DISTURBANCE ATTENUATION CONTROL FOR A CLASS OF HYBRID SYSTEMS

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Abstract

In this paper, a disturbance attenuation problem for a class of hybrid control systems is posed when the switching from a vector field (also called mode) to another is controlled. Using the equivalence between this problem and a linear-quadratic differential game with piecewise deterministic dynamics, a solution is proposed to this problem.

1 Introduction

Recently, there has been an increasing interest in systems where both continuous and discrete variables influence the dynamic behaviour [1], [2], [3], [4], [5]. Such hybrid systems include a fairly large class of physical systems in control engineering applications. One of the reasons that justifies the study of this class of systems is its ability for addressing problems related to the hybrid nature of physical processes. Some of them are not new like those arising in continuous systems including relays with hysteresis. Today, one has also to deal with hybrid phenomena due to the fact that most of the physical plants are now controlled via computers and a considerable interest in looking for a theoritical framework for hybrid systems emerged recently in the control engineering literature [6], [7], [8].

A general description of hybrid systems uses differential equations to describe the continuous dynamics and a discrete event system to model the discrete dynamics. The discrete dynamics plays generally the role of a supervisor of the continuous part. A strong interaction between these two type of variables gives rise to hybrid phenomena. In [7], a relevant description of some representative hybrid phenomena is given. Usually, when a discrete event occurs, it can lead to a change in the continuous dynamics. Indeed, discontinuities in the vector field and/or in the continuous state may appear.

In this paper, we consider hybrid systems described by piecewise linear vector fields where controlled switching phenomena are allowed [7]. The problem considered is to find a hybrid control strategy such that a L_2 gain of the closed loop system is less than or equal to some specified positive level γ. By hybrid strategy we mean to find the switching time sequence, the corresponding mode (vector field) sequence as well as state feedback controllers such that the influence of some exogeneous disturbance $w(t)$, is to be minimized in the worst case. To this end, we use the equivalence between a disturbance hybrid attenuation problem and a linear-quadratic differential game with piecewise deterministic dynamics. In this contribution, we essentially focus on the determination of the switching time sequence, the corresponding vector field sequence as well as the state feedback controllers.

Recently, a problem of finding a switching rule that minimizes a performance index has been formulated in [9]. Given a continuous plant, a collection of output feedback non-linear controllers and a time period T, the authors look for a strategy for switching from one basic controller to another to achieve a level of performance index. The results are based on the existence of suitable solutions to a Riccati algebraic equation and a dynamic programming equation. In [10], piecewise deterministic differential games with hybrid controls has been considered. The changes from one vector field (mode) to another are governed by a finite-state Markov process. One of the associated difficulties is that the corresponding dynamic programming equations cannot be solved explicitly, and may even not admit continuously-differentiable value functions. Relevant results on the existence and uniqueness of viscosity solutions associated with the dynamic programming equations are proposed. Analytical solutions are obtained only when the continuous state has dimension one and a computation algorithm is proposed for the general case.

The outline of the paper is as follows: In section 2, differential game with piecewise deterministic dynamics is presented. The hybrid disturbance attenuation problem which is the main point of the paper is introduced. In the third section, to get a solution, we use the link between the disturbance attenuation problem and the corresponding differential game with piecewise deterministic dynamics. The switching time sequence determination is discussed in the case of controlled switching phenomena. An illustrative example is proposed in section 4 before a general conclusion.

2 Hybrid differential games

By hybrid differential games, we mean games where the dynamics are described by piecewise deterministic vector fields of the form

$$\dot{x}(t) = f(t, x(t), k(t), u_k(t), w(t)), \qquad x(t_0) = x_0 \quad (1)$$

where $x \in \mathbb{R}^n$ is the continuous state. The index $k \in \mathcal{K} = \{1, ..., K\}$ is the discrete state (also called the mode k). In general, its dynamic is given by a transition function

$$k(t) = \phi(t, x(t), k(t^-), d(t)), \quad k(t_0) = k_0 \quad (2)$$

where $d(t) \in \mathcal{D} = \{1, ..., D\}$ is a discrete input. $(u_k(t), d(t)) \in \mathcal{U}_k \times \mathcal{D}$ is the hybrid control associated to the first player and $w(t) \in \mathcal{W}$ is the control associated to the second player. $\mathcal{U}_k$ and $\mathcal{W}$ are assumed to be compact sets.

Here, we are interested in controlled switching hybrid phenomena [7] and the discrete state k is now given by

$$k(t) = \phi(t, x(t), k(t^-), d(t)) \equiv d(t)$$

The transition function $\phi(., ., ., .)$ is of no use in our problem since the discrete control is k itself and we do not deal with autonomous jumps. This means that we are allowed to switch from an operating mode to another one at any time. There are several theoretical and practical interesting problems concerning the use of such a hybrid strategy. For example, given a collection of possible system configurations, one may be interested in associating a control law to each configuration and providing an optimal switch fashion between them ensuring a performance improvement over a fixed control law.

Consider a fixed time t_f and let $[0, \tau_1, ..., \tau_i, ..., t_f]$ be the sequence of switching time and $[k_0, k_1, ..., k_i, ..., k_q]$ ($k_i \in \mathcal{K}$) the corresponding mode sequence. The following criteria is associated with the hybrid dynamic system (1)-(2)

$$J = \sum_{i \geq 0} \int_{\tau_i}^{\tau_{i+1}} L\big(t, x(t), k_i, u_{k_i}(t), w(t)\big) dt \quad (3)$$

Hence, different criteria are allowed with respect to each mode. The functions f and L are assumed to be of Carathéodory type, that is measurable in t, Lipschitz-continuous in x and continuous in (u_k, w).

Let a hybrid zero-sum differential game with state feedback information defined for the hybrid system (1)-(2) with the criteria (3) and where (u_k, k) is the minimizer and w the maximizer. We consider games where feedback strategies are allowed of the form

$$u_k(t) = \varphi_k(t, x(t)), \quad \text{with} \quad \varphi_k : [0, t_f] \times \mathbb{R}^n \mapsto \mathcal{U}_k$$

$$w(t) = \psi(t, x(t)), \quad \text{with} \quad \psi : [0, t_f] \times \mathbb{R}^n \mapsto \mathcal{W}$$

Define the Hamiltonian function associated to each mode k by:

$$H(t, x, \lambda, k, u_k, w) = \lambda^T f(t, x, k, u_k, w) + L(t, x, k, u_k, w)$$

The following proposition follows from dynamic game theory and dynamic programing arguments. The proof uses similar arguments as in the proof of Theorem 4.3. in [11].

Proposition 1 *If there exists a piecewise $\mathcal{C}^1$ function $V(t, x, k)$ such that $V_k = V(., ., k) \in \mathcal{C}^1$ for all $k \in \mathcal{K}$ and satisfying the following Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation*

$$-\frac{\partial V_k}{\partial t} = \max_{w} \min_{u_k} H(t, x, \lambda, k, u_k, w) \quad (4)$$

$$V(t_f, x, k) = 0 \quad (5)$$

with the following transversality conditions satisified at time corresponding to a controlled switching time from the mode k to the mode j:

$$\lambda_j^+ = \lambda_k^- \quad (6)$$

$$H_j^+ = H_k^- \quad (7)$$

then the hybrid zero-sum dynamic game has a saddle point solution. Moreover, the hybrid control law given by

$$(\varphi_k^*(t, x(t)), k^*(t)) \in Arg \min_{u_k, k} H(t, x, -\frac{\partial V_k}{\partial t}, k, u_k, \psi^*) \quad (8)$$

$$\psi^*(t, x(t)) \in Arg \max_{w} H(t, x, -\frac{\partial V_k}{\partial t}, k^*, \varphi_k^*, w) \quad (9)$$

is optimal.

3 Hybrid disturbance attenuation

The main contribution of this paper deals with hybrid systems described by:

$$\dot{x}(t) = A_k x(t) + B_k u_k(t) + E_k w(t), \quad x(0) = x_0 \quad (10)$$

$$z(t) = C_k x(t) + D_k u_k(t), \quad k(0) = k_0 \quad (11)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^{m_k}$ is the continuous control, $w \in \mathbb{R}^l$ is the disturbance, $z(t) \in \mathbb{R}^r$ is the controlled output and the index k denotes the mode $k \in \mathcal{K} = \{1, ..., K\}$.

The hybrid control law (u_k, k) is composed of a discrete control k and a continuous control u_k. The discrete part allows to select which mode is active during a period formed by two switching times. The continuous part is the continuous control relative to that mode during this period. Consider a fixed time t_f, the disturbance attenuation hybrid control synthesis problem addressed in this paper consists in finding a switching time sequence

$$[0, \tau_1, \tau_2, ..., \tau_i, ..., t_f]$$

the corresponding mode sequence

$$[k_0, k_1, ..., k_i, ..., k_q], \quad k_j \in \mathcal{K}, \;\; j = 0, 1, ..., i, ...q$$

and the relative state feedback continuous controls

$$u_{k_j}(t) = \varphi_{k_j}(t, x(t)) \qquad \tau_j \le t < \tau_{j+1}$$

such that the resulting closed loop system has a L_2 gain less than or equal to some specified level γ. Our aim is to get a hybrid control law that minimizes the effect of the disturbance w in the worst case. This naturally leads us to consider a hybrid differential game with a criteria J_γ, indexed by the real positive number γ, defined by:

$$J_\gamma(t, x, k, u_k, w) = \int_0^{t_f} (\|z(t)\|^2 - \gamma^2 \|w\|^2) dt \qquad (12)$$

To simplify the exposition, the following assumption is made

$$\begin{pmatrix} C'_k \\ D'_k \end{pmatrix} \begin{pmatrix} C_k & D_k \end{pmatrix} = \begin{pmatrix} Q_k & \mathbf{0} \\ \mathbf{0} & R_k \end{pmatrix}$$

where $Q_k \ge \mathbf{0}$ and $R_k > \mathbf{0}$ are assumed to be matrices of appropriate dimensions. The results presented in this paper can be adapted to the case $C'D \neq \mathbf{0}$.

The performance index J_γ reduces to

$$J_\gamma = \sum_{i \ge 0} \int_{\tau_i}^{\tau_{i+1}} \left(x^T Q_k x + u_k^T R_k u_k - \gamma^2 w^T w \right) dt \qquad (13)$$

A solution to the disturbance attenuation problem is obtained applying Proposition 1. In fact, a mode k_i operating during a time interval $[\tau_i, \tau_{i+1})$, belongs to an optimal hybrid strategy if $\forall k \in \mathcal{K}, k \neq k_i$:

$$H(t, x, \lambda, k_i, u_{k_i}, w) < H(t, x, \lambda, k, u_k, w) \qquad (14)$$

Satisfying the HJBI equation (4) consists in finding $P_{k_i}(t)$ solution of the following differential Riccati equation

$$\begin{array}{c} \dot{P_{k_i}} + P_{k_i} A_{k_i} + A_{k_i}^T P_{k_i} - P_{k_i}(B_{k_i} R_{k_i}^{-1} B_{k_i}^T - \\ \gamma^{-2} E_{k_i}^T E_{k_i}) P_{k_i} + Q_{k_i} = \mathbf{0}, \end{array} \qquad (15)$$

In this case, the optimal continuous control law is given by

$$\varphi_{k_i}^*(t, x) = -R_{k_i}^{-1} B_{k_i}^T P_{k_i}(t) x(t), \qquad \tau_i \le t < \tau_{i+1} \qquad (16)$$

which corresponds to the worst case disturbance action

$$\psi^*(t, x) = \gamma^{-2} E_{k_i}^T P_{k_i}(t) x(t), \qquad \tau_i \le t < \tau_{i+1} \qquad (17)$$

At time τ_{i+1}, the following limit conditions have to be satisfied:

$$\begin{array}{rcl} P_{k_{i+1}}(\tau_{i+1}) x(\tau_{i+1}) & = & P_{k_i}(\tau_{i+1}) x(\tau_{i+1}) \qquad (18) \\ H_{k_{i+1}} & = & H_{k_i} \qquad (19) \end{array}$$

At the final time, as $x(t_f)$ is free, the following must hold

$$P_k(t_f) x(t_f) = \mathbf{0}$$

Now, solving the hybrid disturbance attenuation problem consists in finding the discret control k, namely the switching time sequence $[0, \tau_1, \tau_2, ..., \tau_i, ..., t_f]$, and the corresponding mode sequence

$$[k_0, k_1, ..., k_i, ...k_q], \qquad k_j \in \mathcal{K}, \;\; j = 0, 1, ..., i, ..., q$$

such that

- (14), (15), (16) and (17) are satisfied for all k_j on $[\tau_j, \tau_{j+1})$,
- (18)-(19) are satisfied at the switching time τ_{j+1}.

The switching time sequence is determined as follows. Assume that the operating mode is k_i, which means that

$$H_{k_i} < H_k \;\; \forall k \in \mathcal{K}, k \neq k_i \qquad (20)$$

A switch time will occur at the first instant where a mode $k_{i+1} \in \mathcal{K}, k_{i+1} \neq k_i$ leads to

$$\nu_{k_i, k_{i+1}} = H_{k_i} - H_{k_{i+1}} = 0$$

Hence, in the case of controlled switching, the discrete part of the hybrid control law is obtained using the following theorem.

Theorem 1 *The function $\nu_{k_i,k_{i+1}} = H_{k_i} - H_{k_{i+1}}$ satisfies an homogeneous linear ODE with constant coefficients defined by the characteristic polynomial S_{k_i} where S_{k_i} is the minimal polynomial of $M_{k_i} \oplus M_{k_i}$ with*

$$M_{k_i} = \begin{pmatrix} A_{k_i} & B_{k_i} R_{k_i}^{-1} B_{k_i}^T - \gamma^2 E'_{k_i} E_{k_i} \\ Q_{k_i} & -A_{k_i}^T \end{pmatrix}$$

Proof: The hamiltonian system corresponding to the saddle point solution is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{pmatrix} A_{k_i} & B_{k_i} R_{k_i}^{-1} B_{k_i}^T - \gamma^2 E'_{k_i} E_{k_i} \\ Q_{k_i} & -A_{k_i}^T \end{pmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

The function $\nu_{k_i,k_{i+1}}(t)$ becomes

$$\nu(t) = \left\langle \begin{bmatrix} x \\ \lambda \end{bmatrix}, \Delta \begin{bmatrix} x \\ \lambda \end{bmatrix} \right\rangle$$

where

$$\Delta = \begin{pmatrix} Q_{k_{i+1}} - Q_{k_i} & 0 \\ A_{k_i} - A_{k_{i+1}} & \tilde{\Delta} \end{pmatrix}$$

with

$$\tilde{\Delta} = B_{k_i} R_{k_i}^{-1} B_{k_i}^T - B_{k_{i+1}} R_{k_{i+1}}^{-1} B_{k_{i+1}}^T - \gamma^2 (E_{k_i}^T E_{k_i} - E_{k_{i+1}}^T E_{k_{i+1}})$$

Differentiating $\nu_{k_i,k_{i+1}}(t)$ successively with respect to t, following the same developments in [12], one gets:

$$\frac{d^n\nu}{dt^n}(t) = \left\langle \begin{bmatrix} x \\ \lambda \end{bmatrix} \otimes \begin{bmatrix} x \\ \lambda \end{bmatrix}, (M_{k_i}^T \oplus M_{k_i})^n \; col(\Delta) \right\rangle$$

Using the Caley-Hamilton theorem allows to state the result given in Theorem 1. ■

Theorem 1 helps in the determination of the switching time sequence. In fact, starting with some given intial conditions x_0 and P_0, the mode k_0 is the one satisfying the inequality

$$H_{k_0} < H_k, \quad \forall k \in \mathcal{K}, k \neq k_0$$

A switch to a mode k_1 will occur at the instant $t = \tau_1$ where

$$\nu_{k_0,k_1}(\tau_1) = 0$$

with $k_1 \in \mathcal{K}$, $k_1 \neq k_0$. According to Theorem 1 this instant can be easily determined since one knows the initial conditions as well as the roots of the homogenous linear ODE. The next switching times and modes are determined by reproducing the previous operations.

Remark 1 *To satisfy the transversality condition (18), at the switching time $t = \tau_{i+1}$, one has to choose $P_{k_{i+1}}$ such that $x \in Ker(P_{k_{i+1}} - P_{k_i})$. One may choose $P_{k_i} = P_{k_{i+1}}$ even if P_{i+1} is not unique at this time. In fact, if there are two candidates $P^1_{k_{i+1}}$ and $P^2_{k_{i+1}}$ after the switching time τ_i, $x(t) \in Ker(P^1_{k_{i+1}} - P^2_{k_{i+1}})$ since the solution of the hamiltonian system is unique.*

Remark 2 *One can see that in the previous the control goal is to attenuate the disturbances in a finite time and not necessary to transfer the state to zero. Letting $t_f \to +\infty$ one can follow the same steps to solve the disturbance attenuation problem for the infinite time horizon.*

4 Illustrative example

To illustrate the proposed hybrid disturbance attenuation control, we consider a hybrid system given by (10)-(11) with two operating modes ($K = 2$) characterized by

$$A_1 = \begin{bmatrix} -1 & 4 \\ -3 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1.7 & 0 \\ 0 & 2 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1.4 & 0 \\ 0 & 1.4 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.7 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.707 & 0.707 \\ 0 & 0.707 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1.414 & 0.354 \\ 0 & 0.935 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We consider a disturbance attenuation problem with

$$t_f \to +\infty$$

If the two modes are considered separately, that is to solve independently two state feeedback linear time invariant optimal H_∞ control problems associated with the two LTI systems

$$\dot{x}(t) = A_i x(t) + B_i u(t) + E_i w(t), \qquad i = 1, 2$$

the corresponding optimal attenuation level values are

$$\gamma_1^* = 0.7545, \quad \gamma_2^* = 0.6519$$

Using Theorem 1, we apply the following algorithm for a given value of γ to find a periodic switching sequence.

- i: Fix x_0, P_0, $i = 0$.
- ii: Evaluate $\nu_{1,2}(\tau_i) = H_1 - H_2$. If $\nu_{1,2}(\tau_i) > 0$ then $k_i = 2$ else $k_i = 1$.
- iii: Find the time τ_{i+1} such that $\nu(\tau_{i+1}) = 0$ by solving the homogenous linear ODE of Theorem 1. Compute $x_{i+1} = x(\tau_{i+1})$ and $P_{i+1} = P(\tau_{i+1})$ and switch to the other mode.
- iv: Normalize x_{i+1} and let $i \leftarrow i + 1$. Go to step iii until one gets a periodic sequence $(x_{i+1}, P_{i+1})_{i>k}$.

Using this algorithm, we look for a periodic solution which may not exit in general. For the proposed example, we get the following sequence

$$\begin{array}{lllllllll} \tau_0, & \tau_1, & .... & \tau_i, & \tau_{i+1}, & .... & \tau_{i+T}, & \tau_{i+T+1}, & .... \\ x_0, & x_1, & .... & x_i, & x_{i+1}, & .... & x_{i+T}, & x_{i+T+1}, & .... \\ P_0, & P_1, & .... & P_i, & P_{i+1}, & .... & P_{i+T}, & P_{i+T+1}, & .... \\ k_0, & k_1, & .... & k_i, & k_{i+1}, & .... & k_{i+T}, & k_{i+T+1}, & .... \end{array}$$

which means that from the rank $i + 1$ to the rank $i + T$, the values $x_{i+1}, x_{i+2}, ..., x_{i+T}$ define lines in the state space which indicate where a switch from a mode to another mode has to occur. Moreover, each mode k_i is active during a time

$$t_{k_i} = \tau_{i+1} - \tau_i$$

after which one has to swich the mode k_{i+1}.

Applying this algorithm for $\gamma = 0.7$, we get a periodic solution leading to the state trajectory depicted in figure 1. The line L_{12} indicates where to switch from the mode 1 to the mode 2. The corresponding time duration is $t_1 = 1.6009s$. The line L_{21} indicates where to switch from the mode 2 to the mode 1.

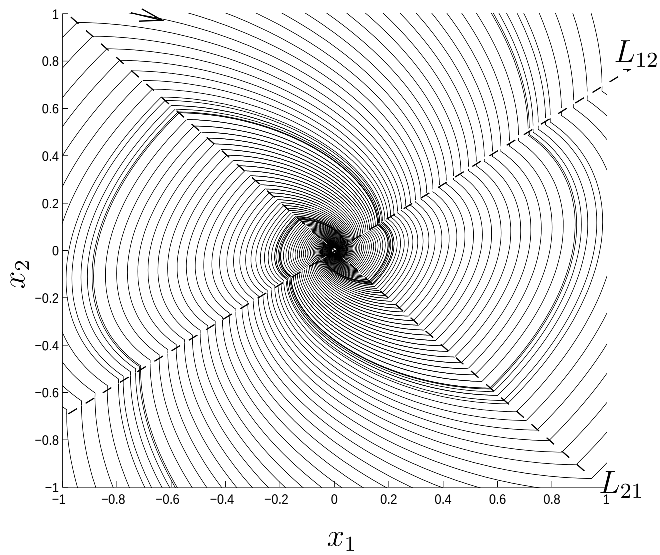

Figure 1: State trajectory $\gamma = 0.7$

The corresponding time duration is $t_2 = 0.4529s$.

Our goal is to find a controlled switching sequence and the corresponding mode sequence such that the closed loop system has a L_2 gain less than or equal to some specified level γ^* better than γ_1^* and γ_2^*. Decreasing the value of the performance level to $\gamma = 0.6089$ and using the previous algorithm the obtained solution leads to the state trajectory depicted in figure 2. The corresponding time durations are

$$t_1 = 1.7327s, \quad \text{and} \quad t_2 = 0.3366s$$

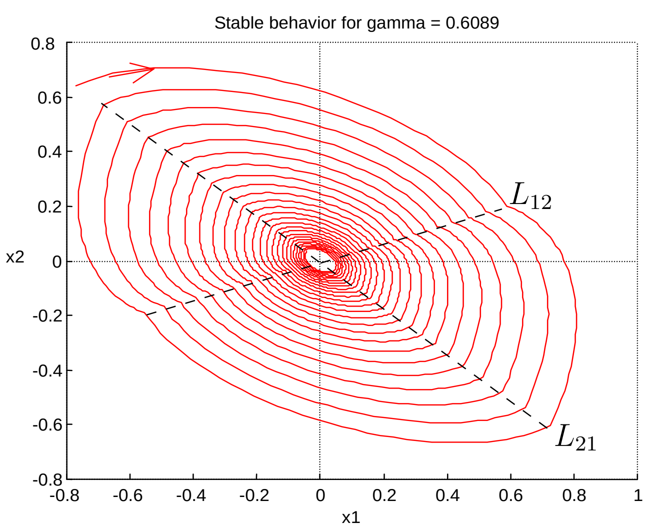

Figure 2: State trajectory $\gamma = 0.6089$

A limit cycle behaviour is obtained using the previous algorithm for $\gamma^* = 0.60874$ (Figure 3). The corresponding time durations are

$$t_1 = 1.7400s, \qquad t_2 = 0.3252s$$

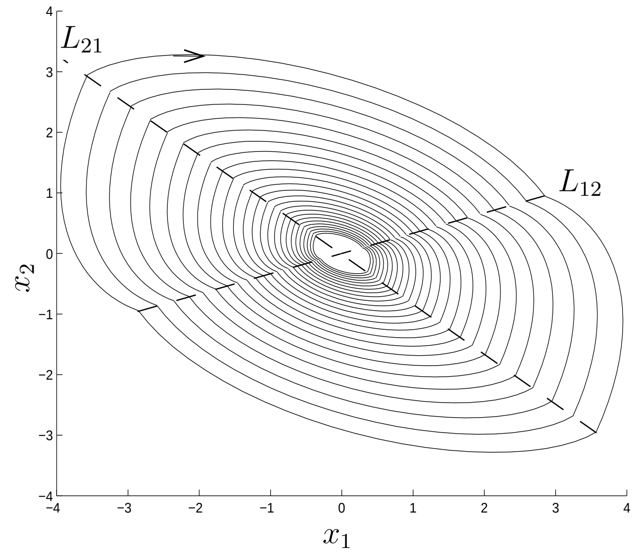

Figure 3: State trajectory $\gamma = 0.60874$

One can consider other values for γ less than 0.60874. However, numerical problems arise under this critical value

$$\gamma = 0.60874$$

5 Conclusion

In this paper, a disturbance attenuation problem for a class of hybrid systems where controlled switching phenomena may occur has been considered. The proposed solution uses the equivalence between this problem and a linear-quadratic differential game with piecewise deterministic dynamics. The results are proposed in the fixed finite time case but still hold for the infinite time horizon case. In general, the proposed solution depends on the initial hybrid state (x_0,k_0), and its main drawback is that there is no systematic algorithm in the general case. Development of efficient and reliable numerical algorithms is the purpose of future work.

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