{\boldsymbol{\omega}}^*$. A local optimization step can be added for $\mathcal{J}(\boldsymbol{\omega})$, in order to obtain more "refined" estimates of u, v and w.

By using a linear transformation of the parameters of interest, the proposed algorithm transforms a continuous parameter search problem into a search over a discretized low dimensional grid with a step λ, inside a bounded domain.

4. SIMULATIONS

In this section we compare in numerical simulations the performances of the proposed algorithm and of the method using the direct local minimization of (7) with the signed modulus of the entries of $\hat{\tilde{\mathbf{p}}}$ as initial point. We used a Nelder-Mead simplex algorithm for the all the local optimization procedures. For the proposed method we plotted the results with and without the refinement stage (see step 5 of the algorithm in section 3.2).

A direct comparison with the algorithm given in [1] is not possible because, for the array configuration used in [1], the matrix $\mathbf{A}$ is singular. Furthermore, in this case the change of variables is pointless, as the periodicities in the phase of the Poynting vector are already well-determined. Also, the method proposed in [1] cannot be applied to the scenario considered in this paper. From this perspective, the proposed approach and the one in [1] can be considered as complementary.

For the experiments presented next we considered two narrowband far-field sources with different central frequencies and of DOA parameters $(\theta_1, \phi_1) = (120°, 40°)$ et $(\theta_2, \phi_2) = (30°, 50°)$. We plotted the composite root mean-square-error (RMSE) for direction-cosines for both sources, estimated with 30 trials per point. The ESPRIT algorithm introduced in [4] was used to estimate the source steering vectors $\tilde{\mathbf{a}}$ from the simulated mixture.

Fig. 2 represents the RMSE for the three algorithms vs. SNR, for a fixed spatial displacement of the antennas. For the inter-antenna spacing we chose $\Delta X = \Delta Y = \Delta Z = 8\lambda$, which guarantee that the conditions of *Proposition 1* are fulfilled[2]. One can see that, at low SNR, our method presents a performance gain of about 15 dB compared to the direct optimization procedure. This can be explained by the fact that at low SNR the modulus of $\hat{\tilde{\mathbf{p}}}$ yields a bad estimate of the direction-cosines and consequently, the local optimization algorithm converges almost systematically to a local and not to the global minimum of the cost function (7). Also, in this case, it seems that the "refinement" procedure is useless in the presence of strong noise but improves slightly the direction-cosines estimates at high SNR. For fig.3, we fixed the SNR= 10 dB and we plotted the RMSE w.r.t. the inter-antenna spacing Δ/λ. The distances between the array elements and their symmetry center are set to $\Delta X = \Delta Y = \Delta Z = \Delta$. One can observe that for very low Δ/λ ratios the accuracy of our "refined" approach is similar to that of the direct minimization method. This corresponds to inter-antenna spacing inferior to $\lambda/2$ in which case there is no cyclical ambiguity

[1] A negative value of one of the direction-cosines can be accounted for by adding a phase-shift of π to the corresponding entry, *e.g.* if $u < 0$ then its corresponding entree in (9) becomes $e^{-j(\frac{2\pi}{\lambda}\overline{u}+\pi)}$.

[2] In this section λ denotes the minimum value between the wavelengths of the two sources, λ_1 et λ_2.

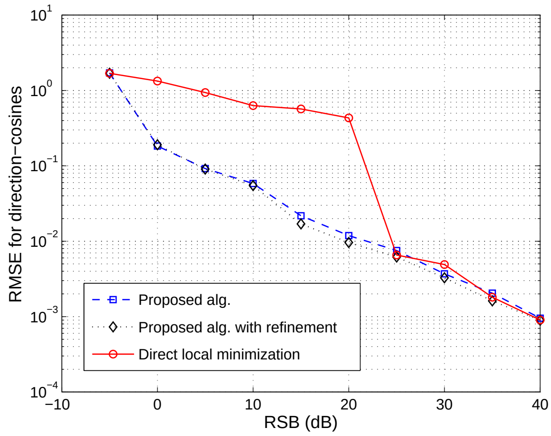

Fig. 2. The composite root mean-square-error of the direction-cosines estimates for two incident sources vs. SNR (dB)

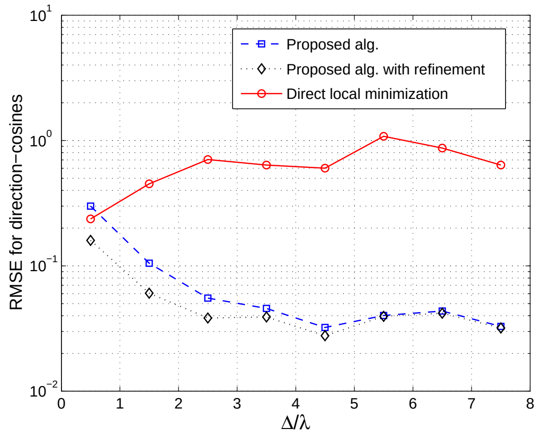

Fig. 3. The composite root mean-square-error of the direction-cosines estimates for two incident sources vs. Δ/λ

for the phase estimates of $\tilde{\mathbf{p}}(u, v, w)$. In this case, the change of variables does not present any interest because the direct minimization of (7) will always converge to the global minimum. Meanwhile, when the distance between components becomes significant (several wavelengths), the direct minimization procedure fails systematically. The reason is that the number of local minima of (7) grows as the distance between the array components increases and therefore, the probability to converge towards a local minimum becomes more important. This is not the case for the proposed method which continues to yield reliable estimates of the direction-cosines parameters. Moreover, the increase of Δ results in an improved DOA estimation accuracy which can be attributed to a larger spatial array aperture. However, the size of the search grid increase with the inter-antenna spacing and consequently the computational burden of the algorithm grows. Nevertheless, for reasonable array apertures ($\approx 10\lambda$) the computation time remains acceptable (several seconds on a MacBook Pro, 2.66 GHz, 4Go RAM).

5. CONCLUSIONS

The non-collocating vector-sensor array configuration introduced in this paper generalizes the acquisition scheme introduced in [1], while still allowing the use of the vector cross-product direction-finding approach. We formulated the direction-cosines estimation problem as a non-linear objective function minimization and we showed that its global minimum can be attained by performing a change of variables and a discrete search over a low dimensional grid.

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