Stabilizing switched control design and pole placement

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Abstract

In this paper, we consider switched linear discrete-time systems. We propose a method to design a switched state feedback control which allows to place the poles of each subsystem in desired locations. In addition, this method guarantee that the switched system is asymptotically stable using Lyapunov arguments. The design of such a switched state feedback control law is formulated in terms of $\mathcal{L}$inear $\mathcal{M}$atrix $\mathcal{I}$nequalities ($\mathcal{LMI}$). The method is illustrated using a simple example.

Key words

Switched Discrete Time Systems, Switched Lyapunov functions, Switched control design, Pole placement, $\mathcal{L}$inear $\mathcal{M}$atrix $\mathcal{I}$nequalities ($\mathcal{LMI}$).

1 Introduction

In recent years, the study of hybrid dynamical systems has received a growing attention in control theory and practice [1], [2]. Switched systems are an important class of hybrid systems consisting of a family of continuous (or discrete) time subsystems and a rule that orchestrates the switching between them. Among the large variety of problems encountered in practice, one can study the existence of a switching rule that ensures stability of the switched system. One can also assume that the switching sequence in not known a priori and look for stability results under arbitrary switching sequences. One can also be interested by certain useful class of switching sequences. A survey of basic problems in stability and design of switched systems is given in [3].

In this paper, we propose a modification of the $\mathcal{L}$inear $\mathcal{M}$atrix $\mathcal{I}$nequalities ($\mathcal{LMI}$) stability conditions developed in [4] for discrete-time switched systems under arbitrary switching. The goal of such a modification is to ensure, in addition of stability of the closed loop switched system, that the closed loop subsystems have their poles in desired location. To our knowledge, there is no result in the literature which combines stability and pole placement for hybrid systems. The motivations of this problem are numerous. It is well known that it is possible to design a switched control which stabilizes a set of unstable systems. In the opposite, a set of stable systems can give an unstable hybrid system by an inappropriate switching sequence. By state feedback, you can imagine to place the poles of each subsystem and get an in-conditionally stable switched system. The advantages are obvious : the dynamic of each subsystem is fixed so

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as to get a correct behavior when a model keeps constant for any long time. When the switching is activated because of variation in the parameters, or in the operating point, or by a supervisor which selects the best control strategy, the stability is ensured whatever the switching control can be.

This paper is organized as follows. The next section gives the problem formulation. In section 3, the design of stabilizing switched control with pole placement is explained. We end the paper by a simple example which illustrates the technique proposed in this paper and a conclusion.

Notations: We use standard notations throughout the paper. M^T is the transpose of the matrix M. $M > \mathbf{0}$ ($M < \mathbf{0}$) means that M is positive definite (negative definite). $\mathbf{0}$ and $\mathbf{I}$ denote the null matrix and the identity matrix with appropriate dimensions. X is the state space $X \subset \mathbb{R}^n$.

2 Problem formulation

Consider the switched system defined by:

$$x_{k+1} = A_\alpha x_k + B_\alpha u_k \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input and $y_k \in \mathbb{R}^p$ is the output vector. $\{(A_i, B_i) : i \in \mathcal{E}\}$ are a family of matrices parameterized by an index set $\mathcal{E} = \{1, 2, ..., N\}$ and $\alpha : X \times \mathbb{N} \to \mathcal{E}$ is a switching signal ($i = \alpha(x_k, k)$). We assume that the switching signal is unknown a priori but real time available.

The classical stabilizing switched state feedback control problem is to find

$$u_k = K_\alpha x_k, \tag{2}$$

ensuring stability of the corresponding closed loop switched system

$$x_{k+1} = \big(A_\alpha + B_\alpha K_\alpha\big) x_k \tag{3}$$

Here, we are interested in assigning the eigenvalues of the closed loop matrices $(A_i + B_i K_i)$, $\forall i \in \mathcal{E}$, to prescribed locations of the complex plane while ensuring stability of the closed loop switched system (3). Hence, we are looking for a switched state feedback such that

- the closed loop dynamic spectra satisfies

$$spec(A_\alpha + B_\alpha K_\alpha) = \{\lambda_{\alpha 1}, \ldots, \lambda_{\alpha n}\} \tag{4}$$

  where the $\lambda_{\alpha q}$, $q = 1, ..., n$, denote desired eigenvalues

- the closed loop switched system is globally asymptotically stable.

3 Control design

The classical switched state feedback design reduces to the computation of the gain matrices K_i, $i \in \mathcal{E}$, ensuring the asymptotic stability for the switched system (3) under arbitrary switching signal. A solution has been proposed to this problem in [4]. It is based on the use of switched quadratic Lyapunov functions and stability conditions from [5]. this solution is recalled in the following theorem.

Theorem 1 *If there exist symmetric matrices S_i, matrices G_i, and R_i, $\forall i \in \mathcal{E}$, such that*

$$\begin{pmatrix} G_i + G_i^T - S_i & (A_iG_i + B_iR_i)^T \\ A_iG_i + B_iR_i & S_j \end{pmatrix} > \mathbf{0} \qquad \forall (i,j) \in \mathcal{E} \times \mathcal{E} \tag{5}$$

then the state feedback control given by (2) with

$$K_i = R_i G_i^{-1} \qquad \forall i \in \mathcal{E} \tag{6}$$

stabilizes asymptotically the system (3).

Proof: see Theorem 2 in [4] □

Now, independently from stability considerations, the problem of computing matrix gains $\hat{K}_i$ which ensure that each matrix $(A_i + B_i\hat{K}_i)$, $i \in \mathcal{E}$ has $\{\lambda_{i1}, \ldots, \lambda_{in}\}$ as its eigenvalues has been considered in [6]. The procedure leads to the computation of the matrix gains $\hat{K}_i$ in two steps :

1. For $q = 1, \ldots, n$ compute bases

$$\begin{bmatrix} M_{iq} \\ N_{i_q} \end{bmatrix} \quad \text{of the null-space of} \quad \begin{bmatrix} A_i - \lambda_{iq}\mathbf{I} & B_i \end{bmatrix} \tag{7}$$

2. For arbitrary column vectors v_{iq} of dimension p compute the gains

$$\hat{K}_i \triangleq \hat{R}_i \hat{G}_i^{-1} \tag{8}$$

where

$$\hat{R}_i \triangleq \begin{bmatrix} N_{i1}, & \ldots, & N_{in} \end{bmatrix} diag_{q=1}^{n} v_{iq}, \qquad \hat{G}_i \triangleq \begin{bmatrix} M_{i1}, & \ldots, & M_{in} \end{bmatrix} diag_{q=1}^{n} v_{iq}$$

The vectors v_{iq} are chosen so that the matrix $\hat{G}_i$ is invertible. To consider complex eigenvalues, this procedure has to be modified. As complex eigenvalues occur in conjugate pairs $(\lambda_{ir}, \lambda_{i(r+1)})$ with $\lambda_{i(r+1)} = \lambda_{ir}^*$, the null-space computation must be replaced with

$$\begin{bmatrix} M_{ir} \\ N_{i_r} \\ M_{i(r+1)} \\ N_{i(r+1)} \end{bmatrix} \quad \text{null-space of} \quad \begin{bmatrix} A_i - Re(\lambda_{ir})\mathbf{I} & B_i & Im(\lambda_{ir})\mathbf{I} & \mathbf{0} \\ -Im(\lambda_{ir})\mathbf{I} & \mathbf{0} & A_i - Re(\lambda_{ir})\mathbf{I} & B_i \end{bmatrix} \tag{9}$$

The matrix gains K_i are given by (8) with:

$$\hat{R}_i = N_i\Psi_i, \qquad \hat{G}_i = M_i\Psi_i \tag{10}$$

where

$$\begin{aligned} M_i &\triangleq \big[\ldots, M_{iq}, \ldots, M_{ir}, M_{i(r+1)}, \ldots\big] \\ N_i &\triangleq \big[\ldots, N_{iq}, \ldots, N_{ir}, N_{i(r+1)}, \ldots\big] \\ \Psi_i &\triangleq diag\big(\ldots, v_{iq}, \ldots, v_{ir}, v_{i(r+1)}, \ldots\big), \quad v_{iq} \in \mathbb{R}^{p\times 1}, \quad v_{ir} = v_{i(r+1)} \in \mathbb{R}^{2p\times 1} \end{aligned}$$

with the subscripts q for real eigenvalues and r, $r+1$ for complex eigenvalues.

To answer the problem considered in this paper, we look for gain matrices K_i, $i \in \mathcal{E}$ such that (4) is satisfied while ensuring global asymptotic stability of the switched closed loop system by enforcing the Lyapunov-type constraints given in Theorem 1. As in [7] for the linear time invariant case, we notice that in the framework of switched linear discrete systems, the stabilizing switched control structure (6) is similar to the matrix gains formulas (8). We can then select among the stabilizing switched controls the one ensuring that the closed loop matrices $(A_i + B_i K_i)$, $i \in \mathcal{E}$ have $\{\lambda_{i1}, \ldots, \lambda_{in}\}$ as eigenvalues. This can be done by replacing in the LMIs of Theorem 1 the unknowns R_i and G_i by those of (10). The matrix Ψ_i, which gathers all degrees of freedom, is used to satisfy the Lyapunov constraints of Theorem 1. This is summarized in the following Theorem.

Theorem 2 *Assume that M_i and N_i have been computed as indicated previously. If there exist symmetric matrices S_i and matrices Ψ_i solutions of:*

$$\begin{bmatrix} M_i\Psi_i + \Psi_i^T M_i^T - S_i & \Psi_i^T M_i^T A_i^T + \Psi_i^T N_i^T B_i^T \\ A_i M_i \Psi_i + B_i N_i \Psi_i & S_j \end{bmatrix} > 0, \quad \forall (i,j) \in \mathcal{E} \times \mathcal{E}, \tag{11}$$

then a stabilizing switched state feedback exists and the resulting gains K_i are given by

$$K_i = (N_i\Psi_i)(M_i\Psi_i)^{-1}$$

Moreover, such gains ensure that the closed loop dynamic is characterized by $spec(A_i + B_i K_i) = \{\lambda_{i1}, \ldots, \lambda_{in}\}$.

Proof : Assume that the LMIs (11) are feasible. This is equivalent to the feasibility of the LMIs (5) with

$$R_i = N_i\Psi_i, \qquad G_i = M_i\Psi_i$$

and the closed loop switched system obtained with $K_i = R_i G_i^{-1} = (N_i\Psi_i)(M_i\Psi_i)^{-1}$ is asymptotically stable. To complete the proof, the following computation allows to check that $\{\lambda_{i1}, \ldots, \lambda_{in}\}$ is a set of closed loop eigenvalues of the matrices $(A_i + B_i K_i)$, $i \in \mathcal{E}$:

$$\begin{aligned} (A_i + B_i K_i)^T M_{iq} v_{iq} &= (A_i + B_i R_i G_i^{-1}) M_{iq} v_{iq} \\ &= \{(A_i + B_i[N_{i1}, \ldots] diag(v_{i1}, \ldots,).([M_{i1}, \ldots] diag(v_{i1}, \ldots,))^{-1})\}. \\ &\quad\ [M_{i1}, \ldots] diag(v_{i1}, \ldots,) e_{iq} \\ &= A_i M_{iq} v_{iq} + B_i N_{iq} v_{iq} \\ &= \lambda_{iq} M_{iq} v_{iq} \end{aligned}$$

where e_{iq} denotes the canonical basis of $\mathbb{R}^n$. □

The success of the proposed procedure is related to the availability of degrees of freedom in excess with respect to a pure pole placement. These degrees of freedom, in matrices Ψ_i, are related to the dimensions of the null-spaces in (7) and (9). As was mentioned in [7], for single input systems these dimensions drop to 1 and 2 respectively so that there is no longer freedom for other constraints. Hence, this approach is only valid for multi-input systems. However, in case of partial pole placement which is a case of practical interest, this procedure is also valid for single input systems. Hence, only a subset of column vectors G_i and R_i have to satisfy the subspace inclusions defined in (7) and (9). The remaining freedom can then be used to meet additional constraints. In this case, the procedure becomes:

1. Compute for $q = 1, \ldots, l$ with $l < n$ bases

$$\begin{bmatrix} M_{iq} \\ N_{iq} \end{bmatrix} \text{ of the null-space of } [A_i - \lambda_{iq}\mathbf{I} \quad B_i]$$

2. The matrix gains are given by

$$K_i \triangleq R_i G_i^{-1}, \quad R_i \triangleq N_i \Psi_{N_i}, \quad G_i \triangleq M_i \Psi_{M_i}$$

with

$$M_i \triangleq \left[M_{i1}, \ldots, M_{il}, \underbrace{\mathbf{I}, \mathbf{I}, \ldots, \mathbf{I}}_{n-l \ \ times}\right], \qquad N_i \triangleq \left[N_{i1}, \ldots, N_{il}, \underbrace{\mathbf{I}, \mathbf{I}, \ldots, \mathbf{I}}_{n-l \ \ times}\right]$$

$$\begin{array}{l} \Psi_{M_i} \triangleq diag(v_{i1}, \ldots, v_{il}, s_{i1}, \ldots, s_{i(n-l)}), \quad v_{iq} \in \mathbb{R}^p, \quad v_{ir} = v_{i(r+1)} \in \mathbb{R}^{2p}, \quad s_{ik} \in \mathbb{R}^n \\ \Psi_{N_i} \triangleq diag(v_{i1}, \ldots, v_{il}, t_{i1}, \ldots, t_{i(n-l)}), \quad v_{iq} \in \mathbb{R}^p, \quad v_{ir} = v_{i(r+1)} \in \mathbb{R}^{2p}, \quad t_{ik} \in \mathbb{R}^p \end{array} \tag{12}$$

Theorem 3 *Let Ψ_{M_i}, Ψ_{N_i} matrices given by (12). If there exist symmetric matrices S_i, matrices Ψ_{M_i}, Ψ_{N_i} solutions of:*

$$\left[\begin{array}{cc} M_i\Psi_{M_i} + (M_i\Psi_{M_i})^T - S_i & (M_i\Psi_{M_i})^T A_i^T + (N_i\Psi_{N_i})^T B_i^T \\ A_i M_i \Psi_{M_i} + B_i N_i \Psi_{N_i} & S_j \end{array} \right] > 0, \quad \forall (i,j) \in \mathcal{E} \times \mathcal{E}, \tag{13}$$

then a stabilizing switched state feedback exists and the resulting gains K_i are given by

$$K_i = (N_i\Psi_{N_i})(M_i\Psi_{M_i})^{-1}$$

Moreover, such gains ensure that the closed loop dynamic is characterized by $\{\lambda_{i1}, \ldots, \lambda_{il}\} \subset spec(A_i + B_i K_i)$.

Proof: The proof is similar to the proof of Theorem 2 □

4 Illustrative Example

In this section we give a simple example in order to illustrate the steps to follow when one is faced with the implementation of the procedure proposed in this paper.

Consider a switched system given by

$$x_{k+1} = A_\alpha x_k + B_\alpha u_k \tag{14}$$

where $\{A_i : i \in \mathcal{E}\}$ and $\{B_i : i \in \mathcal{E}\}$ are a family of matrices parameterized by an index set $\mathcal{E} = \{1, 2\}$ and

$$A_1 = \left[\begin{array}{cc} 0.0094 & 0.3010 \\ -3.0098 & 0.0094 \end{array} \right], \quad A_2 = \left[\begin{array}{cc} 0.0094 & 3.0098 \\ -0.3010 & 0.0094 \end{array} \right], \quad B_1 = B_2 = \left[\begin{array}{c} 1 \\ 0 \end{array} \right]$$

In the autonomous case ($u_k = 0$), the discrete time invariant subsystems characterized by A_1 and A_2 are stable and the matrices A_1 and A_2 have the same eigenvalues: $0.0094 \pm 0.9518i$. However, if the switching signal is characterized by

$$\alpha = \left\{ \begin{array}{ll} 1 & \text{if } x_k^1 x_k^2 \geq 0, \quad \text{with} \quad x_k = [x_k^1 \ \ x_k^2]' \\ 2 & \text{otherwise} \end{array} \right. \tag{15}$$

then the result is unstable. This can be easily checked by depicting the trajectories of such a switched system (Figure 4).

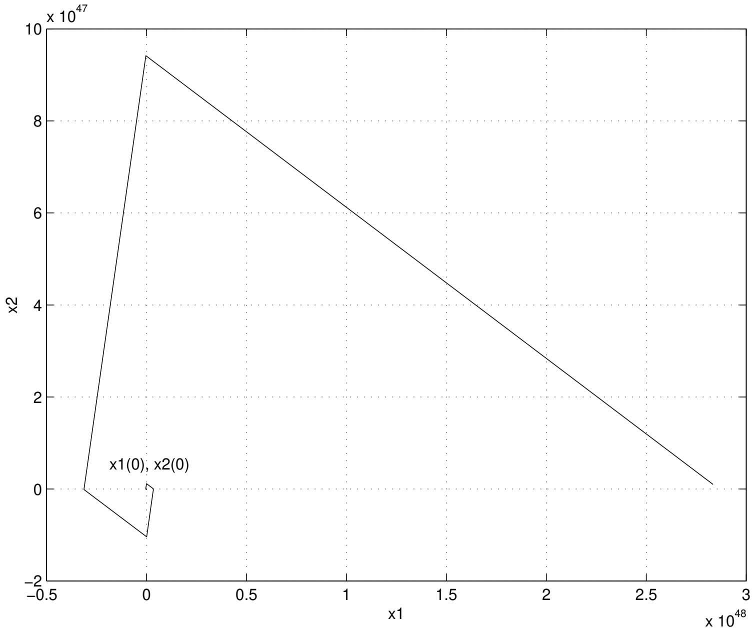

Figure 1: Trajectory of the open loop switched system

One may be interested by designing a switched state feedback that ensures stability of the closed loop system under arbitrary switching signals. In addition, one may be interested by specifying some constraints on the dynamic of each closed loop subsystem. This example is single input and only a partial pole assignment may be achieved. A stabilizing switched state feedback control with the property that each closed loop subsystem has $\lambda = 0$ as one of its eigenvalues can be computed using the procedure proposed in this paper.

Let $i = 1$, the first subsystem, compute for $q = 1, ..., l$ with $l < n$ the null-space of $[A_i - \lambda_{iq}\mathbf{I} \quad B_i]$. Here $n = 2$ and $l = 1$ since only one eigenvalue may be fixed for each subsystem. The result is :

$$\left[\begin{array}{c} 0.0030 \\ 0.9576 \\ -0.2883 \end{array}\right]$$

which means that

$$M_{11} = \left[\begin{array}{c} 0.0030 \\ 0.9576 \end{array}\right], \qquad N_{11} = -0.2883$$

Build the matrices

$$M_i \triangleq [M_{i1}, \ldots, M_{il}, \underbrace{\mathbf{I}, \mathbf{I}, \ldots, \mathbf{I}}_{n-l \quad times}], \qquad N_i \triangleq [N_{i1}, \ldots, N_{il}, \underbrace{\mathbf{I}, \mathbf{I}, \ldots, \mathbf{I}}_{n-l \quad times}]$$

that is

$$M_1 = \left[M_{11} \ \ \mathbf{I}_{2\times 2}\right] = \left[\begin{array}{ccc} 0.0030 & 1 & 0 \\ 0.9576 & 0 & 1 \end{array}\right], \qquad N_1 = \left[N_{11} \ \ \mathbf{I}_{2\times 2}\right] = \left[-0.2883 \ \ 1\right]$$

One has to do similar computations for $i = 2$ the second subsystem and obtain

$$M_2 = \left[\begin{array}{ccc} 0.0098 & 1 & 0 \\ 0.3153 & 0 & 1 \end{array}\right], \qquad N_2 = \left[-0.9490 \ \ 1\right]$$

The next step is to check the feasibility of the Linear Matrix Inequalities of Theorem 3 where the unknowns are the Lyapunov matrices S_i and the matrices

$$\Psi_{M_i} = \begin{bmatrix} v_{i1} & 0 \\ \mathbf{0}_{2\times 1} & s_{i1} \end{bmatrix}, \quad \Psi_{N_i} = \begin{bmatrix} v_{i1} & 0 \\ 0 & t_{i1} \end{bmatrix} \quad \text{with} \quad v_{11} \in \mathbb{R}, \quad s_{11} \in \mathbb{R}^2 \quad \text{and} \quad t_{i1} \in \mathbb{R}$$

for $i = 1$ and $i = 2$; Such LMIs are found feasible and

$$S_1 = 1.0e-03 \begin{bmatrix} 0.0100 & 0.0005 \\ 0.0005 & 0.1912 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.0026 & 0.0462 \\ 0.0462 & 1.4822 \end{bmatrix}$$

$$\Psi_{M_1} = \begin{bmatrix} 0.0066 & 0 \\ 0 & -0.0001 \\ 0 & 2.0158 \end{bmatrix}, \quad \Psi_{N_1} = \begin{bmatrix} 0.0066 & 0 \\ 0 & -0.6067 \end{bmatrix}$$

$$\Psi_{M_2} = \begin{bmatrix} 0.2459 & 0 \\ 0 & 0.0230 \\ 0 & 2.1505 \end{bmatrix}, \quad \Psi_{N_1} = \begin{bmatrix} 0.2459 & 0 \\ 0 & -6.4726 \end{bmatrix}$$

This leads to the following control gains

$$K_1 = (N_1\Psi_{N_1})(M_1\Psi_{M_1})^{-1} = \begin{bmatrix} -0.01786 & -0.30097 \end{bmatrix}$$

$$K_2 = (N_2\Psi_{N_2})(M_2\Psi_{M_2})^{-1} = \begin{bmatrix} -0.0102 & -3.0098 \end{bmatrix}$$

Using the switched control characterized by these values for K_1 and K_2 the closed loop switched system is globally asymptotically stable and the dynamic of each subsystem in the closed loop configuration is characterized by $spec(A_1 + B_1K_1) = \{1.7e-13, \quad 0.93e-3\}$ and $spec(A_1 + B_1K_1) = \{-7e-14, \ 0.0085\}$. The common specified eigenvalue is achieved within the computer accuracy. The closed loop behavior is depicted in Figure 4 with α the switching signal given by (15).

5 Conclusion

Designing a switched state feedback control which allows to place the poles of each subsystem in desired locations and which leads to an in-conditionally stable switched system is proposed. To get this control law, one has to check the feasibility of a number of LMI conditions. This reduces to solve a convex optimization problem for which many tools are available (LMI toolbox by MATALB, Sedumi solver...). To make the method proposed in this paper applicable for a large class of switched systems, the assumption of availability of the all the components of the system state must be dropped. The applicability of the technique proposed in this paper for output feedback control or observer-based control is of interest and will be investigated.

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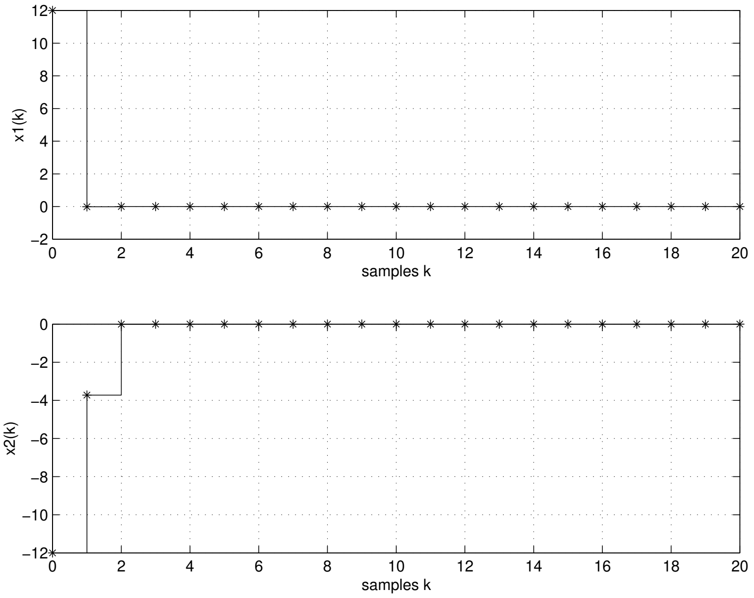

Figure 2: The closed loop behavior

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