On the convergence of linear switched systems

Ulysse Serres, Jean-Claude Vivalda, Pierre Riedinger

On the Convergence of Linear Switched Systems

Ulysse Serres, Jean-Claude Vivalda and Pierre Riedinger

***Abstract*—This paper investigates sufficient conditions for the convergence to zero of the trajectories of linear switched systems. We provide a collection of results that use weak dwell-time, dwell-time, strong dwell-time, permanent and persistent activation hypothesis. The obtained results are shown to be tight by counterexample. Finally, we apply our result to the three-cell converter.**

***Index Terms*—Switched systems, dwell-time, stability, omega-limit set, three-cell converter.**

I. INTRODUCTION

A. Background

SWITCHED systems have attracted a growing interest in recent years [1], [2]. Such systems are common across a diverse range of application areas. As an example, switched systems modeling plays a major role in the field of power systems where interactions between continuous dynamics and discrete events are an intrinsic part of power system dynamic behavior.

In the study of stability of equilibrium points of differential systems, specific results for switched and hybrid systems have been developed: see [3], [4] for multiple Lyapunov based approach, [5] for Lie Algebra based results, [6] for an approach based on dynamical systems techniques, and [7] for a survey of stability criteria for switched and hybrid systems. In the context of switched systems, recent investigations (see [8], [9], [10], [11], [12]) provide interesting contributions leading to extremely general results that require little structure on the family of solutions of the hybrid system ([13], [14]).

Typically, linear switched systems are represented by equations of form

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}_+, \tag{I.1}$$

where σ denotes a piecewise constant signal that actually switches the right-hand-side of the differential equation by selecting different matrices from a finite family $\mathscr{F}$.

In the present paper we aim to find some tight sufficient conditions on the signal σ in order to insure the convergence of the switched system to the origin. Our aim leads us to define several new notions of dwell-time (firstly introduced in [15]) that differ somewhat from the ones introduced in [10], [13], [14]. The main differences rely in the fact that our notions of dwell-time are set for each mode while the ones used in the previously listed references are bearing on the signal itself. We are also led to define the notion of persistent activation which ensures the convergence of the solutions of the system to a minimal invariant set M.

We discuss the asymptotic properties of a switched linear systems whose matrices are only assumed to be stable (not necessarily asymptotically stable). More precisely, a finite family of squared matrices of the same size d, $\mathscr{F} = \{A_1, \dots, A_N\}$ is considered; we assume that there exists a positive definite matrix P such that for every A_i in $\mathscr{F}$,

$$x^{\mathrm{T}}\left(A_i^{\mathrm{T}}P + PA_i\right)x \leqslant 0, \qquad x \in \mathbb{R}^d. \tag{I.2}$$

When inequality (I.2) is strict, it is well known that system (I.1) is globally uniformly (with respect to σ) exponentially stable (*GUES*) at the origin. Conversely, a *GUES* linear system admits a smooth common Lyapunov function (see [16]). Moreover, it has been proved in [17] that the common Lyapunov function can be taken polynomial but that there exists no bound on the degree of the polynomial. When inequality (I.2) is not strict, in the even more general framework of hybrid systems, some very general stability results (which generalize LaSalle's invariance principle) are available in [13] and [14]. The class of systems (I.1) that satisfy (I.2) considered in the present paper being more specific, the obtained results are more sharpened than the ones we could obtain by applying the results proved in the above mentioned literature (for instance, compare Proposition III.8 with [13, Proposition 4.8], and Theorems II.5 and II.10 with [14, Corollary 4.4]).

We reformulate the switched linear system (I.1) as an affine control system

$$\dot{x}(t) = \sum_{i=1}^{N} \alpha_i(t)A_i x(t), \quad x(0) = x_0, \tag{I.3}$$

where $x(t) \in \mathbb{R}^d$, $\alpha_i(t) \in \{0, 1\}$ and $\sum_{i=1}^{N} \alpha_i(t) = 1$. The class of switching signals considered in this work is not equal to the whole $\mathbf{L}^\infty(\mathbb{R}_+, \{1, \dots, N\})$ but we assume that there exists a sequence $([a_n, a_{n+1}))_{n\in\mathbb{N}}$ of consecutive intervals ($0 = a_0 < a_1 < a_2 < \cdots$) whose union is equal to $\mathbb{R}_+$ and such that for every index n, there exists an index $i_n \in \{1, \dots, N\}$ with $\alpha_{i_n}(t) = 1$ if $t \in [a_n, a_{n+1})$ (informally, the a_i's are the switching instants). In this framework, the solutions to (I.1) are infinite products of matrices taken in the family $\{e^{t_1A_1}, \dots, e^{t_NA_N} \mid t_1, \dots, t_N \in \mathbb{R}_+\}$. The convergence of such infinite products has already been considered by different authors (beginning with [18] and then followed by [19], [20], and also [21] in an infinite dimensional context) but only for infinite products of matrices taken in a finite family.

In what follows, we use the symbol δ_n to denote the length of $[a_{n-1}, a_n)$ with the convention that $\delta_0 = 0$.

Université de Lyon, F-69 622 Lyon, France; Université Lyon 1, Villeurbanne; LAGEP UMR CNRS 5007, 43 bd du 11 novembre 1918, 69100 Villeurbanne, France – email: ulysse.serres@univ-lyon1.fr

INRIA Nancy Grand Est CORIDA Project & LMAM UMR CNRS 7122 – Université de Metz – Ile du Saulcy – F-57 045 Metz Cedex 01 – France – email: vivalda@loria.fr

Nancy-Université, CRAN UMR CNRS 7039 – ENSEM – 2, av. de la Forêt de Haye – F-56 516 Vandœuvre-lès-Nancy Cedex– France) – email: pierre.riedinger@ensem.inpl-nancy.fr

Moreover, we shall deal with the scalar product related to matrix P: if $x, y \in \mathbb{R}^d$, we put $\langle x, y\rangle = x^{\mathrm{T}} P y$; also we shall denote by $\|\cdot\|$ the related norm. We shall use the following result see [22].

Theorem I.1. *If A_i satisfies (I.2), then we can split $\mathbb{R}^d$ as $\mathbb{R}^d = V_1^i \oplus V_2^i$ where V_1^i and V_2^i are orthogonal and A_i-invariant, A_i when restricted to V_1^i has all its eigenvalues with negative real parts and A_i restricted to V_2^i is skew-symmetric (with respect to the scalar product $\langle\cdot,\cdot\rangle$).*

In what follows, we assume that the skew-symmetric parts of matrices A_i are zero. In other words, if $\mathcal{B}_k^i$ denotes a basis of V_k^i ($k = 1, 2$), in basis $\mathcal{B}_1^i \cup \mathcal{B}_2^i$, matrix A_i has the following representation

$$A_i = \begin{pmatrix} A_{11}^i & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{(I.4)}$$

where A_{11}^i is a matrix of full rank. We denote by I.2 the following assumption.

Assumption I.2. There exists a positive definite matrix P such that all matrices in family $\mathscr{F}$ satisfy relations (I.2) and (I.4).

Hereafter, we give the precise definition of the ω-limit sets related to system (I.3).

Definition I.3. We shall say that ℓ is an ω-limit point of system (I.3) if there exists a sequence $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that $\ell = \lim_{n\to\infty} x(t_n)$. We denote by $\Omega(x_0)$ the set of ω-limit points of system (I.3) issued from x_0.

B. Preliminaries: paracontracting linear maps

We denote by p_i (resp. q_i) the orthogonal projection on V_1^i (resp. on V_2^i); obviously $p_i + q_i$ is the identity mapping. According to Theorem I.1, we can write

$$e^{t\,A_i}(x) = e^{t\,A_i} \circ p_i(x) + q_i(x) \qquad \text{(I.5)}$$

and

$$\|e^{t\,A_i}(x)\|^2 = \|e^{t\,A_i} \circ p_i(x)\|^2 + \|q_i(x)\|^2. \qquad \text{(I.6)}$$

Following [23] we introduce the notion of paracontracting linear maps.

Definition I.4. A linear mapping $f : \mathbb{R}^d \to \mathbb{R}^d$ is said to be paracontracting with respect to the norm $\|\cdot\|$ if $\|f(x)\| \leqslant \|x\|$ for every $x \in \mathbb{R}^d$ and $\|f(x)\| = \|x\|$ iff $f(x) = x$.

A family of matrices as described in the introduction generates a paracontracting semi-group. More precisely, we have the following result.

Lemma I.5. *If matrix A is of the same type as* (I.4)*, $e^{t\,A}$ is paracontracting for every $t \geqslant 0$.*

Proof: If $t = 0$, the result is obvious; in what follows, we shall assume that $t > 0$. According to formula (I.6), one infers that

$$\|e^{t\,A}(x)\|^2 \leqslant \|p_i(x)\|^2 + \|q_i(x)\|^2 = \|x\|^2.$$

Moreover, $\|e^{t\,A}x\| = \|x\|$ implies that $\|e^{t\,A} \circ p_i(x)\|^2 = \|p_i(x)\|^2$. As $\|e^{s\,A}x\| \leqslant \|e^{s'\,A}x\|$ if $s \leqslant s'$, we have $\|e^{s\,A} \circ p_i(x)\| = \|p_i(x)\|$ for every $s \in [0, t]$ and by analyticity for every $s \geqslant 0$. Hence, $p_i(x)$ is in $\ker A$ and so $p_i(x)$ must be zero and $e^{t\,A}x = x$ follows from formula (I.5). ∎

This lemma allows us to state the following easy result.

Proposition I.6. *The elements of $\Omega(x_0)$ are of the same norm, in other words, there exists $r \geqslant 0$ such that, $\Omega(x_0)$ is included in the sphere centered at the origin of $\mathbb{R}^d$ and with radius r.*

Proof: Since the norm $\|x(t)\|$ is a nonincreasing function of the time, $r = \lim_{t\to\infty} \|x(t)\|$ exists and so $\Omega(x_0)$ is included in the sphere of radius r centered at the origin. ∎

Finally, the following elementary result, given without proof, will be useful for the proofs of results given in the next section.

Proposition I.7. *If matrix A is of the same type as in* (I.4)*, and if $x \notin \ker A$, then for all $\tau > 0$, there exists $\rho \in (0, 1)$ such that $t \geqslant \tau$ implies $\|e^{t\,A}x\| \leqslant \rho\|x\|$.*

II. A CONDITION FOR THE CONVERGENCE TO ZERO

In this section, we deal with the problem of the convergence to zero of the solution of system (I.3). First an easy remark: if the intersection $\bigcap_{i=1}^N \ker A_i$ is not $\{0\}$, then, taking as an initial condition for (I.3) a nonzero element x_0 of this intersection, the solution $x(t)$ is constant and equal to x_0. So, we introduce this first assumption.

Assumption II.1. We say that system (I.3) satisfies the null intersection assumption if $\bigcap_{i=1}^N \ker A_i = \{0\}$.

As we shall see through the following example, Assumption II.1 is not sufficient to ensure the convergence to zero.

Example II.2. In $\mathbb{R}^3$, we consider the following matrices

$$A_1 = \begin{pmatrix} 0 & -1 & a_{13} \\ 1 & 0 & a_{23} \\ -a_{13} & -a_{23} & a_{33} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 & -a_{13} \\ 1 & 0 & -a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

with $a_{33} < 0$ and $|a_{13}| + |a_{23}| \neq 0$. Denoting by $\langle\cdot,\cdot\rangle$ the canonical scalar product in $\mathbb{R}^3$, we have $\langle A_i x, x\rangle = a_{33}\, x_3^2 \leqslant 0$ ($i = 1, 2$) for every $x \in \mathbb{R}^3$. Moreover the characteristic polynomials of A_1 and A_2 are both equal to

$$-X^3 + a_{33}X^2 - (a_{13}^2 + a_{23}^2 + 1)X + a_{33}$$

so, we can see that A_1 and A_2 do not have purely imaginary eigenvalues. We can conclude that these matrices satisfy Assumption I.2. According to Trotter-Kato's formula:

$$\lim_{n\to\infty} \left(e^{t_0\,M/n} \circ e^{t_0\,N/n}\right)^n = e^{t_0\,(M+N)} \qquad \text{(II.1)}$$

where M and N are squared matrices and t_0 is a real number. Define the sequence $(e_k)_{k\in\mathbb{N}} \subset \mathbb{R}^3$ by: $e_0 = (1, 0, 0)^{\mathrm{T}}$, $e_1 = (0, 1, 0)^{\mathrm{T}}$, $e_2 = (-1, 0, 0)^{\mathrm{T}}$, $e_3 = (0, -1, 0)^{\mathrm{T}}$ and for $k \geqslant 4$, e_k will refer to vector e_r where r is the remainder in the Euclidean division of k by 4. We have

$$A_1 + A_2 = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2a_{33} \end{pmatrix}$$

and so, for $t_0 = \pi/4$, we have $e^{t_0\,(A_1+A_2)}(e_k) = e_{k+1}$. Set

$$R = e^{t_0\,(A_1+A_2)}, \quad \varphi_n = \left(e^{t_0\,A_1/n} \circ e^{t_0\,A_2/n}\right)^n.$$

Take $\varepsilon > 0$ and $x_0 = e_0$, from formula (II.1), we know that there exists n_1 such that

$$\|\varphi_{n_1}(x_0) - R(x_0)\| \leqslant \frac{\varepsilon}{2}. \tag{II.2}$$

Assume now that we have built a sequence $(x_1, \ldots, x_k)$ and we have found integers $(n_1, \ldots, n_k)$ such that $x_p = \varphi_{n_p}(x_{p-1})$ and $\|x_p - R(x_{p-1})\| \leqslant \varepsilon/2^p$ for $p = 1, \ldots, k$. Take n_{k+1} such that $\|\varphi_{n_{k+1}}(x_k) - R(x_k)\| \leqslant \varepsilon/2^{k+1}$ and set $x_{k+1} = \varphi_{n_{k+1}}(x_k)$. Consider the sequences $(x_k)_{k\geqslant 1}$ and $(n_k)_{k\geqslant 1}$. Inequality (II.2) can be written $\|x_1 - e_1\| \leqslant \varepsilon/2$ and we shall prove by induction that

$$\|x_k - e_k\| \leqslant \frac{\varepsilon}{2^k} + \cdots + \frac{\varepsilon}{2}. \tag{II.3}$$

By induction hypothesis and because $\|R\| \leqslant 1$, we have

$$\begin{aligned} \|x_{k+1} - e_{k+1}\| &\leqslant \|x_{k+1} - R(x_k)\| + \|R(x_k) - R(e_k)\| \\ &\leqslant \|\varphi_{n_{k+1}}(x_k) - R(x_k)\| + \|x_k - e_k\| \\ &\leqslant \frac{\varepsilon}{2^{k+1}} + \frac{\varepsilon}{2^k} + \cdots + \frac{\varepsilon}{2}. \end{aligned}$$

From (II.3), we get $\|x_k - e_k\| \leqslant \varepsilon$ for every integer k. Consider the switch law $\alpha = (\alpha_1, \alpha_2)$ defined by

$$\alpha_1(t) = \begin{cases} 1 & \text{if } t \in \left[\left(k + \dfrac{2p}{n_{k+1}}\right) t_0, \left(k + \dfrac{2p+1}{n_{k+1}}\right) t_0\right); \\ 0 & \text{otherwise;} \end{cases}$$
$$\alpha_2(t) = 1 - \alpha_1(t);$$

where k and p are integers such that $2p < n_{k+1}$. The switched system defined by matrices A_1 and A_2 and law α is such that $\Omega(e_0) \subset \mathbb{R}^3 \setminus B(0, 1-\varepsilon)$ and so $0 \notin \Omega(e_0)$ if ε is chosen small enough. Moreover, $\Omega(e_0)$ contains at least one point in every open ball $B(e_i, \varepsilon)$ and so, being connected, $\Omega(e_0)$ is an infinite set.

Notice that in the above counter-example the switching law satisfies Assumption III.1 (introduced later on page 5) for every matrix, that is to say the time elapsed on every matrix of the family is infinite. If one relaxes Assumption III.1, one could find a very simple counter-example in dimension $d = 2$.

We state the following definition.

Definition II.3. We shall say that the finite sequence $(i_1, \ldots, i_p)$ of indices (not necessarily distinct) taken in $\{1, \ldots, N\}$ is *compatible* with system (I.3) if there exists p consecutive intervals $[a_n, a_{n+1}), \ldots, [a_{n+p-1}, a_{n+p})$ such that for every $q \in \{1, \ldots, p\}$, $\alpha_{i_q}(t) = 1$ if $t \in [a_{n+q-1}, a_{n+q})$.

We introduce now the following assumption:

Assumption II.4 (Dwell-time)**.** We shall say that the mode i satisfies the condition of dwell-time if there exists a positive number τ and an integer p such that for every compatible sequence $(i_1, \ldots, i_p)$ there exists an index $q \in \{1, \ldots, p\}$ with $i_q = i$ and, if $[a_n, a_{n+1}), \ldots, [a_{n+p-1}, a_{n+p})$ is the related sequence of consecutive intervals, $a_{n+q} - a_{n+q-1} \geqslant \tau$.

We state the following result concerning the modes satisfying the dwell-time condition.

Theorem II.5. *If Assumption II.4 is satisfied for the mode i, then $\Omega(x_0) \subset \ker A_i$.*

Proof: Without loss of generality, we may assume that the dwell-time assumption is satisfied for mode $i = 1$. The solution of system (I.3) at time $t \in [a_n, a_{n+1})$ reads

$$x(t) = e^{(t-a_n)\,A_{i_{n+1}}} \circ e^{\delta_n\,A_{i_n}} \circ \cdots \circ e^{\delta_1\,A_{i_1}}(x_0).$$

Let $\ell \in \Omega(x_0)$, there exists a sequence $(t_k)_{k\in\mathbb{N}}$ such that $\ell = \lim_{k\to\infty} x(t_k)$. For every k, t_k belongs to an interval $[a_{n_k}, a_{n_k+1})$ and there exists an interval $[a_{m_k}, a_{m_k+1})$ with $a_{m_k} > a_{n_k}$ and such that $\alpha_1(t) = 1$ if $t \in [a_{m_k}, a_{m_k+1})$ and $\delta_{m_k+1} \geqslant \tau$. Due to Assumption II.4, the sequence of integers $(m_k - n_k)_{k\in\mathbb{N}}$ is bounded and so even if we have to work with a subsequence of $(t_k)_{k\in\mathbb{N}}$, we can assume that

- the difference $m_k - n_k$ is constant (positive), we denote by r this difference,
- the sequence of switches from t_k to a_{m_k} is independent of k, that is to say, there exists a finite sequence $A_{i_1}, \ldots, A_{i_r}$ of matrices taken in $\{A_1, \ldots, A_N\}$ such that for every k
$$x(a_{m_k}) = e^{u_k^r\,A_{i_r}} \circ \cdots \circ e^{u_k^1\,A_{i_1}}(x(t_k)),$$
- for each $m = 1, \ldots, r$, or the sequence $(u_k^m)_{k\in\mathbb{N}}$ tends to zero as k tends to infinity either there exists $\upsilon^m > 0$ such that $u_k^m \geqslant \upsilon^m$ for all k.

We first prove by induction on r that $\lim_{k\to\infty} x(a_{m_k}) = \ell$. If $r = 1$, we have $x(a_{m_k}) = e^{u_k^1\,A_{i_1}}(x(t_k))$. If $\lim_{k\to\infty} u_k^1 = 0$, the result is obvious, if not, write

$$x(t_k) = p_{i_1}(\ell) + q_{i_1}(\ell) + \bar{x}_k,$$

so we have

$$e^{u_k^1\,A_{i_1}}(x(t_k)) = e^{u_k^1\,A_{i_1}}(p_{i_1}(\ell)) + q_{i_1}(\ell) + e^{u_k^1\,A_{i_1}}(\bar{x}_k),$$

and, as $V_1^{i_1}$ and $V_2^{i_1}$ are orthogonal

$$\begin{aligned} \|e^{u_k^1\,A_{i_1}}(x(t_k))\|^2 &= \|e^{u_k^1\,A_{i_1}}(p_{i_1}(\ell))\|^2 + \|q_{i_1}(\ell)\|^2 \\ &\quad + 2\langle e^{u_k^1\,A_{i_1}}(\ell), e^{u_k^1\,A_{i_1}}(\bar{x}_k)\rangle + \|e^{u_k^1\,A_{i_1}}(\bar{x}_k)\|^2. \end{aligned}$$

Suppose, to reach a contradiction, that $p_{i_1}(\ell) \neq 0$. Then, as $u_k^1 \geqslant \upsilon^1$, there exists $\rho \in (0,1)$ such that $\|e^{u_k^1\,A_{i_1}}(p_{i_1}(\ell))\| < \rho\|p_{i_1}(\ell)\|$. Choose $\varepsilon > 0$, since $\lim_{k\to\infty} \bar{x}_k = 0$ and $\|e^{u_k^1\,A_{i_1}}\| \leqslant 1$, we get for k large enough

$$\begin{aligned} \|e^{u_k^1\,A_{i_1}}(x(t_k))\|^2 &< \rho^2\|p_{i_1}(\ell)\|^2 + \|q_{i_1}(\ell)\|^2 + 2\varepsilon\|\ell\| + \varepsilon^2 \\ &< \|\ell\|^2 - \varepsilon^2, \end{aligned}$$

provided that ε is chosen such that $2\varepsilon^2 + 2\varepsilon\,\|\ell\| < (1-\rho^2)\|p_{i_1}(\ell)\|^2$. But in this case, we could find an element in $\Omega(x_0)$ (a cluster point of the sequence $(x(a_{m_k}))_{k\in\mathbb{N}}$) whose norm is less than $\|\ell\|$, which is impossible. Hence, $p_{i_1}(\ell) = 0$ which implies $\ell = q_{i_1}(\ell)$ and $e^{u_k^1\,A_{i_1}}(x(t_k)) = \ell + e^{u_k^1\,A_{i_1}}(\bar{x}_k)$, since $\|e^{u_k^1\,A_{i_1}}\| \leqslant 1$ and $\lim_{k\to\infty} \bar{x}_k = 0$, we get the result. So, we can write

$$x(a_{m_k}) = e^{u_k^r\,A_{i_r}} \circ \cdots \circ e^{u_k^2\,A_{i_2}}(x(a_{n_k}))$$

with $\lim_{k\to\infty} x(a_{n_k}) = \ell$, applying the induction hypothesis, we conclude that $\lim_{k\to\infty} x(a_{m_k}) = \ell$. Now, we have

$$x(a_{m_k+1}) = e^{(a_{m_k+1} - a_{m_k})\,A_1}(x(a_{m_k}))$$

with $a_{m_k+1} - a_{m_k} \geqslant \tau$. The argument used above in the case where the sequence $(u_k^1)_{k\in\mathbb{N}}$ does not converge to zero proves that $p_1(\ell) = 0$, i.e., $\ell \in \ker A_1$. ■

An easy consequence of Theorem II.5 is stated in the following corollary.

Corollary II.6. *If system* (I.3) *satisfies Assumption II.1 and if every mode satisfies assumption II.4, then $x(t)$, the solution of system* (I.3)*, tends to zero as t tends to infinity.*

Proof: According to Theorem II.5 and Assumption II.1, we have $\Omega(x_0) \subset \bigcap_{i=1}^{N} \ker A_i = \{0\}$. ■

At this step, we could wonder if it were possible to weaken the hypothesis of Theorem II.5. Consider the following hypothesis which is weaker than Assumption II.4.

Assumption II.7 (Weak dwell-time)**.** We shall say that the mode i satisfies the weak dwell-time condition if there exists $\tau > 0$ such that for every $n_0 \in \mathbb{N}$, there exists $n \geqslant n_0$ with $\alpha_i(t) = 1$ if $t \in [a_n, a_{n+1})$ and $a_{n+1} - a_n \geqslant \tau$.

To say that Assumption II.7 is satisfied amounts to say that the sequence of durations during which mode i is activated does not tends to zero. As we shall see through the following example, Assumption II.4 cannot be replaced by Assumption II.7 in Theorem II.5.

Example II.8. In the following family of eight matrices, we assume that the diagonal coefficients $a_{11}, \dots, a_{44}$ are negative, the nondiagonal coefficients a_{ij} $(i \neq j)$ being nonzero.

$$A_1 = \begin{pmatrix} 0 & -1 & a_{13} & 0 \\ 1 & 0 & a_{23} & 0 \\ -a_{13} & -a_{23} & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 & -a_{13} & 0 \\ 1 & 0 & -a_{23} & 0 \\ a_{13} & a_{23} & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & a_{24} \\ 0 & 1 & 0 & a_{34} \\ 0 & -a_{24} & -a_{34} & a_{44} \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -a_{24} \\ 0 & 1 & 0 & -a_{34} \\ 0 & a_{24} & a_{34} & a_{44} \end{pmatrix},$$

$$A_5 = \begin{pmatrix} a_{11} & 0 & -a_{13} & -a_{14} \\ 0 & 0 & 0 & 0 \\ a_{13} & 0 & 0 & -1 \\ a_{14} & 0 & 1 & 0 \end{pmatrix}, \quad A_6 = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ -a_{13} & 0 & 0 & -1 \\ -a_{14} & 0 & 1 & 0 \end{pmatrix},$$

$$A_7 = \begin{pmatrix} 0 & a_{12} & 0 & -1 \\ -a_{12} & a_{22} & 0 & -a_{24} \\ 0 & 0 & 0 & 0 \\ 1 & a_{24} & 0 & 0 \end{pmatrix}, \quad A_8 = \begin{pmatrix} 0 & -a_{12} & 0 & -1 \\ a_{12} & a_{22} & 0 & a_{24} \\ 0 & 0 & 0 & 0 \\ 1 & -a_{24} & 0 & 0 \end{pmatrix}.$$

As in Example II.2, we can easily check that these matrices satisfy Assumption I.2. In this example, we denote by $e_1, \dots, e_4$ the vectors of the canonical basis in $\mathbb{R}^4$ and by $R_1, \dots, R_4$ the matrices $R_i = e^{t_0\,(A_{2i-1}+A_{2i})}$. Taking $t_0 = \pi/4$, we clearly have $R_i(e_i) = e_{i+1}$ (where $e_5 \triangleq e_4$). Finally, for $i = 1, \dots, 4$, we denote by φ_n^i the mappings $\varphi_n^i = \left(e^{(t_0/n)\,A_{2i-1}} \circ e^{(t_0/n)\,A_{2i}}\right)^n$ and we notice that, following Trotter-Kato's formula, we have $\lim_{n\to\infty} \varphi_n^i = R_i$. Choose $\tau > 0$ and consider the following product of matrices

$$P_1 = \prod_{i=5}^{8} e^{\tau\,A_{2i+4}} \circ \varphi_{n_i}^{i-4} \circ \prod_{i=1}^{4} e^{\tau\,A_{2i+3}} \circ \varphi_{n_i}^{i}$$

where we make the convention that if j is greater than 8, matrix A_j is equal to A_r where r is the remainder in the Euclidean division of j by 8 excepted when $r = 0$ in which case, A_j is A_8. We proceed as in Example II.2. Choose $\varepsilon > 0$ and n_1 such that $\|\varphi_{n_1}^1(e_1) - R_1(e_1)\| \leqslant \varepsilon/4$. Then, we have

$$\|e^{\tau\,A_5} \circ \varphi_{n_1}^1(e_1) - e^{\tau\,A_5} \circ R_1(e_1)\| \leqslant \frac{\varepsilon}{4}$$

because $\|e^{\tau\,A_5}\| \leqslant 1$. Notice that, in this inequality, $e^{\tau\,A_5} \circ R_1(e_1) = e_2$. Then, choose n_2 such that

$$\|\varphi_{n_2}^2 \circ e^{\tau\,A_5} \circ \varphi_{n_1}^1(e_1) - R_2 \circ e^{\tau\,A_5} \circ \varphi_{n_1}^1(e_1)\| \leqslant \frac{\varepsilon}{8},$$

and, because $\|R_2\| \leqslant 1$, we have

$$\begin{aligned} &\|\varphi_{n_2}^2 \circ e^{\tau\,A_5} \circ \varphi_{n_1}^1(e_1) - e_3\| \\ &\quad \leqslant \|\varphi_{n_2}^2 \circ e^{\tau\,A_5} \circ \varphi_{n_1}^1(e_1) - R_2 \circ e^{\tau\,A_5} \circ \varphi_{n_1}^1(e_1)\| \\ &\qquad + \|R_2 \circ e^{\tau\,A_5} \circ \varphi_{n_1}^1(e_1) - R_2 \circ e^{\tau\,A_5} \circ R_1(e_1)\| \\ &\quad \leqslant \frac{\varepsilon}{8} + \frac{\varepsilon}{4}. \end{aligned}$$

Proceeding this way, we choose integers $n_3, \dots, n_8$ such that $\|P_1(e_1) - e_1\| \leqslant \varepsilon/2^9 + \cdots + \varepsilon/4 < \varepsilon/2$. We then build P_2, $P_3, \dots$ in the same way as P_1 and we choose the integers n_9, $n_{10}, \dots$ in such a way that $\|P_k(e_1) - e_1\| \leqslant \varepsilon/2^k$. Now reasoning as in Example II.2, we can build a switch law related to the family of matrices $A_1, \dots, A_8$ such that $x(8k(t_0+\tau)) = P_k \circ \cdots \circ P_1(e_1)$. The ω-limit set $\Omega(e_1)$ contains at last one point in the open ball $B(e_1, \varepsilon)$; this ball does not contain the origin of $\mathbb{R}^4$ if ε is chosen small enough and so $x(t)$ does not tend to zero as t tends to infinity. Nevertheless, every mode of this system satisfies Assumption II.7.

This example shows that in Assumption II.4, one cannot cancel the condition on the repartition of switches (excepted in the two-dimensional case as we shall see in the next section). Nevertheless, if we make a stronger assumption on the dwell-times, we can free ourselves from this condition; consider the following assumption.

Assumption II.9 (Strong dwell-time)**.** We shall say that the mode i satisfies the condition of strong dwell-time if there exists $\tau > 0$ such that for every $n_i \in \mathbb{N}$ satisfying $\alpha_i|_{[a_{n_i}, a_{n_i+1})} = 1$, we have $a_{n_i+1} - a_{n_i} \geqslant \tau$. In other words, the sequence of durations during which the mode i is activated has a positive inferior limit.

We have the following result.

Theorem II.10. *If system* (I.3) *satisfies Assumptions II.1 and II.9 for every mode, then $\Omega(x_0) = \{0\}$ for every $x_0 \in \mathbb{R}^d$.*

Proof: We let $t_n = \sum_{k=0}^{n} \delta_k$. As the solution $x(t)$ of (I.3) is bounded, to get the result, it is sufficient to prove that $\Omega(x_0) = \{0\}$. To this end, we have only to prove that every cluster point of the sequence $(x(t_n))_{n\in\mathbb{N}}$ is equal to zero. Let ℓ be such a point, then, ℓ is the limit of a subsequence $(x(t_{n_k}))_{k\in\mathbb{N}}$ of $(x(t_n))_{n\in\mathbb{N}}$. Write $x(t_{n_k}) = e^{\delta_{n_k}\,A(n_k)}(x(t_{n_k-1}))$, where $A(n_k)$ is a matrix in family $\mathscr{F}$. There exists an index $i \in \{1, \dots, N\}$ such that, for infinitely many indices n_k, we have $A(n_k) = A_i$. Even if we have to renumber the matrices of family $\mathscr{F}$, we can suppose that $i = 1$ and, even if we have to work with a subsequence of $(x(t_{n_k}))_{k\in\mathbb{N}}$, we can write $x(t_{n_k}) = e^{\delta_{n_k} A_1}(x(t_{n_k-1}))$. We write $x(t_{n_k-1}) = \ell + \bar{x}_k$ and, in exactly the same manner as in the proof of Theorem II.5, we prove that $p_1(\ell) = 0$.

We now make the following induction hypothesis: $p_1(\ell) = \dots = p_{r-1}(\ell) = 0$. For infinitely many indices k, we can find in the sequence $(x(t_{n_k}))_{k\in\mathbb{N}}$, terms which write $x(t_{m_k}) = e^{\delta_{m_k} A_r} \circ \varphi_k(x(t_{n_k-1}))$ where φ_k is a product of exponentials of matrices taken in the set $\{A_1, \dots, A_{r-1}\}$. Writing $x(t_{n_k-1}) = \ell + \bar{x}_k$, we have $\varphi_k(x(t_{n_k-1})) = \ell + \varphi_k(\bar{x}_k)$ because $\ell \in \bigcap_{i=1}^{r-1} \ker A_i$. Thus, $x(t_{m_k}) = e^{\delta_{m_k} A_r}(\ell) + e^{\delta_{m_k} A_r}(\varphi_k(\bar{x}_k))$. Since $\lim_{k\to\infty} e^{\delta_{m_k} A_r}(\varphi_k(\bar{x}_k)) = 0$, if we suppose that $p_r(\ell) \neq 0$, we are led to a contradiction in the same way as above. We have thus proved that $p_1(\ell) = \dots = p_N(\ell) = 0$, or equivalently, that $\ell \in \bigcap_{i=1}^{N} \ker A_i$. According to Assumption II.1, this set is $\{0\}$ and so $\ell = 0$. ■

To conclude this section, we shall illustrate the different concepts of dwell-time presented in this paper through simple examples. We take a family $\mathscr{F}$ reduced to two matrices and we denote by δ_{2i+1} (resp. δ_{2i}) the lengths of the intervals on which mode 1 (resp. 2) is activated. If we take $\delta_{2i+1} = \tau > 0$, then mode 1 satisfies the strong dwell-time assumption. Consider now a switching law such that $\delta_{2i+1} = 1 + i(-1)^i/(i+1)$; clearly $\liminf_{i\to\infty} \delta_{2i+1} = 0$, so mode 1 does not satisfy the strong dwell-time assumption. Nevertheless, if we take four consecutive intervals of activation, on one of these, the time of activation of mode 1 is greater than one, so mode 1 satisfies the dwell-time assumption. Finally let $T_n = n(n+1)/2$ and consider a command law such that, for $T_n \leqslant i < T_{n+1}$, $\delta_{2i+1} = 1 + n(-1)^n(n+1)/2$. Clearly, mode 1 does not satisfy the strong dwell-time assumption but it satisfies the weak dwell-time assumption. Now this mode does no more satisfy the dwell-time assumption. Indeed, let $K \in \mathbb{N}^*$; if n is large enough and odd, we can find K consecutive intervals of activation whose indices i are such that $T_n < [i/2] \leqslant T_{n+1}$ ($[x]$ denotes the integer part of x); for the intervals with an odd index, we have $\delta_{2i+1} = 1/(n+1)$ which can be made arbitrarily small as n tends to infinity.

III. Some properties of the ω-limit set

A. General considerations

In this section, we introduce the following new assumption.

Assumption III.1 (Permanent activation)**.** We say that the mode i satisfies the permanent activation hypothesis if $\lambda\{t \geqslant 0 \mid \alpha_i(t) = 1\} = \infty$, where λ denotes the Lebesgue measure.

We begin with proving an easy result.

Proposition III.2. *A control law which satisfies Assumption III.1 for every mode being given, the set of points x_0 such that $\Omega(x_0)$ is equal to $\{0\}$ is a subspace of $\mathbb{R}^d$ with dimension at least one. This implies that the set of points x_0 such that $\Omega(x_0)$ does not reduce to $\{0\}$ is either empty or open and dense.*

Proof: Obviously the set of points x_0 such that $\Omega(x_0) = \{0\}$ is a subspace of $\mathbb{R}^d$; moreover, the solution of (I.3) issued from x_0 can be written as $\varphi_t(x_0)$ where φ_t is a product of exponentials of matrices taken in family $\mathscr{F}$. The determinant of φ_t is equal to $\det \varphi_t = e^{\tau_1(t) \operatorname{tr} A_1} \dots e^{\tau_N(t) \operatorname{tr} A_N}$, where $\tau_i(t)$ denotes the measure of the set $\{0 \leqslant s \leqslant t \mid \alpha_i(s) = 1\}$ ($\operatorname{tr} A_i < 0$). Due to Assumption III.1, we have $\lim_{t\to\infty} \tau_i(t) = \infty$, and so $\lim_{t\to\infty} \det \varphi_t = 0$. As, φ_t is bounded, we can find a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\phi = \lim_{n\to\infty} \varphi_{t_n}$ exists. Since $\det \phi = 0$, there exists $x_0 \neq 0$ such that $\phi(x_0) = 0$; for this x_0, we clearly have $\Omega(x_0) = \{0\}$. ■

We consider now the set M related to family $\mathscr{F}$ defined as

$$M = \{x \in \mathbb{R}^d \mid \langle A_i x, x\rangle = 0,\ i = 1, \dots, N\}.$$

For $i = 1, \dots, N$, we also denote by M_i the set $M_i = \{x \in \mathbb{R}^d \mid \langle A_i x, x\rangle = 0\}$. Set M can be regarded as the intersection of the sets of zeros of quadratic forms $x \mapsto \langle A_i x, x\rangle$ which are subspaces of $\mathbb{R}^d$ because these quadratic forms are non positive, so M is a subspace of $\mathbb{R}^d$. Notice than M can contain a nonzero vector even if all the matrices in family $\mathscr{F}$ are of full rank. For instance, consider the two matrices A_1 and A_2 of Example II.2, the related set M is $M = \{x \in \mathbb{R}^3 \mid x_3 = 0\}$.

The two following propositions state that the set of ω-limit points cannot avoid sets M_i.

Proposition III.3. *Suppose that system* (I.3) *satisfies Assumption III.1 for mode i, then there exists $\ell \in \Omega(x_0)$ such that $\langle A_i \ell, \ell\rangle = 0$ (in other words, $\ell \in M_i$).*

Proof: The proof is by contradiction. Assume that for every $\ell \in \Omega(x_0)$, the scalar product $\langle A_i \ell, \ell\rangle$ is nonzero, it is therefore negative. As $\Omega(x_0)$ is compact, this implies that there exists $\mu > 0$, such that $\langle A_i \ell, \ell\rangle \leqslant -\mu$ for every $\ell \in \Omega(x_0)$. We denote by $[a_{k_1}, a_{k_1+1})$, $[a_{k_2}, a_{k_2+1})$, … the intervals of times during which the mode i is activated[1]. For the sake of readability, we denote by x_n the solution of (I.3) at time a_{k_n+1}. We have

$$x_n = e^{\delta_{k_n} A_i} \circ \varphi_n(x_{n-1}) \qquad \text{(III.1)}$$

where φ_n is a product of exponentials of matrices taken in family $\mathscr{F} \setminus \{A_i\}$. It follows from $\|\varphi_n\| \leqslant 1$ that

$$\begin{aligned}\|x_n\|^2 - \|x_0\|^2 &= \sum_{j=1}^{n} \left(\|x_j\|^2 - \|\varphi_j(x_{j-1})\|^2\right) \\ &\quad + \sum_{j=1}^{n} \left(\|\varphi_j(x_{j-1})\|^2 - \|x_{j-1}\|^2\right) \\ &\leqslant \sum_{j=1}^{n} \left(\|x_j\|^2 - \|\varphi_j(x_{j-1})\|^2\right). \qquad \text{(III.2)}\end{aligned}$$

First, we show the result when $\lim_{n\to\infty} \delta_{k_n} = 0$. In this case, we consider the series whose general term is

$$\|x_n\|^2 - \|\varphi_n(x_{n-1})\|^2 = \|e^{\delta_{k_n} A_1} \circ \varphi_n(x_{n-1})\|^2 - \|\varphi_n(x_{n-1})\|^2.$$

If n is large enough, the scalar product $\langle A_i x_n, x_n\rangle$ is far away from zero, more precisely, there exists n_0 such that $\langle A_i x_n, x_n\rangle \leqslant -\mu/2$ as soon as $n \geqslant n_0$. So, as n tends to infinity, the general term of this series is equivalent to $2\langle A_i x_n, x_n\rangle \delta_{k_n}$ which is the general term of a divergent (to $-\infty$) series since the series whose general term is δ_{k_n} is

[1] $\alpha_i(t) = 1$ if t belongs to the union of these intervals

divergent and $\langle A_i x_n, x_n\rangle \leqslant -\mu/2$. So the right-hand side of (III.2) can be made less than $\|\ell\|^2 - \|x_0\|^2$ if n is chosen large enough, which is a contradiction since $\|x_n\| \geqslant \|\ell\|$ for all n.

In the case where we do not have $\lim_{n\to\infty} \delta_{k_n} = 0$, there exists $\tau > 0$ such that for all n_0, there exists $n \geqslant n_0$ with $\delta_{k_n} \geqslant \tau$. So, even if we have to work with a subsequence of $(x_n)_{n\in\mathbb{N}}$, in (III.1) we can assume that $\delta_{k_n} \geqslant \tau$ for every index n and that the sequence $(\varphi_n(x_{n-1}))_{n\in\mathbb{N}}$ is convergent with limit $\ell \in \Omega(x_0)$. But reasoning as in the proof of Theorem II.5, this implies that $A_i\ell = 0$ and so $\ell \in M_i$. ■

We immediately deduce from Proposition I.6 and Proposition III.3 the following corollary.

Corollary III.4. *Suppose that there exists a mode i with $M_i = \{0\}$ and which satisfies Assumption III.1, then $\Omega(x_0) = \{0\}$.*

The next proposition tells us that each ω-limit point belongs to at least one set M_i.

Proposition III.5. *We assume that Assumption III.1 is satisfied by every mode. Then, for every $\ell \in \Omega(x_0)$, there exists a mode $i \in \{1, \dots, N\}$ such that $\ell \in M_i$. In other words, $\Omega(x_0)$ is included in the union $\bigcup_{i=1}^N M_i$.*

Proof: If $\Omega(x_0)$ is a singleton, the result is given by Proposition III.3. If $\Omega(x_0)$ is not a singleton, let $\ell \in \Omega(x_0)$ and take $\varepsilon > 0$ such that $\Omega(x_0)$ contains at least a point outside of the open ball $B(\ell, \varepsilon)$. From the definition of $\Omega(x_0)$, there exists a sequence of times $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x(t_n) = \ell$ and we can assume that $x(t_n) \in B(\ell, \varepsilon)$ for every $n \in \mathbb{N}$. Denote by t'_n the number defined by $t'_n = \inf\{t > t_n \mid x(t) \in \partial B(\ell, \varepsilon)\}$. We claim that there exists $\tau > 0$ such that, for every n, $t'_n - t_n \geqslant \tau$. The proof of the existence of τ is by contradiction, suppose that for every $\tau > 0$, there exists n such that $t'_n - t_n < \tau$, then there exists an increasing sequence of indices $(n_k)_{k\in\mathbb{N}}$ such that $t'_{n_k} - t_{n_k} < 1/k$. As $x(s)$ is bounded and $t'_{n_k} - t_{n_k}$ tends to 0, we have

$$x(t'_{n_k}) - x(t_{n_k}) = \int_{t_{n_k}}^{t'_{n_k}} \sum_{i=1}^N \alpha_i(s) A_i x(s) ds \underset{k\to\infty}{\to} 0,$$

which implies that $x(t'_{n_k})$ tends to ℓ, which contradicts the definition of the sequence $(t'_n)_{n\in\mathbb{N}}$. Assume now that for every $i \in \{1, \dots, N\}$, we have $\langle A_i\ell, \ell\rangle < 0$, then there exists $\mu > 0$ such that $\langle A_i\ell, \ell\rangle < -\mu$ and, if ε is chosen small enough, we have $\langle A_i x(t), x(t)\rangle < -\mu/2$ for every t in the union of intervals $[t_n, t'_n)$. Hence,

$$\begin{aligned}\|x(t'_n)\|^2 - \|x(t_n)\|^2 &= \int_{t_n}^{t'_n} 2\sum_{i=1}^N \alpha_i(t)\langle A_i x(t), x(t)\rangle dt \\ &\leqslant -\mu(t'_n - t_n) \\ &\leqslant -\mu\tau. \end{aligned} \tag{III.3}$$

Up to a subsequence of $(x(t_n))_{n\in\mathbb{N}}$, we can assume that $t_{n-1} < t'_{n-1} < t_n$. In this case, $\|x(t_k)\| \leqslant \|x(t'_{k-1})\|$ and we deduce from (III.3) that

$$\begin{aligned}\|x(t'_n)\|^2 - \|x(t_0)\|^2 &= \sum_{k=0}^n \left(\|x(t'_k)\|^2 - \|x(t_k)\|^2\right) \\ &\quad + \sum_{k=1}^n \left(\|x(t_k)\|^2 - \|x(t'_{k-1})\|^2\right) \\ &\leqslant \sum_{k=0}^n \left(\|x(t'_k)\|^2 - \|x(t_k)\|^2\right) \\ &\leqslant -(n+1)\mu\tau, \end{aligned}$$

which leads to $\|x(t'_n)\|^2$ negative if n is chosen large enough which is impossible. ■

We prove here a proposition announced in the previous section and stating a result of convergence to zero in the two-dimensional case.

Proposition III.6. *In the two-dimensional case ($d = 2$), if every mode of system* (I.3) *satisfies Assumptions II.1 and II.7, then $\Omega(x_0) = \{0\}$ for every $x_0 \in \mathbb{R}^2$.*

Proof: Take $x_0 \in \mathbb{R}^2$, the ω-limit set $\Omega(x_0)$ is included in $\bigcup_{i=1}^N M_i$. Due to Assumption I.2, the sets M_i are zero or one-dimensional subspaces of $\mathbb{R}^2$, so their intersection with S^1 gives a set of isolated points. Assume that $\Omega(x_0) \neq \{0\}$, then it is included on a circle with center the origin and radius $r > 0$; moreover due to Assumption III.1, it is also included in the union $\bigcup_{i=1}^N M_i$. Therefore $\Omega(x_0)$ is included in a finite set of points located on the circle. As $\Omega(x_0)$ is a connected set, we deduce that $\Omega(x_0)$ is a single point that we shall denote by ℓ. Take $i_0 \in \{1, \dots, N\}$, due to Assumption II.7, we can find a sequence $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that we can write $x(t_n) = e^{\tau_n A_{i_0}}(x(t_n - \tau_n))$ with $\tau_n > \tau > 0$. Reasoning as in the proof of Theorem II.5, we deduce that the limit of $x(t_n - \tau_n)$ as n tends to infinity belongs to $\ker A_{i_0}$ but this limit is equal to ℓ. So we proved that $\ell \in \bigcap_{i=1}^N \ker A_i = \{0\}$ (due to Assumption II.1). ■

Remark. One could wonder if Assumption II.7 could be replaced by Assumption III.1 in the above proposition. We do not know the answer to this question.

In order to give a more precise description of the ω-limit set, we shall assume that the different modes are well distributed. Roughly speaking, this means that the contribution of a given mode cannot be neglected with respect to the contributions of the other modes. Below is the precise definition.

Assumption III.7 (Persistent activation)**.** We shall say that the mode i satisfies the "persistent activation" assumption if, for every sequence of intervals $([t_n, t'_n])_{n\in\mathbb{N}}$ such that

- the limit of t_n as n tends to infinity is equal to infinity;
- there exists $\tau > 0$, such that $t'_n - t_n \geqslant \tau$ for every $n \geqslant 0$;
- the limit of the number of commutations occurring in the interval $[t_n, t'_n]$ tends to infinity as n tends to infinity,

we have $\liminf_{n\to\infty} \lambda\{t \in [t_n, t'_n] \mid \alpha_i(t) = 1\} > 0$.

Proposition III.8. *Assume that Assumptions III.1 and III.7 are satisfied by the mode i, then $\Omega(x_0) \subset M_i$.*

Proof: Let $\ell \in \Omega(x_0)$, if $\Omega(x_0)$ reduces to ℓ, the result follows from Proposition III.3; if not, we shall argue by contradiction. So, we suppose that there exists $\ell \in \Omega(x_0)$ such that $\langle A_i\ell, \ell\rangle < -\mu < 0$. As in the proof of Proposition III.5, we take a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x(t_n) = \ell$, we choose $\varepsilon > 0$ and we define the sequence $(t'_n)_{n\in\mathbb{N}}$ by $t'_n = \inf\{t \geqslant t_n \mid \|x(t'_n) - \ell\| = \varepsilon\}$. The positive number ε is chosen small enough to have $\langle A_i x(t), x(t)\rangle \leqslant -\mu/2$ for every $t \in [t_n, t'_n]$ and, up to a subsequence of $(t'_n)_{n\in\mathbb{N}}$, we assume that the limit, denoted by ℓ', of the sequence $(x(t'_n))_{n\in\mathbb{N}}$ exists. As in the proof of Proposition III.5, we can show that there exists $\tau > 0$ with $t'_n - t_n \geqslant \tau$.

Moreover, the number of commutations occurring in the interval $[t_n, t'_n]$ cannot be bounded. If it were the case, up to a subsequence of $([t_n, t'_n])_{n\in\mathbb{N}}$, we could assume that there exists a finite sequence $(i_1, \ldots, i_r)$ of indices taken in $\{1, \ldots, N\}$ such that $x(t'_n) = e^{u_n^1 A_{i_1}} \circ \cdots \circ e^{u_n^r A_{i_r}} x(t_n)$, with $u_n^1, \ldots, u_n^r \geqslant 0$. As in the proof of Theorem II.5, in this case, we could show that $\lim_{n\to\infty} x(t'_n) = \ell$ which is impossible since $\|\ell' - \ell\| = \varepsilon$. So, we can suppose that the limit, as n tends to infinity, of the number of commutations occurring in the interval $[t_n, t'_n]$ is infinite. Therefore, denoting by J_n the set $J_n = \{t \in [t_n, t'_n] \mid \alpha_i(t) = 1\}$, from Assumption III.7, we have $\liminf_{n\to\infty} \lambda(J_n) = \tau_i > 0$. Now we have

$$\begin{aligned} \|x(t'_n)\|^2 - \|x(t_n)\|^2 &= 2\int_{t_n}^{t'_n} \sum_{j=1}^{N} \alpha_j(s)\langle A_j x(s), x(s)\rangle ds \\ &\leqslant 2\int_{J_n} \langle A_i x(s), x(s)\rangle ds \\ &\leqslant -\mu\lambda(J_n). \end{aligned} \tag{III.4}$$

But $\lambda(J_n) \geqslant \tau_i/2$ if n is large enough and so from (III.4), we get

$$\|x(t'_n)\|^2 - \|x(t_n)\|^2 \leqslant -\frac{\tau_i\mu}{2} \tag{III.5}$$

for n large enough. Passing to the limit in (III.5), we get $\|\ell'\|^2 - \|\ell\|^2 \leqslant -\tau_i\mu/2$. A contradiction since $\|\ell'\| = \|\ell\|$. ■

From Proposition III.8 we deduce the following easy consequence.

Corollary III.9. *If Assumptions III.1 and III.7 are satisfied and if $M = \{0\}$, then $x(t)$, the solution of system* (I.3), *tends to zero as t tends to infinity.*

Proof: If Assumption III.7 is satisfied, then $\Omega(x_0)$ reduces to $\{0\}$. ■

Remark. We may wonder if we could weaken the hypothesis in Proposition III.8 by assuming only the permanent activation (Assumption III.1). The following example gives a negative answer to this question.

Example III.10. In this example, we take the matrices A_1 and A_2 from Example II.2 as well as the following matrix A_3 defined as

$$A_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Put $\varphi_n = (e^{t_0\, A_1/n} \circ e^{t_0\, A_2/n} \circ e^{t_0\, A_3/n^2})^n$ with $t_0 = \pi/4$. We have $\lim_{n\to\infty} \varphi_n = e^{t_0\,(A_1+A_2)}$. Given integers $n_1, \ldots, n_4$, set

$$\Theta_{n_1n_2n_3n_4} = \varphi_{n_4} \circ e^{t_0\, A_3} \circ \varphi_{n_3} \circ \varphi_{n_2} \circ e^{t_0\, A_3} \circ \varphi_{n_1}.$$

A positive number ε being given, proceeding as in Example II.2, one can prove that it is possible to choose the integers $n_1, \ldots, n_4$ in such a way that $\|\Theta_{n_1n_2n_3n_4}(e_0) - e_0\| \leqslant \varepsilon/2$; more generally, we can find a sequence $(n_k)_{k\geqslant 1}$ such that

$$\left\| \prod_{i=0}^{k-1} \Theta_{n_{4i+1}n_{4i+2}n_{4i+3}n_{4i+4}}(e_0) - e_0 \right\| \leqslant \frac{\varepsilon}{2} + \cdots + \frac{\varepsilon}{2^k}. \tag{III.6}$$

We choose now a switch law such that the solution of (I.3) (with $x_0 = e_0$) is such that

$$x\left(6kt_0 + \sum_{i=1}^{4k} \frac{t_0}{n_i}\right) = \prod_{i=0}^{k-1} \Theta_{n_{4i+1}n_{4i+2}n_{4i+3}n_{4i+4}}(e_0).$$

For this switch law, inequality (III.6) shows that there exists an ω-limit point ℓ in the open ball $B(e_0, \varepsilon)$ and, if ε is chosen small enough this limit point is such that $\langle A_3\ell, \ell\rangle \neq 0$.

B. *What happens when $\Omega(x_0)$ is a singleton*

The following result is well known, but for the convenience of the reader we shall supply a simple proof.

Lemma III.11. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbf{L}^\infty(\mathbb{R}_+, \mathcal{B})$ where $\mathcal{B}$ is a bounded subset of $\mathbb{R}^N$. If $\varphi_n \overset{*}{\rightharpoonup} \varphi$, then φ takes almost surely values in $\overline{\mathrm{co}}(\mathcal{B})$ (the closed convex hull of $\mathcal{B}$).*

Proof: Let $\mathscr{Q}$ denote the set of affine forms of $\mathbb{R}^N$ with rational coefficients. Set $\mathscr{L} = \{L \in \mathscr{Q} \mid L(\mathcal{B}) \subset \mathbb{R}_+\}$. We have $\overline{\mathrm{co}}(\mathcal{B}) = \bigcap_{L\in\mathscr{L}} L^{-1}(\mathbb{R}_+)$. Take $L \in \mathscr{L}$. Since φ_n takes values in $\mathcal{B}$, $\int_A L(\varphi_n(t))dt \geqslant 0$, for any measurable subset $A \subset \mathbb{R}_+$. Since $\varphi_n \overset{*}{\rightharpoonup} \varphi$ and L is continuous, we get for any measurable set $A \subset \mathbb{R}_+$

$$\begin{aligned} \int_A L(\varphi(t))\,dt = L\left(\int_A \varphi(t)dt\right) &= \lim_{n\to\infty} L\left(\int_A \varphi_n(t)dt\right) \\ &= \lim_{n\to\infty} \int_A L(\varphi_n(t))\,dt \geqslant 0, \end{aligned}$$

which implies that $L \circ \varphi$ is almost surely nonnegative. In other words, the set $I_L = \{t \geqslant 0 \mid L\circ\varphi(t) < 0\}$ has zero measure for every $L \in \mathscr{L}$. Using the countability of $\mathscr{L}$, it follows that

$$\lambda\{t \geqslant 0 \mid \varphi(t) \notin \overline{\mathrm{co}}(\mathcal{B})\} = \lambda\left(\bigcup_{L\in\mathscr{L}} I_L\right) \leqslant \sum_{L\in\mathscr{L}} \lambda(I_L) = 0,$$

or, equivalently that $\varphi(t) \in \overline{\mathrm{co}}(\mathcal{B})$ almost surely. ■

The next proposition is a consequence of Lemma III.11 to linear switched systems. Define the sets $\Delta_{\geqslant 0}$ and $\Delta_{>0}$ by

$$\Delta_{\geqslant 0} = \left\{\alpha \in \mathbb{R}_+^N \mid \alpha_1 + \cdots + \alpha_N = 1\right\},$$

and

$$\Delta_{>0} = \{\alpha \in \Delta_{\geqslant 0} \mid \alpha_i > 0, \quad \forall\, i = 1, \ldots, N\}.$$

Proposition III.12. *If $\Omega(x_0) = \ell$, then there exists $\alpha \in \Delta_{\geqslant 0}$ such that $\sum_{i=1}^N \alpha_i A_i \ell = 0$.*

Proof: Let $\mathcal{B} = \{e_1, \ldots, e_N\}$ denote the canonical basis of $\mathbb{R}^N$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers tending to infinity. Put $\beta(t) = e_i$ if $\sigma(t) = i$, and set $\varphi_n(t) = \beta(t_n + t)$. For all $n \geqslant 0$, we have $\varphi_n \in \mathbf{L}^\infty(\mathbb{R}_+, \mathcal{B})$. It follows from Alaoglu's theorem and Lemma III.11 that φ_n admits a converging subsequense $\varphi_{n_k} \overset{*}{\rightharpoonup} \varphi \in \mathbf{L}^\infty(\mathbb{R}_+, \overline{\mathrm{co}}(\mathcal{B}))$. Thus, $\varphi(t) = \sum_{k=1}^N \alpha_i(t)e_i$, where the α_i's are non negative measurable functions. To each φ_{n_k} corresponds a nonautonomus vector field $X_t^{n_k}$ defined by

$$X_t^{n_k}(x) = \sum_{i=1}^N \langle \varphi_{n_k}(t), e_i\rangle A_i x. \tag{III.7}$$

Consequently, $X_\cdot^{n_k}(x) \overset{*}{\rightharpoonup} X_\cdot(x) = \sum_{i=1}^N \alpha_i(\cdot)A_i x$ for every $x \in \mathbb{R}^d$, as k goes to infinity. In particular, for every $t \geqslant 0$,

$$\int_0^t X_s^{n_k}(x)ds \underset{k\to\infty}{\to} \int_0^t X_s(x)ds. \tag{III.8}$$

It is easy to see that this convergence is indeed uniform with respect to (t, x) on every compact subset of $\mathbb{R}_+ \times \mathbb{R}^d$ (we prove this fact in Lemma III.13 after the present proof), and because the $X_t^{n_k}$ are linear (in x), the same property of uniform convergence holds for all derivatives with respect to x. Let P_k^t and P^t denote the flows of $X_t^{n_k}$ and X_t, respectively. Thus, according to [24, Lemma 8.10], $P_k^t(x)$ goes to $P^t(x)$ as $k \to \infty$, uniformly with respect to (t, x) on every compact subset of $\mathbb{R}_+ \times \mathbb{R}^d$. In particular, for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k \geqslant k_0$

$$\begin{aligned}\left\|P^t(x(t_{n_k})) - x(t + t_{n_k})\right\| &= \left\|P^t(x(t_{n_k})) - P_k^t(x(t_{n_k}))\right\| \\ &< \varepsilon,\end{aligned}$$

which, as k goes to infinity, shows that $\|P^t(\ell) - \ell\| < \varepsilon$, for every $\varepsilon > 0$ and every $t \geqslant 0$. Hence, $P^t(\ell) = \ell$, which, upon differentiating with respect to t, gives the result. ∎

Lemma III.13. *The convergence in relation (III.8) is uniform with respect to (t, x) on every compact subset of $\mathbb{R}_+ \times \mathbb{R}^d$.*

Proof: In view of relation (III.7), and because the mappings $x \mapsto A_i x$ are linear, it is sufficient to show that the convergence (which holds true since $\varphi_{n_k} \overset{*}{\rightharpoonup} \varphi$)

$$\int_0^t \varphi_{n_k}(s)ds \underset{k\to\infty}{\to} \int_0^t \varphi(s)ds$$

is uniform with respect to t on $[0, T]$. Fix $\varepsilon > 0$ and $p \in \mathbb{N}$ such that $5/2^p \leqslant \varepsilon$. Set $I_p^q = [qT/p, (q + 1)T/p)$. Since $\varphi_{n_k} \overset{*}{\rightharpoonup} \varphi$, there exists $k_0(p, q)$ such that

$$\left|\int_0^T \chi_{I_p^q}(s)(\varphi_{n_p} - \varphi)(s)ds\right| \leqslant \frac{1}{k2^{i+1}}, \quad \forall\ k \geqslant k_0(p, q).$$

Let $k_0 = \max\{k_0(p, 1), \ldots, k_0(p, p)\}$. Denote by q_t the index for which $t \in I_p^{q_t}$. For all $k \geqslant k_0$, we have,

$$\begin{aligned}\left|\int_0^t (\varphi_{n_k} - \varphi)(s)ds\right| &\leqslant \sum_{q=0}^{p-1} \left|\int_0^t \chi_{I_p^q}(s)(\varphi_{n_k} - \varphi)(s)ds\right| \\ &= \sum_{q\neq q_t} \left|\int_0^T \chi_{I_p^q}(s)(\varphi_{n_k} - \varphi)(s)ds\right| \\ &\quad + \left|\int_0^t (\varphi_{n_k} - \varphi)(s)ds\right| \\ &\leqslant \sum_{q=0}^{p-1} \left|\int_0^T \chi_{I_p^q}(s)(\varphi_{n_k} - \varphi)(s)ds\right| \\ &\quad + \frac{2}{p}\|\varphi_{n_k} - \varphi\|_\infty \leqslant \varepsilon,\end{aligned}$$

from which the uniform convergence (III.8) follows. ∎

Definition III.14. We shall say that a subset $I \subset \{1, \ldots, N\}$ is ℓ-minimal if there exists a unique α^0 in $\Delta_{>0}$ such that $\sum_{i\in I} \alpha_i^0 A_i \ell = 0$.

The next result shows that Assumption III.7 (and thus Assumption III.1) is necessary if $\{1, \ldots, N\}$ is ℓ-minimal.

Theorem III.15. *Assume that $\Omega(x_0) = \{\ell\}$. If $\{1, \ldots, N\}$ is ℓ-minimal, then Assumption III.7 is satisfied.*

Proof: Let α^0 be the unique element of $\Delta_{>0}$ such that $\sum_{i=1}^N \alpha_i^0 A_i \ell = 0$. According to Lemma III.12, there exists $\alpha \in \Delta_{\geqslant 0}$ such that $\sum_{i=1}^N \alpha_i A_i \ell = 0$. It follows from the ℓ-minimality of $\{1, \ldots, N\}$ that $\alpha = \alpha^0$. In particular, $\alpha_i > 0$ for all $i \in \{1, \ldots, N\}$. Keeping the same notation as in the proof of Lemma III.12, we thus have proved that all the (weak star) convergent subsequences of φ_n converge to $\sum_{i=1}^N \alpha_i^0 e_i$. Consequently, $\varphi_n \overset{*}{\rightharpoonup} \sum_{i=1}^N \alpha_i^0 e_i$. In particular, $\langle \varphi_n, e_i\rangle \overset{*}{\rightharpoonup} \alpha_i^0$ for all $i \in \{1, \ldots, N\}$. Hence, for any sequence of intervals $[t_n, t_n']$ satisfying the hypothesis of Assumption III.7, we have,

$$\begin{aligned}\lambda\{t \in [t_n, t_n'] \mid \alpha_i(t) = 1\} &= \int_{t_n}^{t_n'} \langle \beta(s), e_i\rangle\, ds \\ &\geqslant \int_0^\tau \langle \varphi_n, e_i\rangle\, ds \to \tau\alpha_i^0 > 0,\end{aligned}$$

which proves that Assumption III.7 is satisfied. ∎

C. Further remarks

In subsection III-A, we have seen that, under Assumptions III.1, the set $\Omega(x_0)$ is included in $\bigcup_{i=1}^N M_i$; moreover in subsection III-B, we have seen that, in the case where $\Omega(x_0)$ reduces to an unique point ℓ, there exists a convex combination of the vectors $A_i\ell$ which vanishes. In this subsection, we shall see what we can say of the convex combinations of the $A_i\ell$'s in the general case. Hereafter, we denote by I_ℓ the set of indices defined by $I_\ell = \{i \in \{1, \ldots, N\} \mid \ell \in M_i\}$.

Definition III.16. We shall say that ℓ is an ordinary point of $\bigcup_{i=1}^N M_i$ if, whenever we have i_1 and i_2 in I_ℓ, either $M_{i_1} \subset M_{i_2}$ or $M_{i_2} \subset M_{i_1}$. We shall say that $\ell \in \bigcup_{i=1}^N M_i$ is an extraordinary point if it is not ordinary.

Notice that if $\ell \in \bigcup_{i=1}^N M_i$ is an ordinary point, there exists an index i_ℓ such that $\bigcup_{i\in I_\ell} M_i = M_{i_\ell}$. Clearly, the set of ordinary points is open and dense in $\bigcup_{i=1}^N M_i$. Moreover, if ℓ is an ordinary point of $\bigcup_{i=1}^N M_i$, there exists an open neighborhood U of ℓ such that $U \cap \bigcup_{i=1}^N M_i = U \cap \bigcup_{i\in I_\ell} M_i$. The following proposition gives an additional condition to be satisfied by a point ℓ in order it belongs to $\Omega(x_0)$.

Proposition III.17. *Assume that $\ell \in \Omega(x_0)$. There exists $\alpha \in \Delta_{\geqslant 0}$ such that*

1) *if ℓ is an ordinary point, $\sum_{i=1}^N \alpha_i A_i \ell \in \bigcup_{i\in I_\ell} M_i$,*
2) *if ℓ is an extraordinary point, there exists $I_\ell' \subset I_\ell$ such that $\sum_{i=1}^N \alpha_i A_i \ell \in \bigcap_{i\in I_\ell'} M_i$.*

Proof: Assume first that ℓ is an ordinary point of $\Omega(x_0)$. Assume that, for every $\alpha \in \Delta_{\geqslant 0}$, the sum $\sum_{i=1}^N \alpha_i A_i \ell$ does not belong to $\bigcup_{i\in I_\ell} M_i$, then there exists a vector u orthogonal to $\bigcup_{i\in I_\ell} M_i = M_{i_\ell}$ and a positive constant c, such that $\langle A_i\ell, u\rangle > c$ for every $i \in I_\ell$. Choose $U = B(\ell, r)$ a small enough open ball around ℓ such that $U \cap \bigcup_{i=1}^N M_i = U \cap \bigcup_{i\in I_\ell} M_i$ and $\langle A_i\ell', u\rangle \geqslant c/2$ for every $\ell' \in U \cap \bigcup_{i\in I_\ell} M_i$. There exists a sequence $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that $\lim_{n\to\infty} x(t_n) = \ell$ and $x(t_n) \in B(\ell, r/2)$ for every $n \in \mathbb{N}$.

If $\Omega(x_0)$ reduces to ℓ, put $\tau_n = 1$. Since $\lim_{n\to\infty} x(t_n + \tau_n) = \ell$, $x(t_n+\tau_n) \in B(\ell, r/2)$ if the index n is large enough. If $\Omega(x_0)$ does not reduce to ℓ, there exists an ω-limit point outside of the ball $B(\ell, r)$ if r is chosen small enough. In this last case, we choose $\tau_n = \inf\{t > t_n \mid x(t) \in \partial B(\ell, r/2)\}$ which is well defined for every index n. In any case, the sequence $(x(t_n+\tau_n))_{n\in\mathbb{N}}$ being bounded, we can assume that it converges to $\bar\ell \in U$. We have

$$\begin{aligned}\langle x(t_n+\tau_n) - x(t_n), u\rangle &= \int_{t_n}^{t_n+\tau_n} \sum_{i=1}^N \alpha_i(s)\langle A_i x(s), u\rangle ds \\ &\geqslant \frac{c}{2}\tau_n, \end{aligned} \tag{III.9}$$

if n is large enough. If $\ell = \bar\ell$, put $\tau_n = 1$ for every index n. If $\bar\ell \neq \ell$, as we cannot have $\liminf_{n\to\infty} \tau_n = 0$, there exists $\tau > 0$ such that $\tau_n > \tau$ as soon as index n is large enough, so from (III.9), we deduce

$$\frac{c}{2}\tau \leqslant \langle x(t_n+\tau_n) - x(t_n), u\rangle. \tag{III.10}$$

As $\bar\ell \in \bigcup_{i\in I_\ell} M_i$, the limit of the right-hand side of (III.10) is 0, which leads to a contradiction.

Assume now that ℓ is an extraordinary point in $\bigcup_{i=1}^N M_i$, if ℓ is the limit of a sequence $(\ell_k)_{k\in\mathbb{N}}$ of ordinary points, we can assume that the sets of indices I_{ℓ_k} are all equal to a set $I_0 \subset \{1, \ldots, N\}$. So, there exists a sequence $(\alpha^{\ell_k})_{k\in\mathbb{N}}$ of elements of $\Delta_{\geqslant 0}$ such that

$$\sum_{i=1}^N \alpha_i^{\ell_k} A_i \ell_k \in \bigcup_{i\in I_0} M_i. \tag{III.11}$$

Now, as $\Delta_{\geqslant 0}$ is compact, we can suppose that the sequence $(\alpha^{\ell_k})_{k\in\mathbb{N}}$ converges to $\alpha \in \Delta_{\geqslant 0}$ and equality (III.11) implies $\sum_{i=1}^N \alpha_i A_i \ell \in \bigcup_{i\in I_0} M_i$. If there exists an open neighborhood U of ℓ such that $U \cap \bigcup_{i=1}^N M_i$ is constituted by extraordinary points, let $(\ell_k)_{k\in\mathbb{N}}$ be a sequence of extraordinary points tending to ℓ, we can assume that all the subsets of indices I_{ℓ_k} are equal to I_0 and we can also assume that all the ℓ_k as well as ℓ belong to a same intersection of subspaces $\bigcap_{i\in I_0} M_i$, reasoning as in the first part of this proof, we get the result stated in the proposition. ∎

D. Nonconvergence to zero of the switched system under some weak hypothesis

The next two results show that Assumption III.7 is not sufficient to ensure the convergence to zero of system (I.3) when the space M is not reduced to zero.

Proposition III.18. *Let $\ell \in M \setminus \{0\}$. Assume that there exists $\alpha \in \Delta_{>0}$ such that $\sum_{i=1}^N \alpha_i A_i \ell = 0$. Then, there exists a trajectory satisfying Assumption III.7 which does not converge to zero.*

Proof: Let $\alpha \in \Delta_{>0}$ be such that $\sum_{i=1}^N \alpha_i A_i \ell = 0$. For any $t \in [0, 1]$, set $\Phi^t = e^{t\alpha_N A_N} \circ \cdots \circ e^{t\alpha_1 A_1}$. The mapping $t \mapsto \Phi^t(\ell)$ has Taylor expansion

$$\Phi^t(\ell) = \ell + t\sum_{i=1}^N \alpha_i A_i \ell + t^2 v_t = \ell + t^2 v_t,$$

with $v_t = O(1)$ (since $[0, 1]$ is compact), which implies that there exists $c_0 \geqslant 0$ such that $\|v_t\| \leqslant c_0$ for all $t \in [0, 1]$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0, 1]$ such that $\sum_{n=0}^\infty t_n^2$ converges. Since $\|\Phi^t(v)\| \leqslant \|v\|$, we get

$$\begin{aligned}\|\Phi^{t_{n+1}} \circ \Phi^{t_n}(\ell) - \ell\| &= \|\Phi^{t_{n+1}}(\ell + t_n^2 v_{t_n}) - \ell\| \\ &\leqslant t_{n+1}^2\|v_{t_{n+1}}\| + t_n^2\|\Phi^{t_{n+1}}(v_{t_n})\| \\ &\leqslant (t_{n+1}^2 + t_n^2)c_0.\end{aligned}$$

Put $\Phi_{p,q} = \Phi^{t_q} \circ \cdots \circ \Phi^{t_p}$. By induction, we get, for $q \geqslant p$, $\|\Phi_{p,q}(\ell) - \ell\| \leqslant c_0 \sum_{n=p}^q t_n^2$, and letting q go to infinity leads to $\lim_{q\to\infty} \|\Phi_{p,q}(\ell) - \ell\| \leqslant c_0 \sum_{n=p}^\infty t_n^2$. As the $\Phi_{p,q}$'s are equibounded, we can select a converging subsequence Φ_{p,q_k}. Set $\Psi_p = \lim_{k\to\infty} \Phi_{p,q_k}$. For p sufficiently large, we have

$$\forall\, \varepsilon > 0 \quad \exists\, p_\varepsilon \quad | \quad \forall\, p \geqslant p_\varepsilon \quad \|\Psi_p(\ell) - \ell\| \leqslant \varepsilon, \tag{III.12}$$

which shows that $\Psi_p(\ell) \neq 0$. Moreover, because $\alpha \in \Delta_{>0}$, the constructed trajectory satisfies Assumption III.7. ∎

Proposition III.19. *Assume that there exists $\alpha \in \Delta_{>0}$ such that $M \subset \ker \sum_{i=1}^N \alpha_i A_i$. Then, for every $\ell \in M$, there exists a trajectory that satisfies Assumptions III.1 and III.7, and such that $\ell \in \Omega(x_0)$.*

Proof: Let $\alpha \in \Delta_{>0}$ be such that $M \subset \ker \sum_{i=1}^N \alpha_i A_i$ and let Ψ_p be defined as in the proof of Theorem III.18 according to which it remains to show that for every $\ell \in M$, there exists a trajectory whose ω-limit set contains ℓ. As $M \subset \ker \sum_{i=1}^N \alpha_i A_i$, relation (III.12) holds for every $\ell \in M$.

To get the result it is sufficient to show that there exists a p such that the image by Ψ_p of an open ball in M centered at the origin contains an open ball. Let S_M be the unit sphere of M that is the boundary of the open unit ball B_M. Let $0 < \varepsilon < 1$. Since all mappings Φ^t are Lipschitzian with constant one, all mappings are Ψ_p Lipschitzian with constant one. Thus the family $(\Psi_p)_{p\in\mathbb{N}}$ is equicontinuous. Relation (III.12) (which holds for every $\ell \in M$) indicates that as p goes to infinity, $\Psi_p \to \mathrm{Id}|_M$, pointwise, thus uniformly. Consequently, there exists p_0 such that $\|\Psi_{p_0}(\ell) - \ell\| < \varepsilon$ holds for all $\ell \in S_M$ from which it follows that $B_M(0, 1-\varepsilon) \subset \Psi_p(B_M)$ (see [25, Lemma 7.23]). This completes the proof. ∎

IV. Application to the three-cell converter

In this section we apply our theoretical result to the continuous-time model of the multilevel converter. Due to their particular importance for high-power industrial applications, multilevel converters have attracted increasing attentions in the last decades. Seeing that our present aim is to discuss neither modeling nor goals of such electronic devices, we refer the reader to [26], [27], [28] for a detailed discussion upon these issues.

For simplicity, we limit ourselves to the case of the three-cell converter although all our results are true for p-cell converters with $p \geqslant 3$. Some particularity of the two-cell converter which follows from Proposition III.6 will be explained at the end of present section.

A. Description of the system

The circuit topology of the three-cell converter is represented in Figure 1.

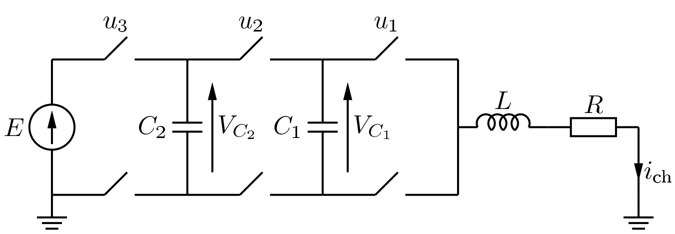

Figure 1. Sketch of the three-cell converter

Our main goal is to estimate the voltage of the capacitors in the case where only the current in the load is measured. Defining $x = (x_1, x_2, x_3) = (V_{C_1}, V_{C_2}, i_{\mathrm{ch}})$ as the state vector, where V_{C_1}, V_{C_2} are the voltages of the corresponding capacitors, i_{ch} is the load current and $y = i_{\mathrm{ch}}$ is the output, the model can be represented by a unique state equation:

$$\begin{cases} \dot{x} = F(u)x + EG(u) \\ y = Cx, \end{cases}$$

where E is the input voltage, $u = (u_1, u_2, u_3) \in \{0,1\}^3$ is the control vector and the matrices $F(u)$, $G(u)$ and C are given by

$$F(u) = \begin{pmatrix} 0 & 0 & \dfrac{u_2 - u_1}{C_1} \\ 0 & 0 & \dfrac{u_3 - u_2}{C_2} \\ \dfrac{u_1 - u_2}{L} & \dfrac{u_2 - u_3}{L} & -\dfrac{R}{L} \end{pmatrix}, \quad G(u) = \begin{pmatrix} 0 \\ 0 \\ \dfrac{u_3}{L} \end{pmatrix},$$
$$C = \begin{pmatrix} 0, & 0, & 1 \end{pmatrix}.$$

In order to achieve our goal, we build a Luenberger switched observer based on the load current measurement (other approaches are possible see for instance [29], [30], [31]). Such an observer takes the form

$$\dot{\hat{x}} = F(u)\hat{x} - L(u)(C\hat{x} - y) + G(u), \quad L(u) \in \mathbb{R}^{3\times 1},$$

and the dynamics of the error $e = \hat{x} - x$ reads

$$\dot{e}(t) = A(u)e(t), \quad A(u) = F(u) - L(u)C,$$

which, putting $i = \sum_{j=1}^{3} u_j 2^{j-1}$, can be rewritten as

$$\dot{e} = \sum_{i=1}^{8} \alpha_i(t) A_i e, \quad \alpha_i(t) \in \{0,1\}, \quad \sum_{i=1}^{8} \alpha_i(t) = 1. \quad \text{(IV.1)}$$

It is easy to show that the gain matrices L_i can be chosen in such a way that family $\mathscr{F} = \{A_1, \ldots, A_8\}$ satisfies Assumption I.2. Straightforward calculations show that necessarily,

$$P = \begin{pmatrix} p_1 & p_3 & 0 \\ p_3 & p_2 & 0 \\ 0 & 0 & p_4 \end{pmatrix}, \quad A_i^{\mathrm{T}} P + P A_i = \begin{pmatrix} 0 & 0 & \xi_i \\ 0 & 0 & \zeta_i \\ \xi_i & \zeta_i & \varsigma_i \end{pmatrix},$$

with $\varsigma_i \leqslant 0$ for $i = 1, \ldots, 8$. The spectrum of $A_i^{\mathrm{T}} P + P A_i$,

$$\mathrm{sp}(A_i^{\mathrm{T}} P + P A_i) = \left\{ 0, \frac{1}{2} \left(\varsigma_i \pm \sqrt{4\xi_i^2 + 4\zeta_i^2 + \varsigma_i^2} \right) \right\}$$

is a subset of nonpositive numbers if and only if $\xi_i = \zeta_i = 0$. In particular, we have $M = \bigcap_{i=1}^{8} M_i = \{x \in \mathbb{R}^3 \mid x_3 = 0\}$. Take P and the gain matrices L_i such that Assumption I.2 holds. We have

$$A_i = \frac{1}{L} \begin{pmatrix} 0 & 0 & (u_2 - u_1)\mu_1 - (u_3 - u_2)\mu_3 \\ 0 & 0 & (u_3 - u_2)\mu_2 - (u_2 - u_1)\mu_3 \\ u_1 - u_2 & u_2 - u_3 & -R_i \end{pmatrix},$$

with $R_i > 0$ and $\mu_j = p_j p_4^2 / \det P$ ($i = 1, \ldots, 8,\ j = 1, 2, 3$). We want to know under which condition solutions to (IV.1) converge to zero. All results from Section IV leading to convergence to zero being dwell-time based, we may wonder which type of dwell-time hypothesis are satisfied by the multi-cell chopper. Unfortunately, the different modes of the three-cell converter do not admit any dwell-time, only the switches do have one. In other words, one may switch from mode A_i to mode A_j ($i \neq j$) in an arbitrarily small time, but one has to wait a positive minimum time between two switches of the same the switch. We thus consider the following assumption.

Assumption IV.1 (Switch dwell-time)**.** The time elapsed between two commutations of the same switch has a positive inferior limit.

Notice that Assumption IV.1 implies that there exists at least one mode which satisfies the weak dwell-time Assumption II.7, but it turns out that its does not imply stronger assumptions on modes.

One may naturally wonder if Assumption IV.1 implies the convergence to zero of the solution to system (IV.1). The answer is negative as we shall see in the next section.

Notice moreover that although $\ker \sum_{i=1}^{8} A_i/8 = M$ (which shows that the three-cell converter satisfies the hypothesis of Proposition III.19), the counter example given by Proposition III.19 is no more valid since the dwell-time on switches has not been taken into account.

B. The ω-limit set of a trajectory of the 3-cell converter observer is not necessarily a singleton

In the present section, we construct a trajectory of the 3-cell converter observer (IV.1) whose ω-limit set is not a singleton. Moreover, we shall see that the trajectory can be constructed in such a way that Assumptions II.7, III.1, III.7 and IV.1 are satisfied.

First of all, let us rewrite system (IV.1) in a simpler way. Notice that up to the change of coordinates $x \mapsto P^{1/2}x$ and the time reparametrization $t \mapsto Lt$ (two transformations that do not change the topology of ω-limit sets), we may assume that all the A_i's have the form

$$A_i = \begin{pmatrix} 0 & 0 & a_1^i \\ 0 & 0 & a_2^i \\ -a_1^i & -a_2^i & -a_3^i \end{pmatrix},$$

with $(a_1^i, a_2^i) \in \{0, \pm 1\}^2 \setminus \{\pm(1,1)\}$ and $a_3^i > 0$. We now rewrite the system (IV.1) using spherical coordinates $z = (r, \theta, \varphi)$ defined by:

$$x_1 = r\cos\theta\cos\varphi, \quad x_2 = r\sin\theta\cos\varphi, \quad x_3 = r\sin\varphi.$$

It is easy to check that:

$$\dot{r} = -a_3^i r \sin^2\varphi \tag{IV.2}$$

$$\dot{\theta} = (a_2^i\cos\theta - a_1^i\sin\theta)\tan\varphi \tag{IV.3}$$

$$\dot{\varphi} = -a_3^i\cos\varphi\sin\varphi - (a_1^i\cos\theta + a_2^i\sin\theta), \tag{IV.4}$$

which shows in particular that $\dot{r} = o(\dot{\theta})$ as t goes to infinity since $\varphi(t)$ goes to zero as t goes to infinity. Heuristically speaking, this means that, approaching infinity, a trajectory of system (IV.1) looses less in norm than it can win in angular position θ, which encourages us to believe that we can build a trajectory of system (IV.1) whose ω-limit set is not a singleton.

Before beginning the construction of the trajectory, let us fix the set $\mathcal{K}$ in which the trajectory will lie and some notation. We set:

- $\mathcal{K} = [0, r_0] \times [\theta_0, \theta_f] \times [0, \varepsilon_0] \subset \{z \mid \cos\theta > \sin\theta > 0\}$,
- $\alpha = \inf_{\{z\in\mathcal{K},\ i\neq 7,8\}} \left\{|\dot{\varphi}(i,z)|, |a_2^i\cos\theta - a_1^i\sin\theta|\right\}$,
- $\beta = \sup_{\{z\in\mathcal{K},\ i\neq 7,8\}} \left\{|\dot{\varphi}(i,z)|, |a_2^i\cos\theta - a_1^i\sin\theta|\right\}$,
- $\alpha_1^i = \inf_{z\in\mathcal{K}} \left\{-a_1^i\cos\theta - a_2^i\sin\theta\right\}$,
- $\alpha_2^i = \sup_{z\in\mathcal{K}} \left\{\pi(-a_1^i\cos\theta - a_2^i\sin\theta)/2\right\}$,
- If $(v_n)_{n\in\mathbb{N}}$ is a sequence we denote by $(S_n^v)_{n\in\mathbb{N}}$ the sequence of its partial sums, i.e., $S_n^v = \sum_{k=0}^n v_k$.

Fix $\pi/4 \geqslant \varepsilon_0 > 0$ so small that we have $\alpha > 0$, $\beta > 0$ and $\alpha_j^i \neq 0$ for every $j \in \{1,2\}$ and every $i \neq 7, 8$.

1st step: Construction of the switching trajectory

We shall construct here a trajectory of system (IV.1) which lies in $\mathcal{K}$ for all positive time t. In order to choose an order for the concatenation of the modes, consider the following sign table of velocities in $\mathcal{K}$.

signs in $\mathcal{K}$	1	2	3	4	5	6	7	8
$\dot{\theta}$	+	−	−	+	+	−	0	0
$\dot{\varphi}$	+	−	+	−	+	−	0	0

(IV.5)

According to the sign table (IV.5), one sees that the forward and backward motions in θ are given by the modes 1, 4, 5 and 2, 3, 6 respectively. For simplicity, we shall only use the modes 2, 3, 4 and 5 for the construction of the trajectory.

From (IV.4), one infers that $-a_3^i\varphi + \alpha_1^i \leqslant \dot{\varphi} \leqslant 2(-a_3^i\varphi + \alpha_2^i)/\pi$. Hence, as long as $\theta(t)$ stays in $[\theta_0, \theta_f]$, we have

$$y_1^i(t) \leqslant \varphi(t) \leqslant y_2^i(2t/\pi), \tag{IV.6}$$

where $y_j^i(t)$ is the solution of the Cauchy problem $\dot{y} = -a_3^i y + \alpha_j^i$, $y(t_0) = \varphi(t_0)$, i.e.,

$$y_j^i(t) = e^{-a_3^i t}\varphi(t_0) + \frac{\alpha_j^i}{a_3^i}\left(1 - e^{-a_3^i(t-t_0)}\right). \tag{IV.7}$$

Construction of the ε_0-forward motion from θ_0 to θ_f

According to (IV.2)-(IV.4), and for every A_i in family $\mathscr{F}$, the projection onto (θ, φ) of the pushforward by the diffeomorphism $x \mapsto z$ of the field A_i is a well-defined nonlinear autonomous vector field in the variables (θ, φ) which we denote by $\bar{A}_i$. Let $e^{t\bar{A}_i}$ denote the flow of $\bar{A}_i$ and $\bar{\mathcal{K}}$ the projection of $\mathcal{K}$ onto (θ, φ).

Set $z_0 = (r_0, \theta_0, 0)$ to be the initial condition and $t_0 = 0$ to be the initial time. We begin with following the mode A_5. Define $\tau_1 = \inf\{t > 0 \mid e^{t\bar{A}_5}(\theta_0, 0) \notin \bar{\mathcal{K}}\}$. Necessarily, $\tau_1 < \infty$; if not, we would have $\theta(t) < \theta_f$ for all positive time, which, according to (IV.7) and (IV.6), would imply that

$$\varphi(t) \geqslant y_1^5(t) \underset{t\to\infty}{\to} \frac{\alpha_1^5}{a_3^5} > 0,$$

which is impossible since $\lim_{t\to\infty}\varphi(t) = 0$.

First case: $\boldsymbol{\theta(\tau_1) = \theta_f}$. For the same reason as for τ_1, we have $\inf\{t > 0 \mid e^{-t\bar{A}_4}(\theta_f, 0) \notin \bar{\mathcal{K}}\} < \infty$, which implies that there must exist δ_1, $\delta_2 > 0$ such that $e^{\delta_1\bar{A}_5}(\theta_0, 0) = e^{-\delta_2\bar{A}_4}(\theta_f, 0)$, i.e., such that $e^{\delta_2\bar{A}_4} \circ e^{\delta_1\bar{A}_5}(\theta_0, 0) = (\theta_f, 0)$.

Second case: $\boldsymbol{\theta(\tau_1) < \theta_f}$. Define $\tau_2 = \inf\{t > 0 \mid e^{t\bar{A}_4} \circ e^{\tau_1\bar{A}_5}(\theta_0, 0) \notin \bar{\mathcal{K}}\}$. For the same reason as for τ_1, we have $\tau_2 < \infty$. As in the first case, if $\theta(\tau_2) = \theta_f$, there must exist δ_1, $\delta_2 > 0$ such that $e^{\delta_1\bar{A}_5}(\theta_0, 0) = e^{-\delta_2\bar{A}_4}(\theta_f, 0)$. Indeed, the nonexistence of such δ_1, δ_2 would imply that

$$\{e^{t\bar{A}_4}(\theta_f, 0) \mid t < 0\} \bigcap \{e^{t\bar{A}_4} \circ e^{\tau_1\bar{A}_5}(\theta_0, 0) \mid t > 0\} \neq \emptyset,$$

which contradicts the uniqueness theorem for solutions to ODEs. If $\theta(\tau_2) < \theta_f$, we set $\delta_1 = \tau_1$ and define $\tau_3 = \inf\{t > 0 \mid e^{t\bar{A}_5} \circ e^{\tau_2\bar{A}_4} \circ e^{\delta_1\bar{A}_5}(\theta_0, 0) \notin \bar{\mathcal{K}}\}$. Then, as for τ_1, we go back to the distinction between first and second case and so on. Step by step, we iteratively construct a sequence $(t_n)_{n\in\mathbb{N}} = (S_n^\delta)_{n\in\mathbb{N}}$ of switching times such that $\theta(t_n) \in [\theta_0, \theta_f]$ and $\varphi(t_n) \in [0, \varepsilon_0]$ for all n. We next show that this process must stop. That is, we reach θ_f after finitely many commutations. To show this let us first evaluate the time elapsed between two consecutive commutations.

Evaluation of $\boldsymbol{t_{n+1} - t_n}$

According to the mean value theorem, for every $\xi \in (t_n, t_{n+1}]$, there exists $c \in (t_n, \xi)$ such that $|\varphi(\xi) - \varphi(t_n)| = |\dot{\varphi}(c)|(\xi - t_n)$. Consequently, according to the definition of α and β,

$$0 < \alpha \leqslant \frac{|\varphi(\xi) - \varphi(t_n)|}{\xi - t_n} \leqslant \beta, \quad \forall\, \xi \in (t_n, t_{n+1}]. \tag{IV.8}$$

Suppose to reach a contradiction that $\theta(t_n) < \theta_f$ for infinitely many n. In this case, we have $|\varphi(t_{n+1}) - \varphi(t_n)| = \varepsilon_0$. At this point, integrating (IV.3) we can evaluate the covered distance in θ. According to (IV.3), (IV.8), and as $\dot{\theta}$, $\varphi \geqslant 0$, and $\varphi(t_{2k}) = 0$, we have

$$\begin{aligned} |\theta(t_{2n+1}) - \theta_0| &= \sum_{k=0}^{2n} \int_{t_k}^{t_{k+1}} \dot{\theta}(\xi)d\xi \\ &\geqslant \sum_{k=0}^{2n} \int_{t_k}^{t_{k+1}} \alpha\varphi(\xi)d\xi \end{aligned}$$

$$\begin{aligned}
&\geqslant \sum_{k=0}^{n}\int_{t_{2k}}^{t_{2k+1}}\alpha\left|\varphi(\xi)-\varphi\left(t_{2k}\right)\right|d\xi\\
&\geqslant \sum_{k=0}^{n}\int_{t_{2k}}^{t_{2k+1}}\alpha^{2}\left(\xi-t_{2k}\right)d\xi\\
&= \sum_{k=0}^{n}\frac{\alpha^{2}}{2}\left(t_{2k+1}-t_{2k}\right)^{2}\\
&\geqslant n\frac{\alpha^{2}\varepsilon_{0}^{2}}{2\beta^{2}}\underset{n\to\infty}{\rightarrow}\infty, \qquad\text{(IV.9)}
\end{aligned}$$

which contradicts the fact that $\theta(t_n)<\theta_f$ for infinitely many n. Let N_0 denote the number of switching times during the constructed ε_0-forward motion. Notice that N_0 is even if the initial time ($t_0=0$) is counted as the first switching time. Once θ_f has been reached, in the same manner as for the ε_0-forward motion, we use the flows e^{tA_2} and e^{tA_3} to construct an ε_1-backward motion (with $\varepsilon_1\leqslant\varepsilon_0$) to go back to θ_0. Step by step, we iteratively construct a switching trajectory of system (IV.1) which is a concatenation of ε_n-motions where $(\varepsilon_n)_{n\in\mathbb{N}}$ is a chosen sequence of positive numbers decreasing to zero. Let N_n denote the number of commutations during an ε_n-motion. By construction, for every $n\in\mathbb{N}$, we have

- $\left|\theta\big(t_{S_{n+1}^N}\big)-\theta\left(t_{S_n^N}\right)\right|=\theta_f-\theta_0$;
- $\varphi(t_{2k})=0, \quad \forall\, k\in\mathbb{N}$;
- $\left|\varphi\left(t_{k+1}\right)-\varphi\left(t_k\right)\right|=\varepsilon_n, \quad \forall\, k\in\{S_{n-1}^N+1,S_n^N-2\}$;
- $\left|\varphi\left(t_{S_n^N}\right)-\varphi\left(t_{S_n^N-1}\right)\right|=\gamma_n, \quad$ with $0<\gamma_n\leqslant\varepsilon_n$.

In the last equality γ_n corresponds to the value of φ at the last switching time of the ε_n-motion.

2$^{\text{nd}}$ **step: evaluation of the number of switching times during an** ε_n**-motion**

One easily repeats on the time interval of an ε_n-motion a computation similar to (IV.9) to conclude that

$$\theta_f-\theta_0=\theta\left(t_{S_n^N+1}\right)-\theta\left(t_{S_n^N}\right)\geqslant\left(\frac{N_n-2}{2}\varepsilon_n^2+\gamma_n^2\right)\frac{\alpha^2}{2\beta^2},$$

from which it follows that

$$N_n\leqslant\frac{4\beta^2(\theta_f-\theta_0)}{\alpha^2\varepsilon_n^2}+\frac{2(\varepsilon_n^2-\gamma_n^2)}{\varepsilon_n^2}\leqslant\frac{C}{\varepsilon_n^2}, \qquad\text{(IV.10)}$$

with $C=\left(4\beta^2(\theta_f-\theta_0)+2\alpha^2\varepsilon_0^2\right)/\alpha^2$.

3$^{\text{rd}}$ **step: evaluation of the loss in norm**

The last thing we have to do is to evaluate the loss in norm along the whole trajectory. Let us first estimate the loss in norm during an ε_n-motion. Denote by $[b_0^n,b_{N_n}^n]$ the time intervals corresponding to this motion. Set $a=\max\{a_3^1,\ldots,a_3^8\}$. According to (IV.2), (IV.8) and (IV.10), one infers that

$$\begin{aligned}
r\left(b_0^n\right)-r\left(b_{N_n}^n\right)&=\int_{b_0^n}^{b_{N_n}^n}|\dot r(\xi)|\,d\xi\leqslant\int_{b_0^n}^{b_{N_n}^n}ar_0\varphi^2(\xi)d\xi\\
&\leqslant ar_0\varepsilon_n^2\int_{b_0^n}^{b_{N_n}^n}d\xi=ar_0\varepsilon_n^2\sum_{k=1}^{N_n}\left|b_k^n-b_{k-1}^n\right|\\
&\leqslant ar_0\varepsilon_n^2\sum_{k=1}^{N_n}\alpha^{-1}\left|\varphi\left(b_k^n\right)-\varphi\left(b_{k-1}^n\right)\right|\\
&\leqslant N_nar_0\alpha^{-1}\varepsilon_n^3\leqslant\widetilde C\varepsilon_n,
\end{aligned}$$

where $\widetilde C=ar_0C/\alpha$. Consequently, the loss in norm up to time $b_{N_n}^n$ is

$$r_0-r\left(b_{N_n}^n\right)=\sum_{k=0}^{n}\left|r\big(b_0^k\big)-r\big(b_{N_k}^k\big)\right|\leqslant\widetilde C\sum_{k=0}^{n}\varepsilon_n,$$

and, passing to the limit as n tends to infinity, we get

$$r_0-\lim_{t\to\infty}r(t)=r_0-\lim_{n\to\infty}r\left(b_{N_n}^n\right)\leqslant\widetilde C\sum_{k=0}^{\infty}\varepsilon_n,$$

which can be made strictly less than r_0. In such a case, $\lim_{t\to\infty}r(t)=r_f>0$ and by construction, $\Omega(z_0)=\{r_f\}\times[\theta_0,\theta_f]\times\{0\}$ which is not a singleton.

4$^{\text{th}}$ **and last step: fulfilling Assumption IV.1**

The constructed trajectory violates Assumption IV.1 (switch dwell-time). We show here that we can slightly modify this trajectory so that it will respect the dwell-time on switches. Notice that during a forward motion the commutation from A_5 to A_4 involves only the use of switch u_1. Thus, at every switching time t_{2k}, instead of switching back to A_5 we can switch to A_8 using the switch u_3. Since $x(t_{2k})\in\ker A_8$, we have $e^{\tau A_8}\left(x(t_{2k})\right)=x(t_{2k})$ for every positive τ. Choosing τ greater than the switch dwell-time shows that any forward ε_{2k}-motion can be made respecting Assumption IV.1. The reader can easily check that the same can also be done for every backward ε_{2k+1}-motion. Consequently, we can assume that the constructed trajectory satisfies Assumption IV.1. Even more, by complicating somewhat the way we construct the trajectory, each matrix of family $\mathscr{F}$ can be employed in such a way that Assumption III.7 (persistent activation) is satisfied.

Final remarks. It is easy to see that one can construct trajectories of system (IV.1) whose ω-limit sets do not reduce to a singleton, and which satisfy both the weak dwell-time and the switch dwell-time assumptions. Such trajectories can be constructed as soon as the number of cells in the converter is greater or equal to three. When the number of cells is smaller or equal to two, Proposition III.6 shows that the weak dwell-time (Assumption II.7) is sufficient to insure the convergence of the trajectories to zero.

A signal which satisfies the condition of dwell-time for every mode (Assumption II.4) is contained in the set denoted by $\mathcal{S}_{\text{average}}[\tau_D,N_0]$ in [10], [13], [14]. Notice that the signal in the above example can be adapted in such a way that it belongs to the set $\mathcal{S}_{\text{average}}[\tau_D,N_0]$. So the dwell-time notion introduced in [10], [13], [14] is weaker than ours but it does not imply the convergence of the state to the origin.

Acknowledgments

This work was written while the first author was member of the IECN laboratory in Nancy, it was supported in part by the CPER AOC-SC2 and by the ANR project ArHyCo, Programme “Systèmes Embarqués et Grandes Infrastructures” ARPEGE, contract number ANR-2008 SEGI 004 01-30011459.

REFERENCES

[1] R. A. Decarlo, M. S. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," in *Proceedings of the IEEE*, vol. 88, no. 7, 2000, pp. 1069–1082.

[2] D. Liberzon, *Switching in systems and control*, ser. Systems & Control: Foundations & Applications. Boston, MA: Birkhäuser Boston Inc., 2003.

[3] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. Automat. Control*, vol. 43, no. 4, pp. 475–482, 1998, hybrid control systems.

[4] J. Daafouz, P. Riedinger, and C. Iung, "Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach," *IEEE Trans. Automat. Control*, vol. 47, no. 11, pp. 1883–1887, 2002.

[5] D. Liberzon, J. P. Hespanha, and A. S. Morse, "Stability of switched systems: a Lie-algebraic condition," *Systems Control Lett.*, vol. 37, no. 3, pp. 117–122, 1999.

[6] P. E. Kloeden, "Nonautonomous attractors of switching systems," *Dyn. Syst.*, vol. 21, no. 2, pp. 209–230, 2006.

[7] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability criteria for switched and hybrid systems," *SIAM Rev.*, vol. 49, no. 4, pp. 545–592, 2007.

[8] A. Bacciotti and L. Mazzi, "An invariance principle for nonlinear switched systems," *Systems Control Lett.*, vol. 54, no. 11, pp. 1109–1119, 2005.

[9] V. Chellaboina, S. P. Bhat, and W. M. Haddad, "An invariance principle for nonlinear hybrid and impulsive dynamical systems," *Nonlinear Anal.*, vol. 53, no. 3-4, pp. 527–550, 2003.

[10] J. P. Hespanha, "Uniform stability of switched linear systems: extensions of LaSalle's invariance principle," *IEEE Trans. Automat. Control*, vol. 49, no. 4, pp. 470–482, 2004.

[11] J. Lygeros, K. H. Johansson, S. N. Simić, J. Zhang, and S. S. Sastry, "Dynamical properties of hybrid automata," *IEEE Trans. Automat. Control*, vol. 48, no. 1, pp. 2–17, 2003.

[12] J. L. Mancilla-Aguilar and R. A. García, "An extension of LaSalle's invariance principle for switched systems," *Systems Control Lett.*, vol. 55, no. 5, pp. 376–384, 2006.

[13] R. G. Sanfelice, R. Goebel, and A. R. Teel, "Invariance principles for hybrid systems with connections to detectability and asymptotic stability," *IEEE Trans. Automat. Control*, vol. 52, no. 12, pp. 2282–2297, 2007.

[14] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Invariance principles for switching systems via hybrid systems techniques," *Systems Control Lett.*, vol. 57, no. 12, pp. 980–986, 2008.

[15] A. Morse, "Supervisory control of families of linear set-point controllers part i. exact matching," *IEEE Trans. Automat. Control*, vol. 41, no. 10, pp. 1413–1431, Oct 1996.

[16] W. P. Dayawansa and C. F. Martin, "A converse Lyapunov theorem for a class of dynamical systems which undergo switching," *IEEE Trans. Automat. Control*, vol. 44, no. 4, pp. 751–760, 1999.

[17] P. Mason, U. Boscain, and Y. Chitour, "Common polynomial lyapunov functions for linear switched systems," *SIAM J. Control Optim.*, vol. 45, no. 1, pp. 226–245, 2006.

[18] I. Daubechies and J. C. Lagarias, "Sets of matrices all infinite products of which converge," *Linear Algebra Appl.*, vol. 161, pp. 227–263, 1992.

[19] W.-J. Beyn and L. Elsner, "Infinite products and paracontracting matrices," *Electron. J. Linear Algebra*, vol. 2, pp. 1–8 (electronic), 1997.

[20] M. Neumann and H. Schneider, "The convergence of general products of matrices and the weak ergodicity of Markov chains," *Linear Algebra Appl.*, vol. 287, no. 1-3, pp. 307–314, 1999.

[21] L. Mate, "On the infinite product of operators in Hilbert space," *Proc. Amer. Math. Soc.*, vol. 126, no. 2, pp. 535–543, 1998.

[22] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, ser. Translated from the French and revised. Amsterdam: North-Holland Publishing Co., 1970.

[23] S. Nelson and M. Neumann, "Generalizations of the projection method with applications to SOR theory for Hermitian positive semidefinite linear systems," *Numer. Math.*, vol. 51, no. 2, pp. 123–141, 1987.

[24] A. A. Agrachev and Y. L. Sachkov, *Control theory from the geometric viewpoint*, ser. Encyclopaedia of Mathematical Sciences. Berlin: Springer-Verlag, 2004, vol. 87, control Theory and Optimization, II.

[25] W. Rudin, *Real and complex analysis*, 3rd ed. New York: McGraw-Hill Book Co., 1987.

[26] T. Meynard, M. Fadel, and N. Aouda, "Modeling of multilevel converters," *IEEE Trans. Ind. Electron.*, vol. 44, no. 3, pp. 356–364, 1997.

[27] T. Meynard, H. Foch, P. Thomas, J. Courault, R. Jakob, and M. Nahrstaedt, "Multicell converters: basic concepts and industry applications," *IEEE Trans. Ind. Electron.*, vol. 49, no. 5, pp. 955–964, 2002.

[28] J. Rodríguez, S. Bernet, B. Wu, J. Pontt, and S. Kouro, "Multilevel voltage-source-converter topologies for industrial medium-voltage drives," *IEEE Trans. Ind. Electron.*, vol. 54, no. 6, pp. 2930–2945, 2007.

[29] K. Benmansour, J. de Leon, and M. Djemaï, "Adaptive observer for multi-cell chopper," in *Second International Symposium on Communications, Control and Signal Processing ISCCSP, Marrakech, Maroc*, 2006.

[30] K. Benmansour, M. Djemaï, M. Tadjine, and M. Boucherit, "On sliding mode observer for hybrid three cells converter : Experimental results," *International Workshop on Variable Structure Systems, 2008. VSS '08*, pp. 373–377, June 2008.

[31] R. Bensaid and M. Fadel, "Floating voltages estimation in three-cell converters using a discrete-time Kalman filter," *Power Electronics Specialists Conference, 2001. PESC. 2001 IEEE 32nd Annual*, vol. 1, pp. 327–332 vol. 1, 2001.

Ulysse Serres received the Ph.D. degree in mathematics from the University of Burgundy, Dijon, France, in 2006.

He is now Assistant Professor since 2009 at the Department of Processes and Electrical Engineering at the University Claude Bernard, Lyon, France and researcher at the LAGEP Laboratory. His research interests include geometric control, observers theory and stabilization of switched systems.

Jean-Claude Vivalda received the HDR degree in mathematics from the University Paul Verlaine of Metz, France, in 2002.

He has been research manager at Metz with the French National Institute for Research in Computer Science and Control (INRIA) since 2003; he is a member of the CORIDA project (a research team of INRIA). His research interests span theoretical control, observers theory and feedback stabilization.

Pierre Riedinger received his M.Sc. degree in applied mathematics from the University Joseph Fourier, Grenoble, France, in 1993 and the Ph.D. degree in Automatic Control from the Institut National Polytechnique de Lorraine, Nancy-Université, Nancy, France, in 1999.

He is now Associate Professor since 2001 at the engineering school ENSEM and researcher at the CRAN Laboratory. His current research interests include analysis, control (optimal, predictive) and observation of complex systems (switched, hybrid).