Suboptimal switched controls in context of singular arcs

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***Abstract*— In this paper suboptimal control in the case of hybrid systems is addressed. After a brief recall of necessary conditions on the optimal hybrid trajectory, it is shown that many hybrid optimal problem cannot have an hybrid solution. Nevertheless, we show in context of switched systems that there is some possibility to take advantage of the study of a convex embedding problem for which a solution exists. Indeed the presence of singular arcs in this solution explains why the original switched system has no solution and how suboptimal chattering solutions can be found.**

I. INTRODUCTION

In the last years, necessary conditions for optimal control of hybrid system have been addressed [1], [2], [3]. These results melt both discrete and continuous classical necessary conditions on the optimal control. The discrete dynamic corresponding to the switching times leads to a dynamic programming type approach [4]. In the other hand, between these a priori unknown discrete values of time, optimization of the continuous dynamic is performed using the maximum principle [5], [6] or Hamilton Jacobi Bellmann equations [7]. In addition at the switching instants the link between the different continuous subsystems is ensured by the tranversality conditions.

Today, there are extensive efforts to develop efficient method to solves such problems [8], [9], [10], [11], [12], ... In the more simple cases as hybrid systems with a vector field depending on a fixed partition of the state space or linear switched systems (at any time any discrete state is available), some results on state or output feedback have been obtained (see [13], [14], [15] [16], [17], [18],) but in general, when the continuous dynamics really affects the discrete dynamics the problem seems reasonably not tractable in term of computational efforts. In this paper we show for a very simple time optimal control problem (a two linear modes switched system) that despite the existence of trajectories from an initial state to a final state the optimal one doesn't exist. So, in order to understand this basic fact, we define an embedding convex problem for which classical existence theorem works. In this situation the optimal solution includes singular arcs and this explains why such a problem doesn't have a solution. Nevertheless, we propose to build a suboptimal control by chattering on singular arcs. A theorem shows that we can approach near at will to the optimal solution if an average condition is respected.

II. SWITCHED-HYBRID OPTIMAL CONTROL

A. A Class of hybrid systems

The class of hybrid systems under consideration in this paper is defined as follows :

For a given finite set of discrete state $\underline{Q} = \{1, ..., Q\}$, there is an associated collection of continuous dynamics defined by differential equations

$$\dot{x}(t) = f_q(x(t), u(t), t) \qquad \text{(II.1)}$$

where $q \in \underline{Q}$, the continuous state $x(.)$ takes its values in $\mathbb{R}^{n_q}$ ($n_q \in \overline{\mathbb{N}}$), the continuous control $u(.)$ takes its values in a control set U_q included in $\mathbb{R}^{m_q}$ ($m_q \in \mathbb{N}$), the vector fields f_q are supposed defined on $\mathbb{R}^{n_q} \times \mathbb{R}^{m_q} \times [a, b]$, $\forall q \in \underline{Q}$.

Here, f_q, $q \in \underline{Q}$ meet classical hypothesis that guarantee the existence and uniqueness of the solution (i.e. f_q is globally lipschitz continuous). Note that the state space as well as the control space depend on the discrete state q and have variable dimensions with respect to it.

The discrete dynamic is defined using a transition function ν of the form:

$$q(t^+) = \nu(x(t^-), q(t^-), d(t), t) \qquad \text{(II.2)}$$

with $q(.)$ the discrete state ($q(t) \in \underline{Q}$) and $d(.)$ the discrete control ($d : [a, b] \to \underline{D}$ where $\underline{D} = \overline{\{1, ..., D\}}$ is a finite set). ν is a map from $X \times \underline{Q} \times \underline{D} \times [a, b]$ to $\underline{Q}$ where X is a subset of $\mathbb{R}^{n_1+\cdots+n_Q}$. More precisely, the set X takes the form: $X = \bigcup_{j=1}^{Q} \{0\}^{\sum_{k=1}^{j-1} n_k} \times \mathbb{R}^{n_j} \times \{0\}^{\sum_{k=j+1}^{Q} n_k} \subset \mathbb{R}^{n_1+\cdots+n_Q}$. For convenience, we should replace $x(t)$ in (II.2) by $\tilde{x}(t) = (0, \ldots, 0, x(t), 0, \ldots, 0) \in X$ with $x(t) \in \mathbb{R}^{n_q}$.

The discrete variable $q(.)$ is a piecewise constant function of the time. This is indicated by t^- and t^+ in (II.2) meaning just before and just after time t.

The value of the transition function ν depends on two kinds of discrete phenomena which can affect the evolution of $q(.)$: changes in the discrete control $d(.)$ and boundary conditions on (x, t) of the form $C_{(q,q')}(x, t) = 0$ which modify the set of attainable discrete states. These boundary conditions can represent thresholds, hysteresis, saturations, time delay between two switches, ... and refer to the manner the continuous dynamic interacts with the discrete part.

A set of jump functions $\Phi_{(q,q')} : \mathbb{R}^{n_q} \times [a,b] \rightarrow \mathbb{R}^{n_{q'}}$ $\forall (q,q') \in \underline{Q}^2$ such that

$$x(t^+) = \Phi_{(q,q')}(x(t^-),t) \quad \text{(II.3)}$$

which resets the continuous state when a discrete transition occurs from q to q' is also considered.

Now starting from a position (x_0, q_0), the continuous state $x(.)$ evolves in $\mathbb{R}^{n_{q_0}}$ according to the continuous control $u(.)$ and the state equation $\dot{x}(t) = f_{q_0}(x(t), u(t), t)$. If at time t_1, $x(.)$ reaches the boundary condition $C_{(q_0,q_1)}(x(t_1), t_1) = 0$ and/or a change in the discrete control $d(.)$ is produced which leads to a new discrete state $q_1 = \nu(x(t_1^-), q_0, d(t_1), t_1)$ then the continuous state jumps in $\mathbb{R}^{n_{q_1}}$ to $x(t_1^+) = \Phi_{(q_0,q_1)}(x(t_1^-), t_1)$ and evolves with a new vector field $f_{q_1}(x(t), u(t), t)$. And so on. Then, a hybrid trajectory on time interval $[a, b]$ can be viewed as the data of a piecewise constant function $q(.)$ and a piecewise continuous function $\widetilde{x}(.)$ in $X \subset \mathbb{R}^{n_1+\cdots+n_Q}$ obtained according to equations (II.1)(II.2) and (II.3).

Equations (II.1)(II.2) and (II.3) denote a hybrid system and must be understood as a causal and consistent dynamical system. So, we do not deal with the well-posedness of the problem [19]. And we assume that the system is designed in such a way that it neither tolerates several discrete transitions at a given time nor zeno phenomena i.e. an infinite switching accumulation points. This hybrid model covers a very large class of hybrid phenomena such as systems with switched dynamics, jumps on the states and variable continuous state space dimension. It takes into account autonomous and/or controlled events.

B. Problem formulation

Consider a hybrid system (II.1)(II.2) and (II.3) under the following assumptions :

1) For each q, the control domain U_q is a bounded subset of $\mathbb{R}^{m_q}$.
2) For each q, the vector fields $f_q(x,u,t)$ and $L_q(x,u,t)$ are continuous functions on the direct product $\mathbb{R}^{n_q} \times \overline{U}_q \times [a,b]$ and continuously differentiable with respect to the state variable and the time variable ($\overline{U}$ denotes the closure of the set U).
3) $\forall (q,q') \in \underline{Q}^2$ the jump functions $\Phi_{(q,q')}(.,.)$ and the boundary constraint $C_{(q,q')}(.,.)$ are continuous and continuously differentiable.
4) (x_0, q_0) is chosen within a given set S_0

Let $[t_0 = a, t_1, ..., t_i, ..., t_m = b]$ and $[q_0, q_1, ..., q_i, ..., q_m]$ (recall that $q_i \in \underline{Q}$ and b can be infinite as well as m in this case, obviously accumulation of switchings, i.e. Zeno phenomenon are not allowed) be the sequence of switching times and the associated mode sequence corresponding to the control $(u,d)(.)$ on the time interval $[a,b]$.

Moreover, a hybrid criterion is introduced as:

$$\begin{aligned} J(u,d) &= \int_a^b L_{q(t)}(x(t), u(t), t)dt \\ &= \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} L_{q_i}(x(t), u(t), t)dt \end{aligned} \quad \text{(II.4)}$$

where $q_i \in \underline{Q}$. For all $q \in \underline{Q}$, we assume that L_q is defined and continuous on the direct product $\mathbb{R}^{n_q} \times \bar{U}_q \times [a,b]$ and continuously differentiable in the state variable and in time. L_q depends on the discrete state. So, different criteria associated with each mode are possible.

The optimal control $(u,d)(.)$ is the control that minimizes the cost function J over the time interval $[a,b]$ subject to the condition that the final state $(x,q)(b)$ lies in a given set S_1.

C. Necessary conditions

Our formulation (II.1), (II.2), (II.3) and (II.4) allow a direct use of a smooth version of MP [5],[6] with an additional dynamic programming argument in order to consider discrete transitions. To this purpose, let us define the Hamiltonian function associated to each mode q as:

$$H_q(p, p_0, x, u, t) = p^T f_q(x,u,t) - p_0 L_q(x,u,t) \quad \text{(II.5)}$$

and the Hamiltonian system as

$$\dot{x} = \frac{\partial H_q}{\partial p} \qquad \dot{p} = -\frac{\partial H_q}{\partial x} \quad \text{(II.6)}$$

where p_0 is a positive constant ($p_0 \geq 0$).

Now ,we have the following theorem:

Theorem 2.1: If $(u^*, d^*)(.)$ and $(x^*, q^*)(.)$ are respectively an admissible optimal control and the corresponding trajectory for the problem (II.1), (II.2), (II.3) and (II.4), then there exists a piecewise absolutely continuous curve $p^*(.)$ and a constant $p_0^* \geq 0$, $(p_0^*, p^*(t)) \neq (0, \mathbf{0})$ on $[a,b]$, so that:

1. the sextuplet $(p^*, p_0^*, x^*, q^*, u^*, d^*)(.)$ satisfies the associated Hamiltonian system (II.6) almost everywhere (a.e.)

2. at any time t, the following maximum condition holds for $(p^*, p_0^*, x^*, q^*)(t)$:

$$H_{q^*}(p^*, p_0^*, x^*, u^*, t) = \sup_{u \in U_{q^*}} H_{q*}(p^*, p_0^*, x^*, u, t) \quad \text{(II.7)}$$

3. at switching time t_i, $i = 0, \ldots, m$, the following transversality conditions are satisfied: there exist a vector π_i^* such that

$$\begin{aligned} p^*(t_i^-) &= \left[\begin{array}{cc} [\frac{\partial \Phi_{(q_{i-1}^*, q_i^*)}(x^*(t_i^-), t_i)}{\partial x}]^T & 0 \end{array} \right] \nabla V_{q_i^*} \\ &\quad + [\frac{\partial C_{(q_{i-1}^*, q_i^*)}(x^*(t_i^-), t_i)}{\partial x}]^T \pi_i^* \end{aligned} \quad \text{(II.8)}$$

$$\begin{aligned} H^*(t_i^-) &= -\left[\begin{array}{cc} [\frac{\partial \Phi_{(q_{i-1}^*, q_i^*)}(x^*(t_i^-), t_i)}{\partial t}]^T & 1 \end{array} \right] \nabla V_{q_i^*} \\ &\quad - [\frac{\partial C_{(q_{i-1}^*, q_i^*)}(x^*(t_i^-), t_i)}{\partial t}]^T \pi_i^* \end{aligned} \quad \text{(II.9)}$$

with $\nabla V_{q_i^*} = \begin{bmatrix} p^{*T}(t_i^+) & -H_{q_i^*}(\omega) \end{bmatrix}^T$ and $\omega = (p^*(t_i^+), p_0^*, x^*(t_i^+), u^*(t_i^+), t_i^+)$

Proof: See [2] ■

Remark 2.2: Equations (II.8) and (II.9) must be obviously adapted according to the final and initial constraints under the state (x, q) at time $t = a$ and $t = b$ (not specified in our case).

Remark 2.3: The notations (II.8), (II.9) imply that π_i^* must be equal to zero if t_i is a controlled switching time without boundary conditions.

Remark 2.4: As for the state $x(.)$, the costate $p(.)$ should also be of different dimensions for different subsystems.

Remark 2.5: In practice many problems only require optimal solutions under fixed number of switchings and/or fixed order of active subsystems. As the above necessary conditions deal with a very large formulation of the optimal control problem, the total number of switchings and the order of active subsystems seem to be a priori unknown. However, it is possible to impose them as well as the switching time : it only depends on how the discrete transition function ν is specified. For example if one wants to impose the order of the active subsystems, the automata should be written in such a way that there is only the expected sequence. To specify the number of switchings, one has to write a tree with the good degree of depth. Moreover the boundary constraints can be used to impose the switching times.

Remark 2.6: In a classical optimal control problem, one may need to solve Boundary Value Problem (BVP). At this stage, it can be observed that the above necessary conditions may lead to a multi stage BVP corresponding to the discrete transitions. Transversality conditions give just a relation between initial and final values of the Hamiltonian and of the costate at each switching times without any information about when these switches occur. In fact due to discrete dynamic, the key to compute the solution is dynamic programming. But the task can be very hard to practice since bifurcation in the trajectory must be taken into account each time a discrete transition is allowed i.e. in the regions of the state space and time space for each subsystems. In addition the degree of freedom in the trajectory at each switching time due to the choice of π_i yields also to bifurcations.

Remark 2.7: In the case of switched system, when the dynamics can be described using a single system:

$$\dot{x}(t) = \sum_{q=1}^{Q} \alpha_q(t) f_q(x(t), u(t), t) \in \mathbb{R}^n$$

with $\alpha_q(t) \in \{0, 1\}$ and $\sum_{q=1}^{Q} \alpha_q(t) = 1, \forall t$. At any time, the active subsystem is selected via the values of the α_q's. Then the discrete transition function (II.2) is degenerated to $q(t^+) = \nu(x(t^-), q(t^-), d(t), t) \equiv d(t)$ with the discrete control set $\underline{D} = \overline{Q}$. In which case, the switching strategy must satisfy the following condition: at any time t, the following maximum condition holds for $(p^*, p_0^*, x^*, q^*)(t)$:

$$H_{q^*}(p^*, p_0^*, x^*, u^*, t) = \max_{q \in \underline{Q}} \sup_{u \in U_q} H_q(p^*, p_0^*, x^*, u, t) \tag{II.10}$$

It means that the active subsystem at any time is the one which has the largest Hamiltonian function. This situation has been studied in [13], [20].

Remark 2.8: Note that we can take into account jump costs:

$$\Psi_{(q,q')}(x, t)$$

every time a discrete transition occurs in a straightforward way. To do this, we need only to add the cost Ψ in (II.4) in order to determine new transversality conditions.

III. Existence and sufficient conditions

In the sequel, only switched systems are considered. Switched systems are the most simple hybrid systems : the only hybrid phenomena are controlled switching. So, such systems can also be expressed using a single vector field meaning, $\dot{x}(t) = F(x, u, \alpha) = \sum_{q=1}^{Q} \alpha_q(t) f_q(x(t), u_q(t))$ with $x \in \mathbb{R}^n$, $u(t) = [u_1(t) \cdots u_Q(t)] \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_Q}$, $\alpha(t)$ is a Boolean vector ($\alpha(t) \in \{0, 1\}^Q$) and $\alpha_q(t)$ refers to the q^{th}-component of $\alpha(t)$ so that there is one and only one component of $\alpha(t)$ equal to 1 i.e. $\alpha(t) \in D$ where $D = \{\alpha \in \{0, 1\}^Q : \sum_{q=1}^{Q} \alpha_q = 1\}$.

Here and by switching beetwen the different values of D, the function α plays the role of the discrete control d.

Now as it is mentioned in [21], [22], the existence of a solution for such systems is closely related to the existence of a bang-bang solution to the embedding problem **(EP)** for which the discrete control set D is extended to its convex hull $D_c = co(D) = \{\alpha \in [0, 1]^Q : \sum_{q=1}^{Q} \alpha_q = 1\}$.

Classical theorems [6] p.61 or [23]p. 222 which state sufficient conditions for the existence of an optimal solution under convexity assumptions can be easily applied to the embedding problem.

The main assumption in both theorems is related to convexity and stated as follows :

for all (x, t), the set
$\{[f(x, u, t), L(x, u, t) + \delta] : u \in U, \delta \geq 0\}$ is convex.

The existence question fails in the context of switched (hybrid) systems due to the non-convexity of the control domain D. In fact, it is clear that if the optimal control $\alpha^*(t)$ for the EP is of bang bang type then it is an optimal solution for the corresponding switched system.

It is appealing in this situation to study the EP. In this case however it must not be forgotten that the solutions found for the EP may not be realizable for the physical system. Indeed, physical signification disappears with regard to hybrid phenomena. For example this is the case for a switch in an electrical circuit which can only be on or off.

From the necessary condition of the MP, the only situations where the control α^* is not of bang bang type occur when the maximum condition (II.10) is obtained simultaneously for a subset $\tilde{Q}$ of Q on a set of time $\top \subset [a,b]$ of nonzero measure. This situation refers to singular trajectories.

A. *The singular case*

Here is given a procedure to examine if a no hybrid solution exists.

Let us consider for simplicity a *time optimal control problem*. The switched system is formed by two free modes, we write, $\dot{x}(t) = \alpha(t)f_1(x(t)) + (1-\alpha(t))f_0(x(t)) \quad , \alpha(t) \in \{0,1\}$ with $f_0, f_1 \in C^\infty(\mathbb{R}^n)$. The Hamiltonian function H is given by, $H(p,x,\alpha) = \langle p, \alpha(f_1(x) - f_0(x)) + f_0(x)\rangle - p_0$. For given x and p, it can be seen that H is an affine function of the control α. Thus the maximum is therefore reached for the following bang bang control law:

$$\alpha(t) = 0 \text{ if } \langle p, f_1(x) - f_0(x)\rangle < 0 \qquad \text{(III.1)}$$
$$\alpha(t) = 1 \text{ if } \langle p, f_1(x) - f_0(x)\rangle > 0 \qquad \text{(III.2)}$$

Now if $\langle p, f_1(x) - f_0(x)\rangle = 0$ then the control is a priori undefined. The question is : Is there a control $\alpha(.)$ (candidate to the optimality) for which $\langle p, f_1(x) - f_0(x)\rangle \equiv 0$ on a nonzero measure set of time ?

Differentiating the last expression, we get:

$$\frac{d^m}{dt^m}\langle p, f_1(x) - f_0(x)\rangle = \langle p, ad^m_{f+\alpha g} g\rangle$$

where $f(t) = f_0(x(t))$ and $g(t) = f_1(x(t)) - f_0(x(t))$

Recall that $[f,g](t) = \frac{\partial g}{\partial x}(t)f(t) - \frac{\partial f}{\partial x}(t)g(t)$ and $ad^k_f g(t) = [f, ad^{k-1}_f g](t)$ with $ad^0_f g(t) = g(t)$.

For $0 \leq m \leq 2$, we obtain $\langle p,g\rangle = 0$, $\frac{d}{dt}\langle p,g\rangle = \langle p,[f,g]\rangle = 0$, $\frac{d^2}{dt^2}\langle p,g\rangle = \langle p,[f,[f,g]] + \alpha[g,[f,g]]\rangle = 0$. Next derivative is not tractable since there is a priori no information about the regularity of α.

Nevertheless, these equalities lead to:

1) if $\langle p,[g,[f,g]]\rangle \neq 0$ then the control α is defined uniquely by : $\alpha = -\frac{\langle p,[f,[f,g]]\rangle}{\langle p,[g,[f,g]]\rangle}$ furthermore , α is analytic and satisfies : $\dot{\alpha} = -\frac{\langle p,[f,[f,[f,g]]]\rangle + \alpha\langle p,[g,[f,[f,g]]] + [f,[g,[f,g]]]\rangle + \alpha^2\langle p,[g,[g,[f,g]]]\rangle}{\langle p,[g,[f,g]]\rangle}$
2) if $\langle p,[g,[f,g]]\rangle \equiv 0$ on a nonzero measure set of time then $\langle p,[f,[f,g]]\rangle \equiv 0$ and we can derive these identities to obtain high order conditions on α, that is:$\langle \lambda, [f,[f,[f,g]]] + \alpha\,[g,[f,[f,g]]]\rangle = 0$ and $\langle \lambda, [f,[g,[f,g]]] + \alpha\,[g,[g,[f,g]]]\rangle = 0$. See [24] for more details on this situation.

Example: For example, if $f_0(x) = A_0 x$ and $f_1 = A_1 x$ with $A_0 = \begin{pmatrix} 0.4 & 0.3 \\ -1.3 & 1.1 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0.2 & -1.4 \\ 0.8 & -0.7 \end{pmatrix}$, then two singular arcs are found (using (**??**)) corresponding to the lines D_1 and D_2 (see figure 1) and the associated controls are $\alpha_1(t) \cong 0.2654$ and $\alpha_2(t) \cong 0.5127$. It can be mentioned that line D_1 and D_2 correspond respectively to the eigenvectors which have the largest and the smallest eigenvalues of the pencil $A_0 + \alpha(A_1 - A_0)$. The problem

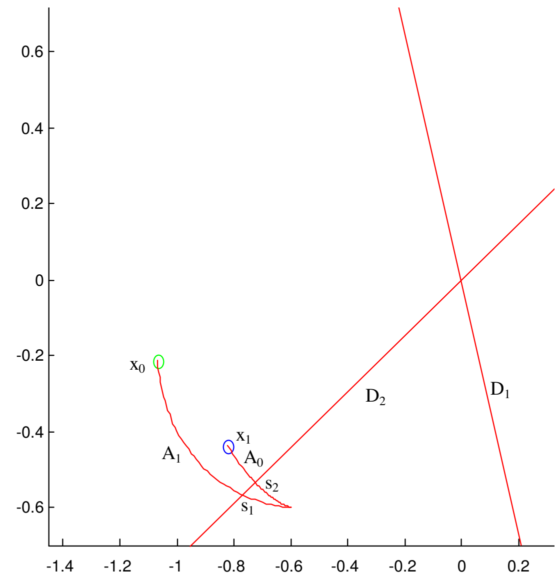

Fig. 1. A one switch trajectory

is to find the optimal time transfer from the initial position $x_0 = (-1.1, -0.2)$ to the final position $x_1 = (-0.8, -0.4)$. In figure 1 an arbitrary trajectory has been plotted which connects $(x_0\ ,\ x_1)$ with one switch and a time $T_1 = 1.4985$. Is there a better one ?

In figure 2, all extremal trajectories (i.e. satisfying equation II.7) finishing to the final position x_1 with a transfer time equal to T_1 are depicted. Therefore, if there is an optimal solution it is included in this set of extremal trajectories. As the initial position is not reached by this set, it can be argued that optimal control for this switched system does not exist. It is known however, that EP has a solution (see the preceding section) and it is obtained with the particular control mentioned above. Indeed, in figure 1 when line D_2 was reached from initial position with control $\alpha(t) = 1$, the control $\alpha_2(t)$ leads to a sliding motion on D_2 until the second intersect point s_2 where the control switch to $\alpha(t) = 0$, is reached. This control gives an optimal transfer time $T_\infty = 1.4304$. Now, sub-optimal trajectories can be obtained by chattering as in figure 3 with transfer time versus the number of switch, equal to $T_1 = 1.4985$, $T_3 = 1.4383$, $T_5 = 1.4333$, $T_{17} = 1.4313$, $T_\infty = 1.4304$. What does this example point out? First, extremal trajectories for the original switched system cannot help us to solve the problem without existence assumptions. Indeed, as it has been shown, it is possible to have a non empty set of candidates without solution. Second, it would be better to solve the embedding problem and to derive optimal or sub-optimal solution for

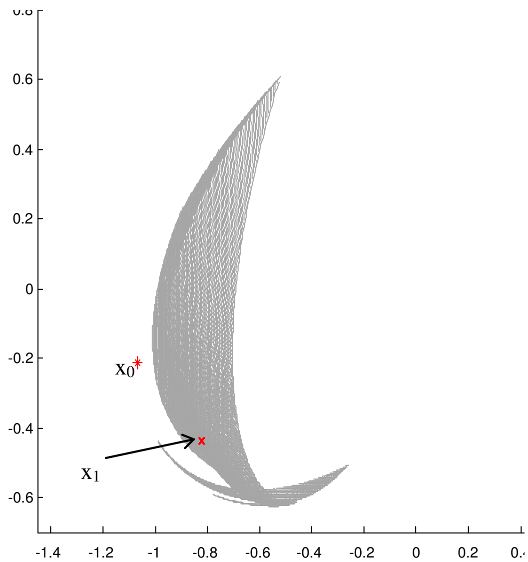

Fig. 2. Extremal attainable set for x_1 in time $t < T_1$

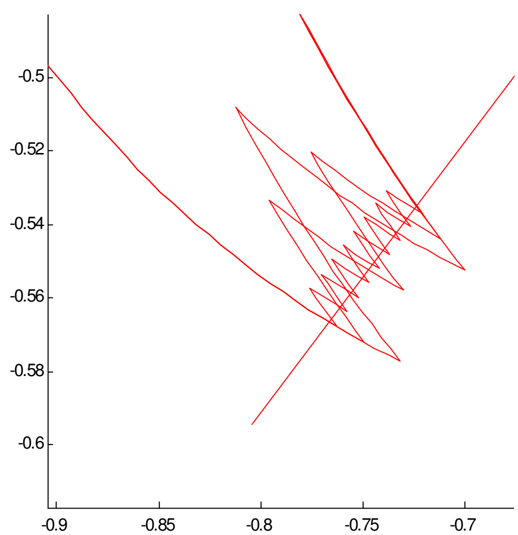

Fig. 3. Zoom on the switching area

hybrid problem.

B. Sub-optimality

As it has been pointed out in the previous section and in [21], existence questions are clearly relevant to the EP formulation. There may be an optimal control for EP without a hybrid solution. Nevertheless, in this case sub-optimal control can be sought after. This purpose is emphasized in the following section. For simplicity, we deal with a two mode switched system but the same results could be obtained with more discrete states.

Consider now an EP defined by two modes

$$\dot{x}(t) = \alpha(t) f_1(x(t), u_1(t)) + (1 - \alpha(t)) f_0(x(t), u_0(t))$$

Consider an interval $[t_1, t_2]$ and an optimal trajectory $\hat{x}(.)$ obtained by a control $\hat{\alpha}$ such that $\hat{\alpha}(t) \in [0, 1]$, $\forall t \in [t_1, t_2]$ and given controls $\hat{u}_0(.)$ and $\hat{u}_1(.)$ for $t \in [t_1, t_2]$.

Under the conditions of theorem 3.1., we prove that there exist bang bang control α which give a solution $x(.)$ near at will from solution $\hat{x}(.)$

Indeed, this trajectory may be approximated using a bang bang control $\alpha(t) \in \{0, 1\}$ $\forall t \in [t_1, t_2]$ and the same controls $\hat{u}_0(.)$ and $\hat{u}_1(.)$ as stated in the following theorem. Let $f^+(x(t), t) = f_1(x(t), u_1(t))$, $f^-(x(t), t) = f_0(x(t), u_0(t))$, $\hat{f}(x(t), t) = \hat{\alpha}(t) f^+(x(t), t) + (1 - \hat{\alpha}(t)) f^-(x(t), t)$

Theorem 3.1: If there are positive numbers ε, C_1, C_2 such that:

i. the initial condition verifies $\|\hat{x}(t_1) - x(t_1)\| \leq C_1 \varepsilon$,
ii. the average $\|\int_{t_1}^{t} (\mathbf{1}_{\{\alpha=1\}}(\hat{\alpha}(\tau) - 1) + \mathbf{1}_{\{\alpha=0\}} \hat{\alpha}(\tau))(f^+ - f^-)(x(\tau), \tau) d\tau\| \leq C_2 \varepsilon$, where symbol $\mathbf{1}_{\{.\}}$ defines the characteristic function
iii. $\hat{f}$ satisfies a Lipschitz condition i.e. $\exists B > 0, \forall x, \forall y, \left\| \hat{f}(x(t), t) - \hat{f}(y(t), t) \right\| \leq B \|x(t) - y(t)\|$

then there is a positive constant C such that $\|\hat{x}(t) - x(t)\| < C\varepsilon$, for all $t \in [t_1, t_2]$

Proof: The proof is obtained by writing the integral equations of $\hat{x}(t)$ and $x(t)$,

$$x(t) = x(t_1) + \int_{t_1}^{t} \mathbf{1}_{\{\alpha=1\}} f^+(x(\tau), \tau) + \mathbf{1}_{\{\alpha=0\}} f^-(x(\tau), \tau) d\tau$$

$$\hat{x}(t) = \hat{x}(t_1) + \int_{t_1}^{t} \hat{f}(\hat{x}(\tau), \tau) d\tau$$

So, the norm of the difference between $x(t)$ and $\hat{x}(t)$ can be estimated. For clarity explicit time dependency is not mentioned in the sequel.

$$\|\hat{x}(t) - x(t)\| \leq \|\hat{x}(t_1) - x(t_1)\| + \int_{t_1}^{t} \left\| \hat{f}(\hat{x}) - \hat{f}(x) \right\| d\tau$$
$$+ \left\| \int_{t_1}^{t} \hat{f}(x) - 1_{\{\alpha=1\}} f^+(x) - \mathbf{1}_{\{\alpha=0\}} f^-(x) d\tau \right\|$$

As $\hat{f}$ is Lipschitz, there is a constant B such that

$$\|\hat{x}(t) - x(t)\| \leq \|\hat{x}(t_1) - x(t_1)\| + \int_{t_1}^{t} B \|\hat{x} - x\| d\tau +$$
$$\left\| \int_{t_1}^{t} \left(\mathbf{1}_{\{\alpha=1\}} (\hat{\alpha} - 1) + \mathbf{1}_{\{\alpha=0\}} \hat{\alpha} \right) (f^+ - f^-)(x) d\tau \right\|$$
$$\|\hat{x}(t) - x(t)\| \leq C_3 \varepsilon + \int_{t_1}^{t} B \|\hat{x} - x\| d\tau$$

with $C_3 = C_1 + C_2$. Applying the Bellman-Gronwall lemma to the last inequality we get

$$\|\hat{x}(t) - x(t)\| \leq C\varepsilon, \qquad \text{(III.3)}$$

with $C = C_3 e^{Bt_2}$ ■

Note that the third condition can be replaced with a weak condition: there is a positive and integrable function $B(.)$ such that $\forall x, \forall y, \forall t, \left\| \hat{f}(x,t) - \hat{f}(y,t) \right\| \leq B(t) \left\| x - y \right\|$.

Corollary 3.2: If optimal control for the EP is not bang bang then we can always define a sub-optimal control for switched systems.

Proof: This holds true if the dimension of the problem is increased by defining the $(n+1)$-vector: $\tilde{x} = [x^T, x_{n+1}]^T$ with $\dot{x}_{n+1} = L(x,u)$. Thus, the average has the additional term $\int_{t_1}^{t} (\mathbf{1}_{\{\alpha=1\}}(\hat{\alpha} - 1)(L^+ - L^-)(x) + \mathbf{1}_{\{\alpha=0\}}\hat{\alpha}(L^+ - L^-)(x))d\tau$ and the conclusion of above theorem implies that the cost functional differs from a number as small as desired.

First, if there is bang bang control leading to a cost which differs about some ε from the optimal of the EP then this control is chosen as sub-optimal. On the other hand, chattering control can be chosen yielding the average condition. So, the reader is referred to sliding modes control [25]. ■

IV. Conclusion

In this paper, after a brief recall on the maximum principle in context of a fairly general hybrid system, we have focussed our attention on the existence of a solution for switched systems. Due to the non convexity of the problem formulation, classical results fail in providing a solution. So, we have proposed to study an embedding problem for which the existence of an optimal control can be assumed. From this solution, it is shown that the presence of singular control explain why the original problem have no solution. Nevertheless, we proposed to derived from the optimal embedding solution a suboptimal one for the switched system.

V. REFERENCES

[1] M. S. Branicky, V. S. Borkar S. K. Mitter, A Unified Framework for Hybrid Control: Model and Optimal Control Theory,*IEEE Trans. on Aut. Cont.,* vol(43)(1), 1998.

[2] P. Riedinger, C. Iung, F. Kratz, An Optimal Control Approach for Hybrid Systems, soumis à European Journal of Control et rapport interne CRAN, 2000.

[3] H.J. Sussmann, A maximum principle for hybrid optimal control problems, proc. of the 38th IEEE Conf. on Decision and Control,pp 425-430, 1999.

[4] D. P. Bertsekas. Dynamic Programming and Optimal Control. Athena Scientific, 2nd edition, 2000.

[5] L.S. Pontryagin,V. G. Boltyanskii, R.V. Gamkrelidze & E. F. Mishchenko (1964), *The Mathematical Theory of Optimal Processes,* Pergamon.

[6] L.D. Berkovitz *Optimal Control Theory,* Applied Mathematical sciences, Springer Verlag 1974.

[7] E.D. Sontag, *Mathematical Control Theory,* Texts in Applied Mathematics 6, Springer Verlag, 1990.

[8] Hedlund S. et Rantzer A., Optimal Control of Hybrid Systems, Proceedings of 38th IEEE Conf. on Decision and Control, Phoenix, 1999.

[9] Bo Lincoln, Anders Rantzer, Relaxed Optimal Control of Piecewise Linear Systems, ADSH 03 Saint Malot, France 2003.

[10] M. S. Branicky, S. K. Mitter (1998), Algorithms for Optimal Control, proc. IEEE Conf. on Decision and Control, pp 2661-2666, 1995.

[11] H. Ye, A. N. Michel & L. Hou (1995), Stability Theory for Hybrid Dynamical Systems, 34th IEEE Conf. on Decision and Control, 2679-2684.

[12] C. G. Cassandras, Q. Liu, K. Gokbayrak, Optimal Control of a two stage hybrid manufacturing system model, proc. of the 38th IEEE Conf. on Decision and Control,pp 450-455, 1999.

[13] P. Riedinger, F. Kratz, C. Iung & C. Zanne (1999), Linear Quadratic Optimization for Hybrid Systems, proc. of the 38th IEEE Conf. on Decision and Control, 3059-3064.

[14] M. Johansson, Piecewise Linear Control Systems, Ph. D. Dissertation, Lund Inst. of Tech. Sweden, 1999.

[15] A. Rantzer, Piecewise Linear Quadratic Optimal Control, *IEEE Transactions on Automatic Control*, April 2000.

[16] J. Daafouz, P. Riedinger, C. Iung, Static Output Feedback Control for Switched Systems, proc. 40th IEEE Conference on Decision and Control, Orlando, 2001.

[17] X. Xu, P. J. Antsaklis, An approach for solving General switched Linear Quadratic Optimal Control Problems, proc. 40th IEEE Conf. on Decision and Control, 2001.

[18] A.Bemporad, D.Corona, A.Giua, C.Seatzu, Optimal State-Feedback Quadratic Regulation of Linear Hybrid Automata, ADSH Saint Malot, France juillet 2003.

[19] Y.L. Lootsma, A.J. van der Schaft & M. K. Camlibel (1999), Uniqueness of solutions of linear relay systems, in Automatica 35(3) 467-478.

[20] P. Riedinger, C. Zanne & F. Kratz (1999), Time Optimal Control of Hybrid Systems, proc. IEEE American Control Conference, 2466-2470.

[21] P. Riedinger, Contribution à la commande optimale des systèmes hybrides, Ph D thesis, Institut National Polytechnique de Lorraine, France 1999.

[22] S. Bengea, R. DeCarlo Conditions for the existence of a solution to a two switched hybrid optimal control problem , submit to ADSH Saint Malot, 2003.

[23] F.H.Clarke, *Optimization and Nonsmooth Analysis,* SIAM, 1990.

[24] B. Jakubczyk & W. Respondek (1998), *Geometry of feedback and optimal control,* Marcel DEKKER.

[25] V.I. Utkin, Sliding Modes in Control Optimization, Springer-Verlag, 1992.