MUSIC Algorithm for Vector-Sensors Array Using Biquaternions

Nicolas Le Bihan, Sebastian Miron, and Jérôme I. Mars

***Abstract*—In this paper, we use a biquaternion formalism to model vector-sensor signals carrying polarization information. This allows a concise and elegant way of handling signals with eight-dimensional (8-D) vector-valued samples. Using this model, we derive a biquaternionic version of the well-known array processing MUSIC algorithm, and we show its superiority to classically used *long-vector* approach. New results on biquaternion valued matrix spectral analysis are presented. Of particular interest for the biquaternion MUSIC (BQ-MUSIC) algorithm is the decomposition of the spectral matrix of the data into orthogonal subspaces. We propose an effective algorithm to compute such an orthogonal decomposition of the observation space via the eigenvalue decomposition (EVD) of a Hermitian biquaternionic matrix by means of a newly defined quantity, the *quaternion adjoint matrix*. The BQ-MUSIC estimator is derived and simulation results illustrate its performances compared with two other approaches in polarized antenna processing (LV-MUSIC and PSA-MUSIC). The proposed algorithm is shown to be superior in several aspects to the existing approaches. Compared with LV-MUSIC, the BQ-MUSIC algorithm is more robust to modelization errors and coherent noise while it can detect less sources. In comparaison with PSA-MUSIC, our approach exhibits more accurate estimation of direction of arrival (DOA) for a small number of sources, while keeping the polarization information accessible.**

***Index Terms*—Biquaternions and biquaternion-valued matrices, Biquaternion MUSIC (BQ-MUSIC), eigenvalue decomposition (EVD) of biquaternionic matrices, vector-sensor array processing.**

I. Introduction

THE vector-sensors are now of common use in different applications such as electromagnetics, communications, seismic sensing, seismology, etc. These sensors record the components of the observed nonisotropic field and allow the recovery of polarization information. Depending on the application and the type of sensors, one can record two (two-component sensors) to six (three components of $\vec{\mathbf{E}}$ and three components of $\vec{\mathbf{B}}$ for electromagnetic wave fields) signals on a collocated sensor. The use of such sensors has proved its advantages in increasing the performances of classical algorithms (due to the redundancy of signals on the different components) and represents at the same time the only possibility to recover polarization information. There is a large number of studies on the extensions of classical signal/array processing techniques to the vector-sensor case (see [1] and references therein). Further, high-resolution array processing algorithms were studied for the multicomponent case, mainly by Nehorai [2], [3], Wong and Zoltowski [1], [4]–[7] and Li ([8]–[11]), for different configurations and both for MUSIC- and ESPRIT-like algorithms. Furthermore, the performances of vector-sensor arrays were analyzed and quantified in [3] and [12]. In all these contributions considering arrays of vector-sensors, the *vector* dimension of the recorded signals was unfolded along the *distance* (related to the number of sensors/aperture of the array) dimension, resulting in the so-called “long-vector” approach. This way of processing data originated from vector-sensors has the main advantage of allowing, together with a rather complicated parametrization of the data, the use of well-known matrix algebra techniques over the real or the complex field. However, the “long-vector” approach has the drawback of destroying locally the vector-type of the signal because of the reorganization of the data into a large vector.

In this paper, we propose an alternative way to process signals from vector-sensor arrays. Instead of reorganizing data into long vectors, we introduce a hypercomplex model for multicomponent signals impinging on vector-sensors. This model is based on biquaternions (quaternions with complex coefficients) and allows the processing of multicomponent signals using linear algebra algorithms over the biquaternions. Consequently, the derivation of high-resolution techniques for vector-sensors array is possible. We illustrate our approach by deriving a Biquaternion MUSIC (BQ-MUSIC)-like algorithm for this type of arrays. The use of biquaternions allows us to skip the parametrization step used in long-vector techniques [3] as it intrinsically includes the vector dimension in the process. The authors previously proposed the use of quaternions to process vector-sensor signals [13], [14]. In [13], a quaternion model for three-components vector-sensor signals was used and a subspace method was derived in the time domain, allowing denoising of polarized waves. In [14], only two-component vector-sensors arrays were considered. A quaternion modelization of the output signals was used and a MUSIC algorithm derived for direction-of-arrival (DOA) and polarization parameters estimation. The proposed technique in this paper is a generalization of the one presented in [14] to the case of three-component vector-sensor arrays. The use of biquaternions for signal modelization leads to new problems, such as the diagonalization of the biquaternionic sample covariance matrix. An original technique is proposed for this task.

Since biquaternions have not been widely studied in literature, there is a lack of known results on matrices with biquater-

Manuscript received February 6, 2006; revised December 25, 2006. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. A. Rahim Leyman.

N. Le Bihan and J. I. Mars are with the GIPSA-Lab, INPG, CNRS, Département Signal-Images, Domaine Universitaire, 38402 Saint Martin d'Hères Cedex, France (e-mail: nicolas.le-bihan@lis.inpg.fr; jerome.mars@lis.inpg.fr).

S. Miron is with the Centre de Recherche en Automatique de Nancy (CRAN), Faculte des Sciences et Techniques, 54506 Vandoeuvre-les-nancy Cedex, France (e-mail: sebastian.miron@cran.uhp-nancy.fr).

Digital Object Identifier 10.1109/TSP.2007.896067

nion coefficients. We present here some novel results about such matrices with emphasis on eigenvalue decomposition. We provide a way to compute the eigenvalue decomposition of a Hermitian biquaternion valued matrix and show its application in a biquaternion MUSIC algorithm. The proposed biquaternion MUSIC algorithm is then compared with the classical long-vector MUSIC and to the Polarization Smoothing Algorithm (PSA)-MUSIC algorithm [15]; its superiority in computational/robustness/resolution issues is demonstrated on numerical examples.

The biquaternion approach developed here is part of a new way of considering vector-sensor signals, whose global underlying philosophy consists in considering that these signals evolve on extended algebraic structures, rather than trying to make the signal fit the already existing algorithms/concepts.[1]

The paper is organized as follows. In Section II, we introduce biquaternions and their basic properties. Then in Section III, we present a detailed study of biquaternion valued matrices with particular attention to the eigenvalue decomposition problem. This decomposition is introduced, and the link with orthogonal decomposition and rank properties are illustrated. In Section IV, the biquaternion model for polarized waves recorded on three-component vector-sensor arrays is introduced. This model, together with the eigenvalue decomposition (EVD) allows the definition of a BQ-MUSIC algorithm described in Section V. Simulation results and comparisons with the long-vector approach and PSA-MUSIC are enlightened in Section VI. Concluding remarks about this work are presented in Section VII.

II. Biquaternions

Biquaternions, also known as "complexified quaternions," are an eight-dimensional (8-D) algebra and consist of quaternion numbers with complex coefficients. They were discovered by Hamilton in 1853 [17]. While Hamilton's (real) quaternions [18] are noted $\mathbb{H}$, the set of complex quaternions is noted $\mathbb{H}_{\mathbb{C}}$ [19].

Definition 1: A complexified quaternion $q \in \mathbb{H}_{\mathbb{C}}$ is given by

$$q = q_0 + q_1\mathtt{i} + q_2\mathtt{j} + q_3\mathtt{k} \tag{1}$$

where $q_0, q_1, q_2, q_3 \in \mathbb{C}_I$ and with elements of $\mathbb{C}_I$ defined as

$$z = \Re(z) + I\Im(z) \Leftrightarrow z \in \mathbb{C}_I \tag{2}$$

with $I = \sqrt{-1}$ and $\Re(z), \Im(z) \in \mathbb{R}$. The following *standard* relations between imaginary quaternion units hold:

$$\begin{aligned} \mathtt{i}^2 &= \mathtt{j}^2 = \mathtt{k}^2 = \mathtt{ijk} = -1 \\ \mathtt{ij} &= -\mathtt{ji} = \mathtt{k} \\ \mathtt{ki} &= -\mathtt{ik} = \mathtt{j} \\ \mathtt{jk} &= -\mathtt{kj} = \mathtt{i} \end{aligned} \tag{3}$$

with, in addition, the following relations between complex imaginary unit I and quaternion imaginary units:

$$\mathtt{i}I = I\mathtt{i};\ \mathtt{j}I = I\mathtt{j};\ \mathtt{k}I = I\mathtt{k} \tag{4}$$

[1]This approach has to be put in parallel with the one developed by Manton [16], who developed the processing of signals evolving on manifolds.

meaning that any complex coefficient commutes with any quaternion imaginary unit.

Thus, biquaternions form an 8-D vector space over $\mathbb{R}$ with basis:

$$\{1, \mathtt{i}, \mathtt{j}, \mathtt{k}, I, \mathtt{i}I, \mathtt{j}I, \mathtt{k}I\}. \tag{5}$$

Biquaternions form an *associative algebra* but *not a normed division algebra*. The only 8-D normed division algebra are the ones isomorphic to Cayley's octonions (this is known as generalized Frobenius and Hurwitz theorems, see [20] for details).

Biquaternions are isomorphic to Clifford algebra $\mathcal{Cl}_3$ (the Clifford algebra built over $\mathbb{R}^3$ with basis $\{e_1, e_2, e_3\}$ and such that $e_m e_n + e_n e_m = 2\delta_{nm}$), with identifications, as follows:

$$\begin{cases} \mathtt{i}I \leftrightarrow -e_3 \\ \mathtt{j}I \leftrightarrow -e_1 \\ \mathtt{k}I \leftrightarrow e_2 \end{cases} \quad \text{and} \quad \begin{cases} \mathtt{i} \leftrightarrow e_1 e_2 \\ \mathtt{j} \leftrightarrow e_2 e_3 \\ \mathtt{k} \leftrightarrow e_1 e_3 \\ I \leftrightarrow e_1 e_2 e_3 \end{cases} \tag{6}$$

where $e_n e_m$ are bivectors and $e_1 e_2 e_3$ is a pseudoscalar [21], [22].

Next, we present a nonexhaustive list of properties for biquaternions. The interested reader will find more material in [19]. Note that (real) Hamilton's quaternions are a special case of biquaternions. As in the case of quaternions, any biquaternion q can be seen as the sum of a scalar and a vector part, both with complex valued coefficients, as follows:

$$q = \mathcal{S}(q) + \mathcal{V}(q) \tag{7}$$

where

$$\begin{cases} \mathcal{S}(q) = q_0 \\ \mathcal{V}(q) = q_1\mathtt{i} + q_2\mathtt{j} + q_3\mathtt{k}. \end{cases} \tag{8}$$

At the same time, q can be seen as the sum of a real and an imaginary part, both being quaternion valued, as follows:

$$q = \Re(q) + I\Im(q) \tag{9}$$

where

$$\begin{cases} \Re(q) = \Re(q_0) + \Re(q_1) + \Re(q_2) + \Re(q_3) \\ \Im(q) = \Im(q_0) + \Im(q_1) + \Im(q_2) + \Im(q_3). \end{cases} \tag{10}$$

This notation of a biquaternion can be seen as an equivalent of the Cayley–Dickson notation for real quaternions [20], and it will be useful in the study of biquaternion valued matrices. Note that a biquaternion with zero scalar part ($\mathcal{S}(q) = 0$) is called *pure*.

Some known properties of complex and quaternions numbers, such as the multiplication and the addition, extend naturally to biquaternions. For some others, the extension is not trivial.

Definition 2: There exist three different conjugations over $\mathbb{H}_{\mathbb{C}}$. Thus, given a complex quaternion q, it is possible to define its conjugations, as follows:

- $\mathbb{C}$-conjugate: $q^* = q_0^* + q_1^*\mathtt{i} + q_2^*\mathtt{j} + q_3^*\mathtt{k} = \Re(q) - I\Im(q)$;
- $\mathbb{H}$-conjugate: $q^\star = q_0 - q_1\mathtt{i} - q_2\mathtt{j} - q_3\mathtt{k} = \mathcal{S}(q) - \mathcal{V}(q)$;
- (Total) $\mathbb{H}_{\mathbb{C}}$-conjugate: $\overline{q} = (q^*)^\star = q_0^* - q_1^*\mathtt{i} - q_2^*\mathtt{j} - q_3^*\mathtt{k}$.

These definitions induce different possible definitions for norms. We mention here a norm and a pseudonorm.

Definition 3: The norm of a biquaternion $q \in \mathbb{H}_{\mathbb{C}}$, noted $|q|$, is given by

$$|q| = \sqrt{|q_0|^2 + |q_1|^2 + |q_2|^2 + |q_3|^2}. \tag{11}$$

Note that $|q| \geq 0 \ \forall q \in \mathbb{H}_{\mathbb{C}}$, and $|q| = 0 \Rightarrow q = 0$; the biquaternions are not a normed algebra under this norm, so in general $|qp| \neq |q||p|$ for $q, p \in \mathbb{H}_{\mathbb{C}}$.

It is possible to define a pseudo-norm satisfying the property that the pseudo-norm of a product of biquaternions is equal to the product of the pseudo-norms of the individuals.

Definition 4: The pseudo-norm of a biquaternion $q \in \mathbb{H}_{\mathbb{C}}$, noted $|q|_p$, is given by

$$|q|_p = q_0^2 + q_1^2 + q_2^2 + q_3^2 \tag{12}$$

and it satisfies the following equality: $|rq|_p = |r|_p |q|_p$ for $r, q \in \mathbb{H}_{\mathbb{C}}$. It has the drawback of being complex valued in general. This involves that the pseudo-norm of a nonzero biquaternion can vanish. For example, for the biquaternion $q = e^{I(\pi/4)} + e^{-I(\pi/4)}\mathtt{i} + e^{I(\pi/4)}\mathtt{j} + e^{-I(\pi/4)}\mathtt{k}$, its norm is $|q| = 2$ while its pseudo-norm is $|q|_p = 0$. This problem forbids a systematic use of this pseudo-norm in biquaternion valued signal processing for obvious reasons (problems in estimating the magnitude or the energy of a signal for example). We also give the following property that will be useful in the sequel.

Property 1: Any complex number $z = z_1 + z_2 I$ with $z_1, z_2 \in \mathbb{R}$, (i.e., $z \in \mathbb{C}_I$) commutes with any biquaternion $q \in \mathbb{H}_{\mathbb{C}}$ defined as in (1), as follows:

$$qz = zq. \tag{13}$$

The proof is straightforward from the multiplication rules in (3) and (4). We now introduce some material on matrices with biquaternion valued coefficients and on their decomposition.

III. Matrices With Biquaternion Coefficients

In this section, we present definitions and properties of biquaternion valued matrices. The study of these matrices was not paid much attention to in literature. In [23], Tian proved the existence of the eigenvalues and the eigenvectors for biquaternion matrices as well as a few other properties. We present in this section the definitions necessary for our purpose and we concentrate mainly on Hermitian biquaternion matrices as they will be of interest in Section V.

A. Vectors and Matrices of Biquaternions

Biquaternions have mainly been used in formulations of electromagnetics [24] and special relativity [19], [25]. However, in such studies, the case of matrices with biquaternions coefficients has not been considered. We present here some results of a study on such matrices with particular attention to the eigendecomposition of Hermitian biquaternion matrices.

1) Biquaternion Valued Vectors: A biquaternion valued vector is an element of $\mathbb{H}_{\mathbb{C}}^N$. Equipped with the classical addition of vectors and the multiplication with a biquaternionic scalar, $\mathbb{H}_{\mathbb{C}}^N$ is a $\mathbb{H}_{\mathbb{C}} - module$ (vector space over the ring $\mathbb{H}_{\mathbb{C}}$). The scalar product of two biquaternion valued vectors $\mathbf{a}, \mathbf{b} \in \mathbb{H}_{\mathbb{C}}^N$ is defined the following way:

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbb{H}_{\mathbb{C}}} = \mathbf{a}^{\dagger}\mathbf{b} = \sum_{n=1}^{N} \overline{a}_n b_n \tag{14}$$

where $\dagger$ stands for total conjugation-transposition. With this definition, two biquaternion valued vectors $\mathbf{a}, \mathbf{b} \in \mathbb{H}_C^N$ are said *orthogonal* iff

$$\langle \mathbf{a}, \mathbf{b} \rangle_{\mathbb{H}_{\mathbb{C}}} = 0. \tag{15}$$

Based on the scalar product definition, the norm of a biquaternion valued vector $\mathbf{b} \in \mathbb{H}_{\mathbb{C}}^N$ is given by

$$||\mathbf{b}|| = \sqrt{S\left(\langle \mathbf{b}, \mathbf{b} \rangle_{\mathbb{H}_{\mathbb{C}}}\right)} \tag{16}$$

where $\mathcal{S}(.)$ is the scalar part defined in (8). We now turn to matrices with biquaternion coefficients.

2) Matrices of Biquaternions: A biquaternion valued matrix with M rows and N columns is an element of $\mathbb{H}_{\mathbb{C}}^{M\times N}$. Given a biquaternion valued matrix $\mathbf{B} = (b_{st}) \in \mathbb{H}_{\mathbb{C}}^{M\times N}$, one can define the following[23], [26]:

- the *dual* matrix of $\mathbf{B}$: $\mathbf{B}^{\triangleleft} = (b_{ts}^{\star}) \in \mathbb{H}_{\mathbb{C}}^{N\times M}$;
- the transpose-conjugate of $\mathbf{B}$: $\mathbf{B}^{\dagger} = (\overline{b}_{ts}) \in \mathbb{H}_{\mathbb{C}}^{N\times M}$.

A matrix $\mathbf{B} \in \mathbb{H}_{\mathbb{C}}^{N\times N}$ is then called *Hermitian* if $\mathbf{B} = \mathbf{B}^{\dagger}$ and unitary if $\mathbf{B}\mathbf{B}^{\dagger} = \mathbf{B}^{\dagger}\mathbf{B} = \mathbf{I}_N$. Invertibility and the definition of the inverse of a biquaternion valued matrix are defined similarly to the real or complex case. Given two matrices $\mathbf{A} \in \mathbb{H}_{\mathbb{C}}^{M\times N}$ and $\mathbf{B} \in \mathbb{H}_{\mathbb{C}}^{N\times P}$, then the following equalities stand [23], [26]:

1) $(\mathbf{A}^{\triangleleft})^{\triangleleft} = \mathbf{A}$, $(\mathbf{A}^{\dagger})^{\dagger} = \mathbf{A}$;
2) $(\mathbf{AB})^{\triangleleft} = \mathbf{B}^{\triangleleft}\mathbf{A}^{\triangleleft}$, $(\mathbf{AB})^{\dagger} = \mathbf{B}^{\dagger}\mathbf{A}^{\dagger}$;
3) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, if $\mathbf{A}$ and $\mathbf{B}$ are invertible;
4) $(\mathbf{A}^{\triangleleft})^{-1} = (\mathbf{A}^{-1})^{\triangleleft}$, $(\mathbf{A}^{\dagger})^{-1} = (\mathbf{A}^{-1})^{\dagger}$ if $\mathbf{A}$ is invertible.

These properties will be of use in the sequel.

3) Quaternionic Adjoint Matrix of a Biquaternion Valued Matrix: In order to compute the eigenvalue decomposition of a biquaternion valued matrix, we now introduce the *quaternionic adjoint matrix* of a given biquaternionic matrix. A similar technique was employed by Lee and Brenner [27] in the study of quaternion matrices. The use of such an "equivalent" quaternion matrix is possible because any Clifford algebra is isomorphic to a complex matrix algebra [28]. Consequently, any biquaternion (and by extension any matrix of biquaternions) is isomorphic to a complex matrix (by extension to a tensor product of complex matrices). (For more details on isomorphisms between complex matrices algebras and Clifford algebras, see [28, Ch. 11].)

Given a biquaternion valued matrix $\mathbf{B} \in \mathbb{H}_{\mathbb{C}}^{M\times N}$ written as $\mathbf{B}_1 + I\mathbf{B}_2$, where $\mathbf{B}_1, \mathbf{B}_2 \in \mathbb{H}^{M\times N}$, then its *quaternionic adjoint matrix*, noted $\gamma_{\mathbf{B}}$, takes values in $\in \mathbb{H}^{2M\times 2N}$ and has the following expression:

$$\gamma_{\mathbf{B}} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ -\mathbf{B}_2 & \mathbf{B}_1 \end{pmatrix}. \tag{17}$$

Consider now the complex matrix $\mathbf{\Psi}_M \in \mathbb{C}_I^{M\times 2M}$ defined as

$$\mathbf{\Psi}_M = (\mathbf{I}_M, -I\mathbf{I}_M) \tag{18}$$

where $\mathbf{I}_M$ is the identity matrix of dimension $M \times M$. It is straightforward that the following equality holds:

$$\mathbf{B} = \frac{1}{2}\mathbf{\Psi}_M \gamma_{\mathbf{B}} \mathbf{\Psi}_N^\dagger. \tag{19}$$

It is also important to notice the two following properties of the matrix $\mathbf{\Psi}_M$ that will be of use in the forthcoming calculations:

$$\mathbf{\Psi}_M \mathbf{\Psi}_M^\dagger = 2\mathbf{I}_M \tag{20}$$

$$\gamma_{\mathbf{B}} \mathbf{\Psi}_N^\dagger \mathbf{\Psi}_N = \mathbf{\Psi}_M^\dagger \mathbf{\Psi}_M \gamma_{\mathbf{B}}. \tag{21}$$

Property (20) can be demonstrated by direct calculation while, for the equality (21), it can be proved by multiplication on the left by $\mathbf{\Psi}_M$ and on the right by $\mathbf{\Psi}_N^\dagger$. Property (20) is then used to fulfill the demonstration.

Lemma 1: The *quaternion adjoint matrix* of a Hermitian biquaternion matrix is also Hermitian.

Proof: Consider a biquaternion valued Hermitian matrix $\mathbf{B} \in \mathbb{H}_{\mathbb{C}}^{N \times N}$

$$\mathbf{B} = \mathbf{B}^\dagger \tag{22}$$

and its quaternion adjoint matrix $\gamma_{\mathbf{B}} \in \mathbb{H}^{2N \times 2N}$. Substituting (19) in (22), one can write

$$\mathbf{\Psi}_N \gamma_{\mathbf{B}} \mathbf{\Psi}_N^\dagger = \left(\mathbf{\Psi}_N \gamma_{\mathbf{B}} \mathbf{\Psi}_N^\dagger\right)^\dagger. \tag{23}$$

Using the fact that for biquaternion valued matrices $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger \mathbf{A}^\dagger$, (23) becomes

$$\mathbf{\Psi}_N \gamma_{\mathbf{B}} \mathbf{\Psi}_N^\dagger = \mathbf{\Psi}_N \gamma_{\mathbf{B}}^\dagger \mathbf{\Psi}_N^\dagger \tag{24}$$

leading to

$$\gamma_{\mathbf{B}} = \gamma_{\mathbf{B}}^\dagger. \tag{25}$$

Thus $\gamma_{\mathbf{B}}$ is Hermitian. ■

In a similar way, using definition (17) and properties (20) and (21), it is possible to prove that the *quaternion adjoint matrix* conserves the unitary property of a biquaternion valued matrix.

Next, we make use of the *quaternion adjoint matrix* for the computation of the eigenvalue decomposition of a biquaternion valued matrix.

B. Eigendecomposition of a Biquaternion Valued Matrix

As in the quaternion case [29], the noncommutativity of biquaternion multiplication leads to two possible eigenvalues, namely the *left* and the *right* eigenvalues. However, in the sequel, we will only consider right eigenvalues. This choice is motivated by the link between biquaternionic right eigenvalues and quaternionic eigenvalues of the *quaternion adjoint matrix*. In the quaternion case, the theory of left eigenvalues is still not complete [30], and this motivates our choice to consider only right eigenvalues, which have been well understood for several years now [31].

After a definition of (right) EVD for biquaternion valued matrices, we present several lemmas and corrolaries that are helpful for effective computation of the eigenelements of a biquaternion matrix.

Definition 5: Given a biquaternion valued matrix $\mathbf{B} \in \mathbb{H}_{\mathbb{C}}^{N \times N}$, then its eigenvalue decomposition is given by

$$\mathbf{B} = \mathbf{U}\mathbf{D}\mathbf{U}^\dagger \tag{26}$$

where $\mathbf{U} \in \mathbb{H}_{\mathbb{C}}^{N \times 2N}$ is a biquaternion valued matrix containing the eigenvectors of $\mathbf{B}$ and $\mathbf{D} \in \mathbb{H}_{\mathbb{C}}^{2N \times 2N}$ is a diagonal matrix containing eigenvalues of $\mathbf{B}$ on its diagonal.

Next, we present some results showing how the eigenelements of a biquaternion matrix can be obtained from the eigenvalue decomposition of its *quaternion adjoint matrix*. First, the (right) eigenvectors of a square biquaternionic matrix $\mathbf{B} \in \mathbb{H}_{\mathbb{C}}^{N \times N}$ can be obtained using the following lemma.

Lemma 2: Given a square biquaternionic matrix $\mathbf{B} \in \mathbb{H}_{\mathbb{C}}^{N \times N}$, then if $\mathbf{u}_q \in \mathbb{H}^{2N}$ is a right eigenvector of its *quaternion adjoint matrix* $\gamma_{\mathbf{B}}$, then $\mathbf{u}_b \in \mathbb{H}_{\mathbb{C}}^{N}$, defined as

$$\mathbf{u}_b = \mathbf{\Psi}_N \mathbf{u}_q \tag{27}$$

is a right eigenvector of $\mathbf{B}$.

Proof: Assume $\mathbf{u}_q$ is a right eigenvector of $\gamma_{\mathbf{B}}$, then the following equality holds:

$$\gamma_{\mathbf{B}} \mathbf{u}_q = \mathbf{u}_q \lambda. \tag{28}$$

Using (19) and (27), one can write

$$\begin{aligned}\mathbf{B}\mathbf{u}_b &= \frac{1}{2}\mathbf{\Psi}_N \gamma_{\mathbf{B}} \mathbf{\Psi}_N^\dagger \mathbf{\Psi}_N \mathbf{u}_q \\ &= \frac{1}{2}\mathbf{\Psi}_N 2\mathbf{I}_N \gamma_{\mathbf{B}} \mathbf{u}_q \\ &= \mathbf{\Psi}_N \gamma_{\mathbf{B}} \mathbf{u}_q.\end{aligned} \tag{29}$$

Substituting (28) in (29) results in

$$\mathbf{B}\mathbf{u}_b = \mathbf{\Psi}_N \mathbf{u}_q \lambda = \mathbf{u}_b \lambda \tag{30}$$

so $\mathbf{u}_b = \mathbf{\Psi}_N \mathbf{u}_q$ is a right eigenvalue of $\mathbf{B}$. ■

As a result, the eigenvalue decomposition of a biquaternion valued matrix can be obtained from the eigendecomposition of a double size quaternion valued matrix, the *quaternion adjoint matrix*. As a consequence, it is possible to use algorithms developed for quaternion valued matrices for this calculation [13]. The following corollary states this fact.

Corollary 1: Consider a biquaternion valued matrix $\mathbf{B} \in \mathbb{H}_{\mathbb{C}}^{N \times N}$ and assume that its *quaternion adjoint matrix* $\gamma_{\mathbf{B}}$ has the following EVD: $\gamma_{\mathbf{B}} = \mathbf{U}\mathbf{D}\mathbf{U}^\dagger$, where $\mathbf{U} \in \mathbb{H}^{2N \times 2N}$ and $\mathbf{D} \in \mathbb{C}_{\mathtt{j}}^{2N \times 2N}$ ($\mathbb{C}_{\mathtt{j}}$ is a subset of $\mathbb{H}$, isomorphic to $\mathbb{C}$, for which the coefficients of the imaginary units $\mathtt{i}$ and $\mathtt{k}$ are null). The eigendecomposition of $\mathbf{B}$ is then given by

$$\mathbf{B} = \mathbf{U}_b \mathbf{D} \mathbf{U}_b^\dagger \tag{31}$$

where $\mathbf{U}_b = (1/\sqrt{2})\mathbf{\Psi}_N \mathbf{U} \in \mathbb{H}_{\mathbb{C}}^{N \times 2N}$ and $\mathbf{D}$ is the diagonal matrix with the eigenvalues of $\gamma_{\mathbf{B}}$ as diagonal elements.

Proof: Assuming the EVD of $\gamma_{\mathbf{B}}$ can be written as

$$\gamma_{\mathbf{B}} = \mathbf{U}\mathbf{D}\mathbf{U}^\dagger \tag{32}$$

with $\mathbf{U} \in \mathbb{H}^{2N\times 2N}$ and $\mathbf{D} \in \mathbb{C}_{\mathtt{j}}^{2N\times 2N}$, then, by replacing (32) in (19), one gets

$$\mathbf{B} = \frac{1}{2}\mathbf{\Psi}_N\mathbf{U}\mathbf{D}\mathbf{U}^\dagger\mathbf{\Psi}_N^\dagger = \frac{1}{2}\mathbf{\Psi}_N\mathbf{U}\mathbf{D}(\mathbf{\Psi}_N\mathbf{U})^\dagger. \quad (33)$$

Knowing that $(1/\sqrt{2})\mathbf{\Psi}_N\mathbf{U} = \mathbf{U}_b \in \mathbb{H}_{\mathbb{C}}^{N\times 2N}$, then it is possible to write

$$\mathbf{B} = \mathbf{U}_b\mathbf{D}\mathbf{U}_b^\dagger \quad (34)$$

where $\mathbf{D}$ is a diagonal matrix, and $\mathbf{U}_b \in \mathbb{H}_{\mathbb{C}}^{N\times 2N}$ contains the eigenvectors of $\mathbf{B}$ on its columns, as previously shown. ∎

The eigenvalues of $\gamma_{\mathbf{B}}$ are also the eigenvalues of $\mathbf{B}$. In the general case, the eigenvalues of a biquaternion valued matrix are quaternion valued. However, the possible values taken by the eigenvalues are either in $\mathbb{C}_{\mathtt{i}}$, $\mathbb{C}_{\mathtt{j}}$, or $\mathbb{C}_{\mathtt{k}}$, which are, in the biquaternion case, degenerate quaternions.[2] This means that in the biquaternion case, the eigenvalues are generally quaternions with two or three null components.

Nevertheless, it is important to notice that the eigenvalues of the *quaternion adjoint matrix* do not appear in conjugate pairs along the diagonal of $\mathbf{D}$, as opposed to the quaternion case where this happens for the eigenvalues of the *complex adjoint matrix* [29], [31]. As a consequence, it is necessary to consider all the $2N$ eigenvectors and their associated eigenvalues to rebuild a whole biquaternionic matrix $\mathbf{B} \in \mathbb{H}_{\mathbb{C}}^{N\times N}$.

Note that in the case of symmetric octonion[3] valued matrices, it has been demonstrated that a 3×3 matrix has six independent eigenvalues [32].

An interpretation to this large number of eigenvalues can be given using isomorphisms. It has been shown that the algebra of complexified quaternions is identical to that generated by Pauli matrices (elements of $\mathbb{C}^{2\times 2}$) [19], [28]. The space of biquaternion valued matrices $\mathbb{H}_{\mathbb{C}}^{N\times N}$ is then isomorphic to $\mathbb{C}^{2\times 2} \otimes \mathbb{R}^{N\times N}$, where $\otimes$ denotes the tensor product of two vector-spaces. As a consequence the dimension of the column vector space of $\mathbb{H}_{\mathbb{C}}^{N\times N}$ is given by

$$dim\left(\mathbb{H}_{\mathbb{C}}^{N\times N}\right) = dim(\mathbb{C}^{2\times 2})dim(\mathbb{R}^{N\times N}) = 2N. \quad (35)$$

1) EVD of a Hermitian Biquaternionic Matrix: The high-resolution vector-sensor array processing algorithm presented in Section V is based on the decomposition of the covariance matrix of the observations into orthogonal subspaces, using a biquaternion model. This covariance matrix is biquaternionic Hermitian. Consequently, we now pay attention to the EVD of a Hermitian biquaternion valued matrix.

A matrix $\mathbf{B} \in \mathbb{H}_{\mathbb{C}}^{N\times N}$ is called Hermitian if $\mathbf{B} = \mathbf{B}^\dagger$. We have already demonstrated (Lemma 1) that the *quaternion adjoint matrix* $\gamma_{\mathbf{B}} \in \mathbb{H}^{2N\times 2N}$ of a Hermitian biquaternionic matrix is also Hermitian. Thus, $\gamma_{\mathbf{B}} = \gamma_{\mathbf{B}}^\dagger$.

As the eigenvalues of $\mathbf{B}$ are the same as the ones of $\gamma_{\mathbf{B}}$, and due to the fact that the eigenvalues of a Hermitian quaternion valued matrix are real valued [29], then the eigenvalues of a Hermitian biquaternion valued matrix are real as well. It is easy to demonstrate (see [33] for the quaternion case) that for Hermitian matrices, the right and left eigenvalues (and associated eigenvectors) are the same. We now prove that an important lemma, well known for the real, complex, and quaternionic case, extends to biquaternions. This is fundamental for the construction of any algorithm based on orthogonal decomposition of the observed data.

Lemma 3: Given a Hermitian biquaternion valued matrix $\mathbf{B}$, then any two of its eigenvectors corresponding to two different eigenvalues are orthogonal.

Proof: Consider two eigenvalues of $\mathbf{B} \in \mathbb{H}_{\mathbb{C}}^{N\times N}$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$ and their associated eigenvectors $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{H}_{\mathbb{C}}^{N}$. Then, one can write

$$\begin{aligned}\lambda_1\left(\mathbf{u}_1^\dagger\mathbf{u}_2\right) &= (\mathbf{u}_1\lambda_1)^\dagger\mathbf{u}_2 = (\mathbf{B}\mathbf{u}_1)^\dagger\mathbf{u}_2 = \mathbf{u}_1^\dagger\mathbf{B}^\dagger\mathbf{u}_2\\ &= \mathbf{u}_1^\dagger(\mathbf{B}\mathbf{u}_2) = \mathbf{u}_1^\dagger(\mathbf{B}\mathbf{u}_2) = \left(\mathbf{u}_1^\dagger\mathbf{u}_2\right)\lambda_2. \quad (36)\end{aligned}$$

As $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$, then equality (36) involves $\mathbf{u}_1^\dagger\mathbf{u}_2 = 0$, which means that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal. ∎

The following numerical example illustrates the link between the rank of a biquaternion valued matrix and its eigenvalue decomposition. Consider a biquaternion valued vector of dimension 3, $\mathbf{s} \in \mathbb{H}_{\mathbb{C}}^{3}$, given as (37), shown at the bottom of the page.

Then, the following matrix is Hermitian:

$$\mathbf{S} = \mathbf{s}\mathbf{s}^\dagger \in \mathbb{H}_{\mathbb{C}}^{3\times 3}. \quad (38)$$

Using the classical definition for the rank of a matrix, by construction $\mathbf{S}$ has a rank equal to 1. The eigendecomposition of $\mathbf{S}$ gives two different non-null real eigenvalues: $\lambda_1 = 5.918$ and $\lambda_2 = 4.166$. The remaining four other eigenvalues are null. The eigenvectors associated to the non-null eigenvalues are $\mathbf{u}_1 \in \mathbb{H}_{\mathbb{C}}^{N}$ and $\mathbf{u}_2 \in \mathbb{H}_{\mathbb{C}}^{N}$ and have the numerical values of (39) and (40), shown at the bottom of the next page.

It can be directly verified by calculation that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal. Thus, the eigendecomposition of $\mathbf{S}$ can be written as

$$\mathbf{S} = \sum_{k=1}^{2}\lambda_k\mathbf{u}_k\mathbf{u}_k^\dagger \quad (41)$$

[2]Note that the biquaternion case is different from the quaternion case; as for the latter, the eigenvalues of quaternion matrices are isomorphic to complex eigenvalues.

[3]Octonions are the only 8-D normed division algebra [20]. They form a nonassociative and noncommutative algebra.

$$\mathbf{s} = \begin{pmatrix} 0.950+0.486I & & 0.456+0.444I & & 0.921+0.405I & & 0.410+0.352I\\ 0.231+0.891I & +\mathtt{i} & 0.018+0.615I & +\mathtt{j} & 0.738+0.935I & +\mathtt{k} & 0.893+0.813I\\ 0.606+0.762I & & 0.821+0.791I & & 0.176+0.916I & & 0.057+0.009I \end{pmatrix}. \quad (37)$$

Now, comparing (41) and (38), one remarks that in order to recover the information contained in $\mathtt{s}$, it is necessary to consider two eigenvalues and their associated biquaternion eigenvectors. This result will be used in the vector-sensor HR array processing algorithm derived in Section V.

We saw that in order to be consistent with the real, complex, and quaternion valued matrix theory, the classical definition of the rank of a matrix needs to be revisited. Thus, the following definition stands for rank definition for biquaternion valued matrices.

Definition 6: The rank of a biquaternionic matrix $\mathbf{B} \in \mathbb{H}_{\mathbb{C}}^{N \times M}$ is given by

$$\text{rank}(\mathbf{B}) = \frac{1}{2}\text{rank}(\gamma_{\mathbf{B}}). \tag{42}$$

Now, with the presented material on spectral decomposition of biquaternion matrices and the matrix algebra tools over $\mathbb{H}_{\mathbb{C}}$, we are ready for developing our biquaternionic model and the algorithm for vector-sensor array processing purpose.

IV. Polarized Signal Model Using Biquaternions

Following the approach proposed in [14] for the processing of signals recorded on two-component vector-sensors, we introduce a biquaternionic model for polarized signals recorded on three-component vector-sensors.

A. Three-Component Vector-Sensor Signals

Consider a three-component vector-sensor, recording the three orthogonal components of an incident vector wave field, yielding the output signals $s_{c_1}(t)$, $s_{c_2}(t)$, and $s_{c_3}(t)$. The three components of the vector-sensor define an orthogonal basis in the Euclidean 3-D space. If $(O, \vec{x}, \vec{y}, \vec{z})$ is the orthonormal basis associated to the vector-sensor, the vector product relations between the unit vectors $\vec{x}, \vec{y}, \vec{z}$ fit perfectly the relationships between the quaternionic units $\mathtt{i}, \mathtt{j}, \mathtt{k}$ [see (3)]. Thus, the idea of using quaternions/biquaternions to model the signals recorded on the three components of a vector-sensor comes naturally.

The associated three-components pure quaternion valued signal $s(t)$ is then given by

$$\mathtt{s}(t) = \mathtt{s}_{c_1}(t)\mathtt{i} + \mathtt{s}_{c_2}(t)\mathtt{j} + \mathtt{s}_{c_3}(t)\mathtt{k}. \tag{43}$$

Defining the Fourier transform of $\mathtt{s}(t)$ as a triplet of complex Fourier transforms applied separately on each of the three components, one gets

$$s(\nu) = s_{c_1}(\nu)\mathtt{i} + s_{c_2}(\nu)\mathtt{j} + s_{c_3}(\nu)\mathtt{k} \tag{44}$$

where $s_{c_\alpha}(\nu) = \mathtt{TF}[\mathtt{s}_{c_\alpha}(t)]$, with $\alpha = 1, 2, 3$ and with the Fourier transform taking values in $\mathbb{C}_I$. Using the modulus-phase representation, (44) can be rewritten as

$$s(\nu) = \eta_1(\nu)e^{I\chi_1(\nu)}\mathtt{i} + \eta_2(\nu)e^{I\chi_2(\nu)}\mathtt{j} + \eta_3(\nu)e^{I\chi_3(\nu)}\mathtt{k} \tag{45}$$

where $\eta_1, \eta_2, \eta_3 \in \mathbb{R}$ are the amplitudes and $\chi_1, \chi_2, \chi_3 \in \mathbb{R}$ are the phases of the signals recorded on the three components. In the following, the frequency argument ν is omitted for clarity as the proposed algorithm is derived for narrowband signals, or it is applied at different frequencies independently. Considering the first component as reference, one can rewrite the biquaternion signal as the product between a pure biquaternion containing the relative amplitude ratios and the phase shifts of the second and the third components with respect to the first component, and a complex number representing the absolute amplitude and phase of the signal on the first component, as follows:

$$s = p(\rho_1, \varphi_1, \rho_2, \varphi_2)\eta_1 e^{I\chi_1}. \tag{46}$$

The expression for $p(\rho_1, \varphi_1, \rho_2, \varphi_2)$ is given by

$$p(\rho_1, \varphi_1, \rho_2, \varphi_2) = \mathtt{i} + \rho_1 e^{I\varphi_1}\mathtt{j} + \rho_2 e^{I\varphi_2}\mathtt{k} \tag{47}$$

with $\rho_1 = \eta_2/\eta_1$, $\rho_2 = \eta_3/\eta_1$ and $\varphi_1 = \chi_2 - \chi_1$, $\varphi_2 = \chi_3 - \chi_1$. In this model, $p(\rho_1, \varphi_1, \rho_2, \varphi_2)$ contains the polarization information of the signal, if we consider the first component as reference.

B. Polarized Plane Waves

Now, given a set of N_x equally spaced three-component vector-sensors, recording the contributions of L polarized plane waves, using the biquaternion model, the recorded signal $\mathbf{x} \in \mathbb{H}_{\mathbb{C}}^{N_x}$ is given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N_x} \end{bmatrix} = \sum_{l=1}^{L} p_l(\rho_{1l}, \varphi_{1l}, \rho_{2l}, \varphi_{2l})\mathbf{a}(\theta_l)s_l + \mathbf{n} \tag{48}$$

where $p_l(\rho_{1l}, \varphi_{1l}, \rho_{2l}, \varphi_{2l})$ is the biquaternion valued polarization coefficient of the l^{th} wave, containing its polarization parameters, $\mathbf{a}(\theta_l)$ is the propagation vector of the l^{th} wave on the array and is given (assuming plane waves contributions only) by

$$\mathbf{a}(\theta_l) = \begin{bmatrix} 1 & e^{-I\theta_l} & \dots & e^{-I(N_x-1)\theta_l} \end{bmatrix}^T. \tag{49}$$

The vector $\mathbf{n} \in \mathbb{H}_{\mathbb{C}}^{N_x}$ contains unpolarized noise contributions on the vector-sensor array. Also, the s_l coefficients correspond

$$\mathbf{u}_1 = \begin{pmatrix} 0.333 \\ 0.347 + 0.366I \\ 0.287 + 0.036I \end{pmatrix} + \mathtt{i}\begin{pmatrix} 0.095I \\ 0.125 - 0.087I \\ 0.1899 - 0.196I \end{pmatrix} + \mathtt{j}\begin{pmatrix} 0.031I \\ 0.021 - 0.011I \\ 0.116 + 0.162I \end{pmatrix} + \mathtt{k}\begin{pmatrix} 0.317I \\ 0.344 - 0.337I \\ -0.030 - 0.258I \end{pmatrix} \tag{39}$$

$$\mathbf{u}_2 = \begin{pmatrix} -0.429 \\ -0.356 - 0.039I \\ -0.307 - 0.290I \end{pmatrix} + \mathtt{i}\begin{pmatrix} -0.306I \\ 0.045 - 0.245I \\ 0.150 - 0.201I \end{pmatrix} + \mathtt{j}\begin{pmatrix} -0.011I \\ 0.013 - 0.033I \\ 0.035 + 0.071I \end{pmatrix} + \mathtt{k}\begin{pmatrix} -0.300I \\ 0.010 - 0.257I \\ 0.259 - 0.236I \end{pmatrix}. \tag{40}$$

to the magnitude contribution of the $l^{\rm th}$ wave (at a fixed frequency). In the following, we use the notation

$$\mathbf{d}_l(\theta_l, \rho_{1l}, \varphi_{1l}, \rho_{2l}, \varphi_{2l}) = p_l(\rho_{1l}, \varphi_{1l}, \rho_{2l}, \varphi_{2l})\mathbf{a}(\theta_l) \quad (50)$$

where $\mathbf{d}_l$ is called the *polarized steering vector* of the $l^{\rm th}$ wave and so that the observations can be written as

$$\mathbf{x} = \sum_{l=1}^{L} \mathbf{d}_l s_l + \mathbf{n}. \quad (51)$$

The biquaternion observation vector $\mathbf{x} \in \mathbb{H}_{\mathbb{C}}^{N_x}$ is built from the observations (in frequency domain) $\mathbf{x}_{c_1}, \mathbf{x}_{c_2}, \mathbf{x}_{c_3}$ on the three components as

$$\mathbf{x} = \mathbf{x}_{c_1}\mathtt{i} + \mathbf{x}_{c_2}\mathtt{j} + \mathbf{x}_{c_3}\mathtt{k}. \quad (52)$$

C. Long-Vector Approach

As a comparison, the long-vector approach classically used in vector-sensor array processing [2], [3] makes use of the concatenated vector $\mathbf{x}_{lv}$ built the following way:

$$\mathbf{x}_{lv} = \left[\mathbf{x}_{c_1}^T | \mathbf{x}_{c_2}^T | \mathbf{x}_{c_3}^T\right]^T \quad (53)$$

with $\mathbf{x}_{lv} \in \mathbb{C}^{3N_x}$. The long-vector approach allows, with additional parametrization, the use of classical matrix algebra algorithms and was used to define MUSIC- and ESPRIT-like algorithms for vector-sensor arrays [3], [8]. However, the use of long vectors has some drawbacks, such as leading to "over computation" and breaking the local polarized structure of the data. This last point has no deep consequences in the presented algorithm but could be of importance in more complicated ones, for example, if higher order statistics (HOS) are used. The use of long vectors in a processing involving HOS would lead to (highly) complicated structures in tensor valued cumulants or cost functions. We claim here that the use of hypercomplex numbers (and more generally the use of geometric numbers/algebras) can lead to easier manipulation of vector valued signals.

V. Biquaternion MUSIC Estimator

The BQ-MUSIC algorithm is based on the decomposition of the biquaternionic spectral matrix of the observation data vector $\mathbf{x}$ into signal and noise orthogonal subspaces. Using the modelization and linear algebra tools previously presented, we derive in the sequel an expression for this new BQ-MUSIC estimator.

A. Biquaternionic Spectral Matrix

Since second-order statistics of the observed data are used in the BQ-MUSIC, we now introduce the biquaternionic spectral matrix. All the biquaternion valued signals are considered centered here.

1) Definition: Considering that the output of the vector-sensor array is $\mathbf{x} \in \mathbb{H}_{\mathbb{C}}^{N_x}$ given in (51), then the spectral matrix is defined as

$$\mathbf{\Lambda} = \mathbb{E}[\mathbf{x}\mathbf{x}^\dagger] \in \mathbb{H}_{\mathbb{C}}^{N_x \times N_x} \quad (54)$$

The mathematical expectation $\mathbb{E}[.]$ is defined naturally over $\mathbb{H}_{\mathbb{C}}$, just like it is done over $\mathbb{C}$ or $\mathbb{H}$ [34]. Substituting (51) in (54) and assuming decorrelation between the different sources (i.e., $\mathbb{E}[s_f \overline{s_v}] = 0$ for $f \neq v$) and between sources and noise (i.e., $\mathbb{E}[s_f \overline{n_v}] = 0 \; \forall f, v$), the biquaternionic spectral matrix takes the following form:

$$\mathbf{\Lambda} = \sum_{l=1}^{L} \sigma_l^2 \mathbf{d}_l \mathbf{d}_l^\dagger + \mathbf{\Lambda}_\mathbf{n} \quad (55)$$

where σ_l^2 are the powers of the L sources on the antenna and $\mathbf{d}_l \in \mathbb{H}_{\mathbb{C}}^{N_x}$ are the *biquaternionic source vectors* describing source contributions on the antenna. The matrix $\mathbf{\Lambda}_\mathbf{n}$ is given by $\mathbf{\Lambda}_\mathbf{n} = \mathbb{E}[\mathbf{n}\mathbf{n}^\dagger] = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_{N_x}^2)$, where $\sigma_f^2 = \mathbb{E}[n_f \overline{n_f}]$ is the power of the noise on the $f^{\rm th}$ sensor. In order to build a MUSIC estimator, it is necessary to decompose the observation data spectral matrix into orthogonal subspaces, using the algorithm derived in Section III-B-1).

B. BQ-MUSIC Estimator

As presented in (50), every polarized wave impinging on the vector-sensor array is parametrized by five parameters, and the proposed version of MUSIC aims to estimate the five of them simultaneously. In order to do so, and as usual in MUSIC approach, a parametrized steering vector is projected onto the *noise* subspace built using the last eigenvectors of the spectral matrix of the observations. The biquaternionic steering vector has the following expression:

$$\mathbf{f}(\Omega) = \frac{1}{\mathcal{N}} \begin{bmatrix} p & pe^{-I\theta} & \ldots & pe^{-I(N_x-1)\theta} \end{bmatrix}^T \quad (56)$$

where

$$\begin{aligned} \Omega &= \{\theta, \rho_1, \rho_2, \varphi_1, \varphi_2\} \\ p &= \mathtt{i} + \rho_1 e^{I\varphi_1}\mathtt{j} + \rho_2 e^{I\varphi_2}\mathtt{k} \\ \mathcal{N} &= \sqrt{N_x(1 + \rho_1^2 + \rho_2^2)}. \end{aligned}$$

Then, the BQ-MUSIC consists of finding the set of parameters Ω that maximizes the following functional:

$$\mathcal{F}(\Omega) = \frac{1}{\mathbf{f}^\dagger(\Omega)\mathbf{\Pi}_\mathbf{N}\mathbf{f}(\Omega)} \quad (57)$$

where $\mathbf{\Pi}_\mathbf{N} = \sum_{p=2L+1}^{2N_x} \mathbf{u}_p \mathbf{u}_p^\dagger$, built with the last $2(N_x - L)$ eigenvectors of $\mathbf{\Lambda}$ is the orthogonal biquaternionic projector on noise subspace. One can see that the use of hypercomplex numbers allows an estimator expression very similar to scalar-valued signal, without any "additional" structure in the projector, except the algebra on which it is expressed. The functional $\mathcal{F}(\Omega)$ has maxima for the values of the parameters corresponding to polarized plane waves that have impinged on the local vector-sensor array. In the case where those parameters are unknown, finding these maxima will consist in finding the extrema of a 5-D surface. The use of a biquaternionic formulation for polarized MUSIC estimator has not been studied for this optimization problem. Consequently, the presented study does not allow to conclude on possible advantages of the proposed approach among others on this aspect of the algorithm. We present next some results for the *long-vector* and the *biquaternion* approaches regarding computational and orthogonality issues.

C. Computational Issues

If the three-component *long-vector* model (53) is used, the spectral matrix is complex of size $3N_x \times 3N_x$. Compared to this long-vector matrix having $9N_x^2$ complex entries, the spectral matrix in the biquaternionic approach has N_x^2 biquaternion-valued coefficients. As a biquaternion is composed of four complex numbers, the biquaternion spectral matrix can thus be represented on $4N_x^2$ complex values. This way, the memory requirements for data covariance representation are reduced by a factor of 4/9, provided that a biquaternion model is used.

D. Orthogonality Issues

As we saw in Section V-B, BQ-MUSIC algorithm is based on the orthogonality between biquaternion-vectors. We show next that this orthogonality constraint implies stronger relationships between the three components of the signal than the long-vector approach does. Consider two biquaternionic vectors $\mathbf{x}, \mathbf{y} \in \mathbb{H}_{\mathbb{C}}^{N_x}$, with their expressions given by

$$\begin{cases} \mathbf{x} = \mathbf{x}_{c_1}\mathtt{i} + \mathbf{x}_{c_2}\mathtt{j} + \mathbf{x}_{c_3}\mathtt{k} \\ \mathbf{y} = \mathbf{y}_{c_1}\mathtt{i} + \mathbf{y}_{c_2}\mathtt{j} + \mathbf{y}_{c_3}\mathtt{k}. \end{cases} \tag{58}$$

The corresponding long-vector representations [see (53)] are $\mathbf{x}_{lv}, \mathbf{y}_{lv} \in \mathbb{C}^{3N_x}$, as follows:

$$\mathbf{x}_{lv} = \begin{pmatrix} \mathbf{x}_{c_1} \\ \mathbf{x}_{c_2} \\ \mathbf{x}_{c_3} \end{pmatrix} \quad \text{and} \quad \mathbf{y}_{lv} = \begin{pmatrix} \mathbf{y}_{c_1} \\ \mathbf{y}_{c_2} \\ \mathbf{y}_{c_3} \end{pmatrix}. \tag{59}$$

By imposing the orthogonality constraint for the biquaternion vectors

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{H}_{\mathbb{C}}} = 0 \tag{60}$$

one gets the following relationships between the complex components:

$$\mathbf{y}_{c_1}^{\dagger}\mathbf{x}_{c_1} + \mathbf{y}_{c_2}^{\dagger}\mathbf{x}_{c_2} + \mathbf{y}_{c_3}^{\dagger}\mathbf{x}_{c_3} = 0 \tag{61}$$

$$\mathbf{y}_{c_3}^{\dagger}\mathbf{x}_{c_2} = \mathbf{y}_{c_2}^{\dagger}\mathbf{x}_{c_3} \tag{62}$$

$$\mathbf{y}_{c_1}^{\dagger}\mathbf{x}_{c_3} = \mathbf{y}_{c_3}^{\dagger}\mathbf{x}_{c_1} \tag{63}$$

$$\mathbf{y}_{c_2}^{\dagger}\mathbf{x}_{c_1} = \mathbf{y}_{c_1}^{\dagger}\mathbf{x}_{c_2}. \tag{64}$$

The orthogonality constraint for the long-vector approach

$$\langle \mathbf{x}_{lv}, \mathbf{y}_{lv} \rangle_{\mathbb{C}} = 0 \tag{65}$$

yields only (61), implying that

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{H}_{\mathbb{C}}} = 0 \quad \Rightarrow \quad \langle \mathbf{x}_{lv}, \mathbf{y}_{lv} \rangle_{\mathbb{C}} = 0. \tag{66}$$

The reciprocal is not always true meaning that the biquaternionic orthogonality imposes stronger constraints between the components of the vector-sensor array, and implicitly between the signal and noise subspaces. This affects in a positive way the robustness of BQ-MUSIC algorithm to different kinds of errors as we show in the next section.

The following section compares some simulation results on the resolution and robustness of the BQ-MUSIC estimator to the long-vector approach and to PSA-MUSIC proposed by Rahamim [15], which uses the polarization information to improve the spectral matrix conditioning.

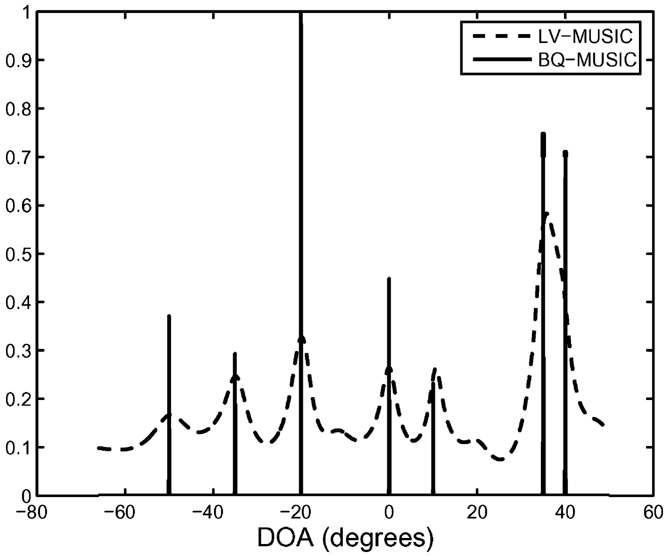

Fig. 1. Robustness to polarization parameters errors.

VI. Simulation Results

By maximizing the functional (57) over the five parameters simultaneously, it is possible to jointly estimate the DOA and the polarization parameters for the sources impinging on the antenna. For computational power reasons, we supposed in this section that the polarization parameters were known or they have been estimated previously and we focused only on the estimation of the direction of arrival parameter θ. In practice, this situation corresponds to DOAs estimation for sources of known polarization, as it is often the case in electromagnetics. Before presenting the simulation results, notice that on an array of N_x three-component vector-sensors, the BQ-MUSIC algorithm allows detection of maximum $N_x - 1$ sources while the long-vector approach (LV-MUSIC) detects a maximum number of $3N_x - 1$ sources. This reduction of the signal subspace dimension is directly related to the fact that a stronger orthogonality constraint is imposed between signal and noise subspaces (as shown in Section V-D). On the other hand, this stronger constraint increases the algorithm robustness to noise, model errors and polarization parameters estimation errors as we show in simulations.

First, we consider an array of 20 vector-sensors and seven sources of known polarization parameters $(\rho_1^{\iota}, \rho_2^{\iota}, \varphi_1^{\iota}, \varphi_2^{\iota})$, $\iota = 1 \ldots 7$ impinging on the antenna. The simulated DOAs for the sources are as follows: $-50°$, $-35°$, $-20°$, $0°$, $10°$, $35°$, $40°$, and the SNR $= 30$ dB. If the polarization parameters are correctly estimated, the two algorithms (BQ-MUSIC and LV-MUSIC) perform identically well. For the plots in Fig. 1, we supposed that the estimated polarization parameters were slightly biased (the perturbation bias has a equal to 5% of the norm of the original vector). The DOA detection results for the two algorithms are presented. The detection curves corresponding to each of the seven sources were superposed in order to have all results on the same plot (Fig. 1). The long-vector approach undergoes a serious loss in resolution power, failing to discriminate sources 6 and 7, while BQ-MUSIC performs a

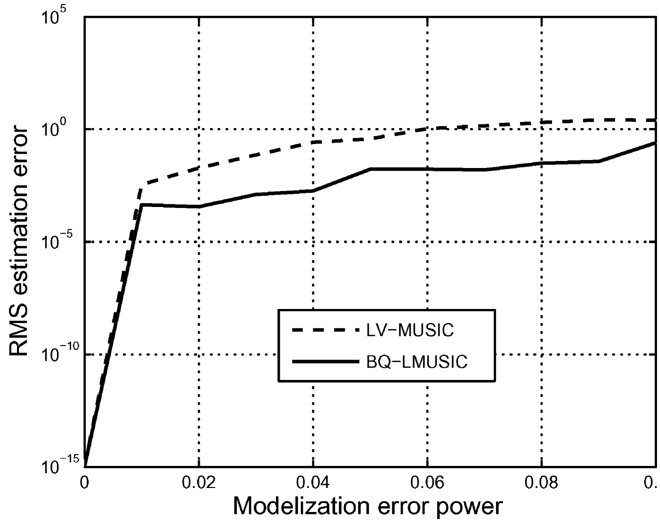

Fig. 2. RMS estimation error for modelization errors.

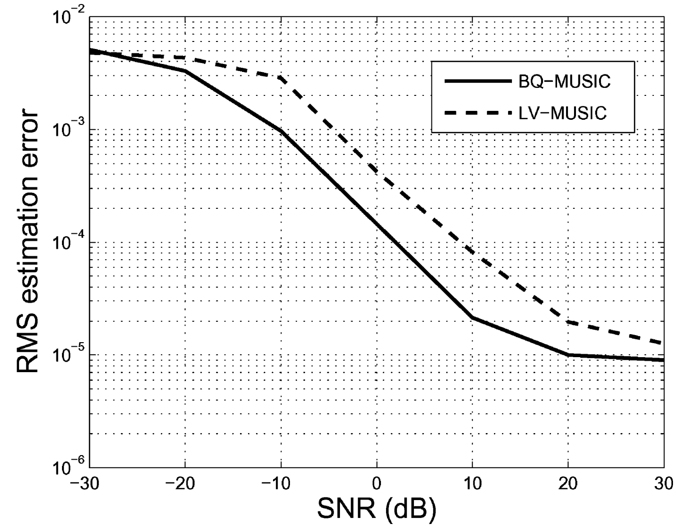

Fig. 4. RMS estimation error for one source in the presence of coherent noise.

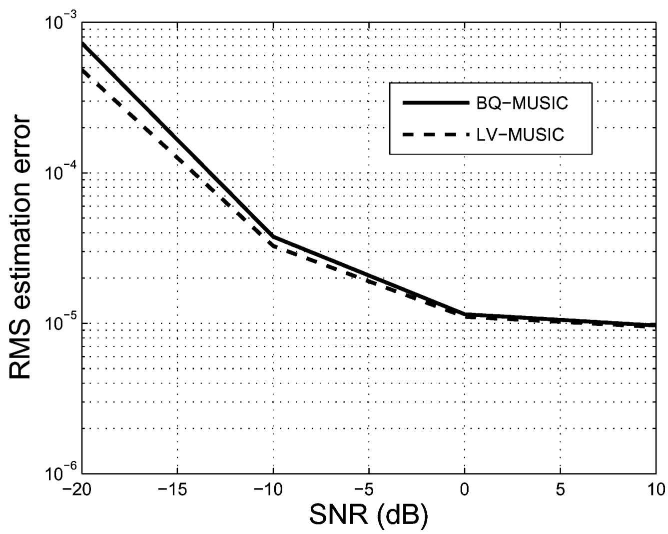

Fig. 3. RMS estimation error for one source in the presence of noncoherent noise.

very accurate detection, proving to be more robust to polarization parameters estimation errors.

The second simulation tests the robustness of the algorithms to modelization errors. The same set of sources is considered as before. We assume that the model used for the source vectors $\mathbf{d}_l$ [(50)] is not accurate and we modeled this lack of knowledge by an additive Gaussian noise of variable power. Fig. 2 plots the root-mean-square (RMS) error for the estimation of the DOA of source number 4 ($\theta = 0^\circ$), versus the energy of the noise corrupting the model. For each point on the image, 100 runs were used. As expected, for a perfectly fitting model, the errors for the two methods approach zero. As the error increases, BQ-MUSIC overperforms the classical approach and seems to be more robust to modelization errors.

Fig. 3 illustrates the behavior of the two algorithms to noncoherent noise on the sensors. A scenario with one source of DOA 10°, impinging on a ten-vector-sensors array was considered. We supposed that the snapshots were corrupted by additive Gaussian, nonpolarized, spatially white noise. The polarization parameters of the source are supposed perfectly known. For each point, 100 runs were used. We plotted the RMS estimation error for the source DOA estimation versus the SNR. The proposed algorithm performs fairly well compared with the long-vector approach, with only a slight loss of accuracy for very low SNR.

In addition, we tested the robustness of the BQ-MUSIC algorithm to coherent noise as it is well known that this is the weak point of MUSIC-like algorithms. We considered the same configuration as before, but this time, the additive noise is coherent along the array and on the three components. Noncoherent noise was also injected with a signal-to-noise ratio of 0 dB. The results of the simulation are presented in Fig. 4 which plots the estimation error for the DOA of the source versus the signal-to-coherent-noise ratio. The BQ-MUSIC algorithm proves to be more robust to this kind of errors than its long-vector version. The strange form of the detection curves for low SNR (<10 dB) can be explained by the fact that when the coherent noise becomes important, it behaves as an interfering source, biasing the signal subspace estimation and strongly perturbing the detection of the targeted source. For high values of SNR, noncoherent noise becomes more important than the coherent one, and we fall into the configuration previously studied.

As we mentioned at the beginning of this section, the "long-vector" approach allows the detection of maximum number of sources almost three times larger than BQ-MUSIC; therefore, the comparison between algorithms is not completely fair. In the sequel, we compare in simulations BQ-MUSIC with PSA-MUSIC [15] a high-resolution technique based on PSA. The idea behind this algorithm is to use the polarization information to improve the estimation of the spectral matrix, by averaging over the three components of the antenna. As a result, the information on the polarization parameters is lost, which is not the case for LV-MUSIC and BQ-MUSIC; the maximum number of detectable sources is $N_x - 1$ (the same as BQ-MUSIC).

We considered two scenarios, the first with six sources impinging on a seven-vector-sensor array (Fig. 5) and the second with only one source (Fig. 6). In the first case, the sources have

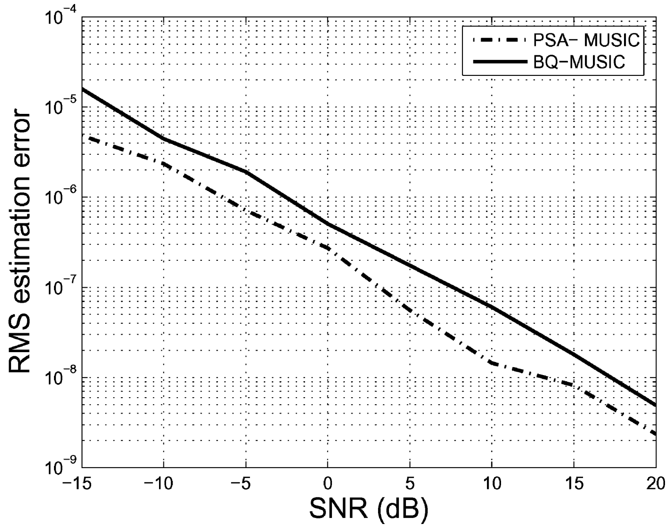

Fig. 5. RMS estimation error in the presence of six sources recorded on seven sensors, for different SNR.

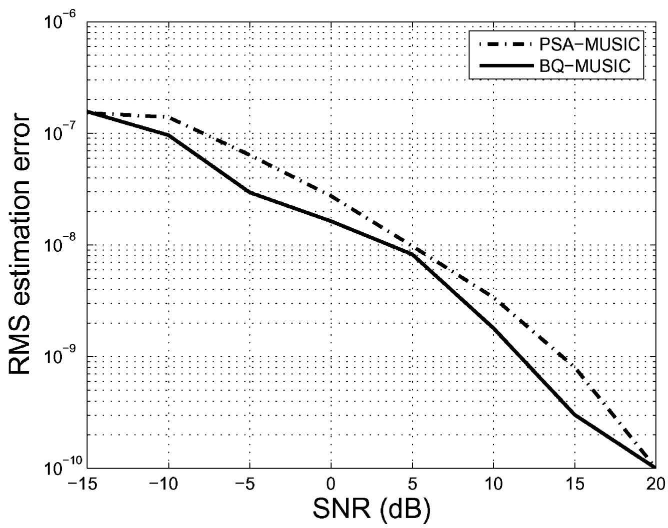

Fig. 6. RMS estimation error in the presence of one source recorded on seven sensors, for different SNR.

different polarizations and their DOAs are -50°, -35°, -20°, 0°, 10°, 35°, 40°. In Fig. 5, we plotted the RMS estimation error for the forth source with respect to the SNR (in decibels). In the second, the same curve was plotted (Fig. 6), assuming the presence of only one source of DOA equal to -20° in the recorded data. For each point on the figures, 100 runs were used, and the spectral matrix was estimated with 50 samples. One can remark that when the number of sources is large (equal to the limit of MUSIC algorithm), PSA-MUSIC performs better than the biquaternion algorithm. This can be explained by the fact that the estimation of the spectral matrix is more accurate in the case of PSA-MUSIC, because the number of samples used for estimation is three times larger than for BQ-MUSIC algorithm (in the case of PSA the three components can be assimilated to three snapshots). However, when the noise subspace dimension grows, the biquaternion orthogonality constraint prevails and BQ-MUSIC behaves better than PSA-MUSIC (Fig. 6). The main advantage of our algorithm over PSA-MUSIC is the preservation of the polarization information of the sources.

Meanwhile, if the polarization parameters are unknown, the performance of BQ-MUSIC is expected to degrade. A version of BQ-MUSIC including the estimation of polarization information will be the focus of future work.

VII. Conclusion

In this paper, we proposed a MUSIC-like algorithm (BQ-MUSIC) for three-component vector-sensor array processing, based on *biquaternions*. The performances of this algorithm are compared in simulations to the classical approach (LV-MUSIC) based on the concatenation of the three components in a *long vector* and with PSA-MUSIC, which performs an average over the three components. Furthermore, we present a technique for the decomposition of biquaternion-valued matrices into eigenelements.

The BQ-MUSIC algorithm is based on a quaternionic model of a polarized source, and it is well adapted to the acquisition geometry. The use of this model preserves the polarization information and imposes a stronger orthogonality constraint between the signal and noise subspaces. As a result, the proposed method proves to be more robust to coherent noise, modelization errors, and polarization parameters estimation errors. Nevertheless, the use of biquaternions provides a more compact and elegant way of handling multicomponent signals.

Also, this paper illustrates the high potentiality of high-dimensional algebras (and especially geometric algebras) to model complex-structured data in signal processing.

References

[1] K. T. Wong and M. D. Zoltowski, "Uni-vector-sensor ESPRIT for multi-source azimuth, elevation and polarization estimation," *IEEE Trans. Antennas Propag.*, vol. 45, no. 10, pp. 1467–1474, 1997.

[2] A. Nehorai and E. Paldi, "Vector-sensor array processing for electromagnetic source localisation," in *Proc. 25th Asilomar Conf. Signals, Systems, Computers*, Pacific Grove, CA, 1991, pp. 566–572.

[3] A. Nehorai and E. Paldi, "Vector-sensor array processing for electromagnetic source localization," *IEEE Trans. Signal Process.*, vol. 42, no. 2, pp. 376–398, Feb. 1994.

[4] K. T. Wong, L. Li, and M. D. Zoltowski, "Root-MUSIC-based direction-finding & polarization-estimation using diversely-polarized possibly-collocated antennas," *IEEE Antennas Wireless Propag. Lett.*, vol. 12, no. 8, pp. 129–132, 2004.

[5] K. T. Wong and M. D. Zoltowski, "Root-MUSIC-based azimuth-elevation angle of arrival estimation with uniformly spaced but arbitrarily oriented velocity hydrophones," *IEEE Trans. Signal Process.*, vol. 47, no. 12, pp. 3250–3260, Dec. 1999.

[6] K. T. Wong and M. D. Zoltowski, "Self-initiating MUSIC direction finding and polarization estimation in spatio-polarizational beamspace," *IEEE Trans. Antennas Propag.*, vol. 48, no. 8, pp. 1235–1245, 2000.

[7] M. D. Zoltowski and K. T. Wong, "ESPRIT-based 2-D direction finding with a sparse array of electromagnetic vector-sensors," *IEEE Trans. Signal Process.*, vol. 48, no. 8, pp. 2195–2204, Aug. 2000.

[8] J. Li and T. Compton, "Angle estimation using a polarization sensitive array," *IEEE Trans. Antennas Propagat.*, vol. 39, pp. 1539–1543, 1991.

[9] J. Li and T. Compton, "Performance analysis for angle and polarization estimation using ESPRIT," in *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing*, Apr. 1992, pp. V417–V420.

[10] J. Li and T. Compton, "Two-dimensional angle and polarization estimation using the ESPRIT algorithm," *IEEE Trans. Antennas Propag.*, vol. 40, no. 5, pp. 550–555, May 1992.

[11] J. Li and T. Compton, "Angle and polarization estimation in a coherent signal environment," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 29, no. 3, pp. 706–716, Jul. 1993.

[12] A. J. Weiss and B. Friedlander, "Performance analysis of diversely polarized antenna arrays," *IEEE Trans. Signal Process.*, vol. 39, no. 7, pp. 1589–1603, Jul. 1991.

[13] N. Le Bihan and J. Mars, “Singular value decomposition of quaternion matrices: A new tool for vector-sensor signal processing,” *Signal Process.*, vol. 84, no. 7, pp. 1177–1199, 2004.
[14] S. Miron, N. Le Bihan, and J. Mars, “Quaternion-music for vector-sensor array processing,” *IEEE Trans. Signal Process.*, vol. 4, no. 54, pp. 1218–1229, 2006.
[15] D. Rahamim, J. Tabrikian, and R. Shavit, “Source localization using vector sensor array in multipath environment,” *IEEE Trans. Signal Process.*, vol. 52, no. 11, pp. 3096–3103, Nov. 2004.
[16] J. Xavier and J. Manton, “On the generalization of AR processes to Riemannian manifolds,” in *IEEE Int. Conf. Acoustics, Speech, Signal Processing*, Toulouse, France, 2006, pp. V–V.
[17] W. Hamilton, “On the geometrical interpretation of some results obtained by calculation with biquaternions,” in *Proc. Roy. Irish Acad.*, 1853, vol. V, pp. 388–390.
[18] W. Hamilton, “On quaternions,” in *Proc. Roy. Irish Acad.*, 1843, pp. 1–16.
[19] J. Ward, *Quaternions and Cayley Numbers, Algebra and Applications*. New York: Kluwer Academic, 1997.
[20] I. Kantor and A. Solodovnikov, *Hypercomplex Numbers, an Elementary Introduction to Algebras*. New York: Springer-Verlag, 1989.
[21] P. Lounesto, *Clifford Algebra and Spinors*, ser. Electrical and computer engineering: Digital signal processing. Cambridge, U.K.: Cambridge Univ. Press, 1997.
[22] D. Hestenes and G. Sobzcyk, *Clifford Algebra to Geometric Calculus: A Unified Language for Physics and Mathematics*. Amsterdam, The Netherlands: Reidel, 1984.
[23] Y. Tian, “Matrix theory over the complex quaternion algebra,” *ArXiv Mathematics e-prints* 2000 [Online]. Available: http://arxiv.org/abs/math/0004005
[24] V. Majernick, “Quaternionic formulation of the classical fields,” *Appl. Clifford Algebras*, vol. 9, no. 1, pp. 119–130, 1999.
[25] J. Edmond, “Nature’s natural numbers: Relativistic quantum theory over the ring of complex quaternions,” *Int. J. Theoret. Phys*, vol. 6, no. 3, pp. 205–224, 1972.
[26] M. Mehta, *Matrix Theory, Selected Topics and Useful Results*. Delhi, India: Hindustan Publishing Corp., 1989, 2nd ed.
[27] L. Wolf, “Similarity of matrices in which the elements are real quaternions,” *Bull. Amer. Math. Soc.*, vol. 42, pp. 737–743, 1936.
[28] I. Porteous, *Clifford Algebra and Classical Groups*. Cambridge, U.K.: Cambridge Univ. Press, 1995.
[29] F. Zhang, “Quaternions and Matrices of Quaternions,” *Linear Algebra Its Appl.*, vol. 251, pp. 21–57, 1997.
[30] L. Huang and W. So, “On left eigenvalues of a quaternionic matrix,” *Linear Algebra Its Appl.*, vol. 323, pp. 105–116, 2001.
[31] H. Lee, “Eigenvalues and canonical forms of matrices with quaternions coefficients,” *Proc. Roy. Irish Acad.*, vol. 52A, no. 2, pp. 253–260, 1949.
[32] S. Okubo, “Eigenvalue problem for symmetric 3×3 octonionic matrix,” *Adv. Appl. Clifford Algebras*, vol. 1, no. 9, pp. 131–176, 1999.
[33] R. Serôdio, E. Pereira, and J. Vitória, “Computing the zeros of quaternion polynomials,” *Comput. Math. With Appl.*, vol. 42, no. 8, pp. 1229–1237, 2001.
[34] P.-O. Amblard and N. Le Bihan, “On properness of quaternion valued random variables,” presented at the 6th IMA Conf. Mathemathics in Signal Processing, Cirenceister, U.K., 2004.

Nicolas Le Bihan was born in Morlaix, France, in 1974. He received the B.Sc. degree in physics from the Universit de Bretagne Occidentale (UBO), Brest, France, in 1997 and the M.Sc. degree and the Ph.D. degree in signal processing from the Institut National Polytechnique de Grenoble (INPG), Grenoble, France, in 1998 and 2001, respectively.

Since 2002, he is Charg de Recherche at the Centre National de la Recherche Scientifique (CNRS) and is working with the Laboratoire des Images et des Signaux (UMR 5083) of Grenoble, France.

His research interests include polarized signal processing using multilinear, geometric algebra and group theory techniques, and applications of signal processing in geophysics.

Sebastian Miron was born in Suceava, Romania, in 1977. He graduated from “Gh. Asachi” Technical University of Iasi, Romania, in 2001 and received the M.Sc. and Ph.D. degree in signal, image, and speech processing from the Institut National Polytechnique of Grenoble, France, in 2002 and 2005, respectively.

He is currently a Maître de Conférence at the University of Nancy, France, and he is conducting research at the Centre de Recherche en Automatique de Nancy (CRAN), Nancy. His current research interests include seismic data processing, vector-sensor array processing, multilinear algebra, and hypercomplex numbers.

Jérôme I. Mars received the M.Sc. degree in geophysics from Joseph Fourier University, Grenoble, France, in 1986 and the Ph.D. degree in signal processing from the Institut National Polytechnique of Grenoble, France, in 1988.

From 1989 to 1992, he was a Postdoctoral Researcher at the Centre d’Étude des Phénomènes Aléatoires et Géophysiques, Grenoble, France. From 1992 to 1995, he was a visiting Lecturer and a Scientist in the Materials Sciences and Mineral Engineering Department, University of California, Berkeley. He is currently Professor of signal processing at the Institut National Polytechnique of Grenoble and is with the Laboratoire des Images et des Signaux, Grenoble, France. His research interests include seismic and acoustic signal processing, wave-field separation methods, time-frequency time-scale characterization, and applied geophysics.

Prof. Mars is a Member of the Society of Exploration Geophysics (SEG) and the European Association of Geoscientisits and Engineers (EAGE).