# Optimal Control for Hybrid Systems: An Hysteresis Example

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**Abstract**

In this paper, the authors present an approach based on the Maximum Principle of Pontryagin to solve optimal control for hybrid dynamical systems (HDS). In a first step a general hybrid optimal control problem is formulated. Secondly, our method can then be used to solve an hysteresis example and get analytic form of the solutions.

## 1 Introduction

In automatic control area, there is a challenge in building systems which include continuous and discrete phenomena. The aim is to design complex and flexible systems which enable some logic decision as well as discontinuous dynamic. The behavior of such systems is generally described by dynamical systems such as automata and differential equations respectively. Their integration into one system ensures interaction between them.

In the last decade, there was a great interest in studying such systems with regard to their potential applications as well as to the theoretical point of view [1]-[6]. Despite an extensive effort, it appears palpably that the wide class of hybrid systems excludes an unified approach but rather several theoretical frameworks according to the objectives and the task [7]. It could be further added that questions relevant to continuous aspect deal more with the non smooth analysis with a poor discrete part. In other words the control of the continuous dynamic is determined by switching between continuous models [8]. On the other hand, verification tools, supervisory control and computer science are used in full like in hybrid automata but with a poor continuous dynamic such as integrators [9].

In this article optimization problem for continuous part of hybrid systems is investigated. This paper is organized as follows. In section II. optimal control problem for hybrid systems is formulated. Such a formulation leads to consider the Maximum Principle [10] as an efficient tool for handling the problem under study. Section III. focuses on a hysteresis example and shows how synthesis optimal control law. Despite the fact this example seems very simple, the behavior of optimal trajectory explains the difficulties which are often encoutered in such problem.

## 2 Optimal Control for Hybrid Systems: A Pontryagin Approach

Let us start by a general description of controlled hybrid systems and associated optimal problems.

We have in mind systems that switch among vector fields over time, in the form

$$\dot{x}(t) = f_k(x(t), u_k(t)), k \in \underline{s} = \{1, ..., s\} \qquad (1)$$

where $x(t) \in R^n$ is the continuous state and $u_k(t) \in U_k$ a bounded set included in $R^{m_k}$, is the continuous control. Here, $f_k$ and $\frac{\partial f_k}{\partial x}$ are assumed to be continuous functions on the direct product $R^n \times \bar{U}_k$. More generally, weaker hypotheses can be considered (Lipschitz conditions and jumps on the states), see [11], [12] and [13] for more details.

The index $k \in \underline{s}$ or mode is referred as the discrete state whose dynamic is obtained using a transition function,

$$k(t) = \phi(x(t), k(t^-), d(t), t) \qquad (2)$$

where $d(t) \in \underline{r} = \{1, ..., r\}$ is a discrete input. Note that the relations (1) and (2) can be described by an hybrid automata.

In order to conciliate the discrete dynamic with the continuous dynamic, the time is chosen as a continuous variable and the discrete variables $k$ and $d$ are considered as piecewise constant functions $t \rightarrow k(t) \in \underline{s}$ and $t \rightarrow d(t) \in \underline{r}$. Thus the transition function

$\phi : R^n \times \underline{s} \times \underline{r} \to \underline{s}$ appears piecewise constant from the right. This mention is indicated by $t^-$ in (2).

Two categories of hybrid phenomena can be represented in this formulation: the autonomous switching and the controlled switching.

The autonomous switching occurs when the vector field changes discontinuously at the time the state $x(.)$ hits certain boundaries (i.e. some equality constraint of the form $C(x(t), t) = 0$). The controlled switching occurs when the vector field changes abruptly in response to a control command (in our case $d(.)$). All these changes are reflected on the discrete state $k(t)$ or mode of the system at the time $t$ using the transition function.

Let $[0, \tau_1, ..., \tau_i, ..., T]$ be the sequence of switching time. It is clear that this sequence depends on the control $(u, d)(.)$. A hybrid criteria can then be introduced as:

$$J = \sum_{i \geqslant 0} \int_{\tau_i}^{\tau_{i+1}} L_{k_i}(x(t), u_{k_i}(t))dt \tag{3}$$

where $k_i \in \underline{s}$, $L_k$, for all $k \in \underline{s}$, is defined and continuous on the direct product $R^n \times \bar{U}_k$ and continuously differentiable in the state variable. So, different criteria with respect to each mode are possible. Then, the optimal control law is the control that minimizes the cost function $J$.

The main advantage of the Maximum Principle (MP) over the calculus of variations is that it has extremely large admissible control classes. Indeed, only measurable control inputs that take values into a bounded set are needed and no differentiability with respect to the control variable is required which is the case here. MP is a principle which can be applied to various optimization problems with different specific and technical conditions (boundary, free final time, constraints, etc.). Hence, according to the regularity assumptions on the data, several versions of MP can be stated.

In our case formulation (1)(2) and (3) allows a direct use of a smooth version of MP with an additional dynamic programming argument in order to consider autonomous switching.

To this purpose, define the Hamiltonian function associated to each mode $k$ as (mention of time is omitted):

$$H_k(\lambda, \lambda_0, x, u) = \lambda^T f_k(x, u_k) - \lambda_0 L_k(x, u_k) \tag{4}$$

and the Hamiltonian system as

$$\dot{x} = \frac{\partial H_k}{\partial \lambda} \qquad -\dot{\lambda} = \frac{\partial H_k}{\partial x} \tag{5}$$

where $\lambda_0$ is a positive constant ($\lambda_0 \geq 0$).

Now ,we have the following theorem:

**Theorem 1** *If $(u, d)(.)$ and $(x, k)(.)$ are respectively an admissible optimal control and the corresponding trajectory for the problem (1)(2) and (3), then there exists a piecewise absolutely continuous curve $\lambda(.)$ and a constant $\lambda_0 \geq 0$, $(\lambda_0, \lambda(t)) \neq (0, 0)$ on $[0, T]$, such that:*

1. *the sixtuplet $(\lambda, \lambda_0, x, k, u, d)(.)$ satisfies the associated Hamiltonian system (5) almost everywhere*
2. *for given $(\lambda, \lambda_0, x, k)(t)$ at given time $t$, the following maximum condition holds if $t$ is not a switching time*

$$H_k(\lambda, \lambda_0, x, u) = \max_{d \in \underline{r}} \left( \sup_{v \in U_m} H_m(\lambda, \lambda_0, x, v) \right) \tag{6}$$

$$\text{where } m = \phi(x, k, d) \tag{7}$$

3. *at switching time $\tau_i$ (i.e. when (6) is not obtained for the k-mode) the following transversality conditions are satisfied:*
   (a) *if $\tau_i$ is an autonomous switching time on the manifold defined by a $p$ components vector ($p < n$), $C_k(x, t) = 0$, from the mode $k$ to the mode $j$:*

$$\lambda_j^+ = \lambda_k^- - \frac{\partial C_k^T}{\partial x}\pi\Big|_{\tau_i} \tag{8a}$$

$$H_j^+ = H_k^- + \frac{\partial C_k^T}{\partial t}\pi\Big|_{\tau_i} \tag{8b}$$

   *with $\pi$ a $p$ dimensional vector.*
   (b) *if $\tau_i$ is a controlled switching time from the mode $k$ to the mode $j$:*

$$\lambda_j^+ = \lambda_k^- \tag{9a}$$

$$H_j^+ = H_k^- \tag{9b}$$

**Remark 2** *It can be noted that $H_k(\lambda, \lambda_0, x, u) = \sup_{v \in U_{m_k}} H_k(\lambda, \lambda_0, x, v)$ is exactly the optimal condition holding for a single differential system. Hence the active available mode is the one which has the largest Hamiltonian.*

**Remark 3** *Trajectories such that the previous necessary conditions are satisfied are called extremals.*

**Proof:** The proof have been made in preceding works for controlled switching [14]-[15]. The case of autonomous switching is obtained with a dynamic programming argument. It is an extension of a result given in [16] with the approach of MP. ■

## 3 An Hysteresis example

In the paper [17], the authors describe an algorithm to design optimal controls of hybrid dynamical systems (HDS). The method proposed here is mainly based on dynamic programming. We would like to comment on one example the authors refer to chapter VIII. This example, which is fully described in [18]-[19], illustrates fairly well the capability of the proposed algorithm to converge towards the optimal criteria in certain cases. Unfortunately, a bifurcation both in the trajectory and the criteria appears with respect to a parameter value in the criteria, then the trajectories found with this approach are very far from the optimal ones. Our method can then be used to solve the example given by Branicky, Borkar and Mitter and get the optimal value of the criteria and the optimal trajectories.

Let us consider the hybrid system defined by

$$\dot{x} = f(x) + u \tag{10}$$

where the multivalued function $f(x)$ is shown in Figure 1. The control problem is:

minimize the criteria

$$J = \frac{1}{2}\int_0^\infty (qx^2 + u^2)e^{-t}dt \tag{11}$$

where $x$ is subject to (10), $x(0) = x_0$ and $u(t) \in R$.

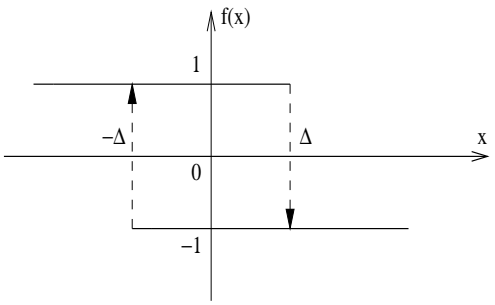


**Figure 1:** Hysteresis

In the sequel we will use the index $i = -1, 1$ to refer to the two models $\dot{x} = f_i(x, u)$ with

$$f_i(x, u) = i + u. \tag{12}$$

Clearly this hybrid system presents a discontinuity in the vector field at interior point $\Delta$ and -$\Delta$. In the mode 1, when the trajectory $x$ meets the boundary $\Delta$ from the left hand side, a switch to the mode $-1$ occurs. In the mode $-1$, there is a switch to the mode 1 if the trajectory $x$ meets the boundary $-\Delta$ from the right. So, the nature of hybrid phenomenon is autonomous switching. The figure 2 shows the associated automata.

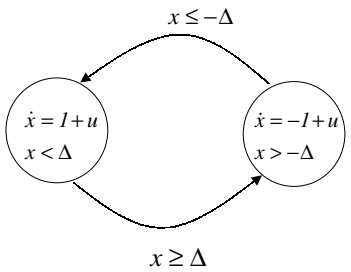


**Figure 2:** Associated Automata

In order to use the Maximum Principle, the Hamiltonian function is defined by:

$$H_i(\lambda, x, u, t) = \lambda^T f_i(x, u) - \frac{1}{2}(qx^2 + u^2)e^{-t}\ . \tag{13}$$

And the Hamiltonian System (HS) can be written as:

$$\dot{x} = \frac{\partial H_i}{\partial \lambda} = f_i(x, u) \tag{14a}$$

$$\dot{\lambda} = -\frac{\partial H_i}{\partial x} = qxe^{-t} \tag{14b}$$

Hence optimality condition stated by the Maximum Principle yields to the control $u(.)$ defined in terms of the adjoin variable $\lambda(.)$,that is:

$$\frac{\partial H_i}{\partial u} \equiv 0 \Leftrightarrow u = \lambda e^t. \tag{15}$$

Thus the Hamiltonian system for extremal trajectories satisfies

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{pmatrix} 0 & e^t \\ qe^{-t} & 0 \end{pmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} i \\ 0 \end{bmatrix}. \tag{16}$$

Recall that a trajectory is to be said *extremal* if it satisfies the maximum condition $\sup_u H_i(\lambda, x, u, t)$ for given $\lambda, x, t$.

A discontinuity in the vector field occurs when the trajectory $x(.)$ hits $\Delta$ or -$\Delta$ at some time $\tau$. This may be expressed by the set of interior boundary conditions,

$$C_i(x(t), t) \equiv x - i\Delta = 0. \tag{17}$$

Thus the transversality conditions are, if $C_i(x(\tau),\tau) = 0$:

$$\lambda_i(\tau^-) = \lambda_{-i}(\tau^+) + \alpha_i \left.\frac{\partial C_i}{\partial x}\right|_{t=\tau} \quad (18a)$$

$$H_i(\tau^-) = H_{-i}(\tau^+) - \alpha_i \left.\frac{\partial C_i}{\partial t}\right|_{t=\tau} \quad (18b)$$

where $\alpha_i$ is a real constant. Note that these transversality conditions are obtained with a dynamic programming argument and extend the transversality conditions which must be satisfied at final and initial points in continuous optimal problem.

As $\frac{\partial C_i}{\partial x} \equiv 1$ and $\frac{\partial C_i}{\partial t} \equiv 0, \forall i$, (18) is rewritten in (19) by:

$$\lambda_i(\tau^-) = \lambda_{-i}(\tau^+) + \alpha_i \quad (19a)$$

$$H_i(\tau^-) = H_{-i}(\tau^+) \quad (19b)$$

In order to determine the number $\alpha_i$, we solve (19) using (13) and (15). Hence, we get easily the polynomial expression in $\alpha_i$ :

$$(\alpha_i + 2ie^{-t})(\alpha_i + 2\lambda_{-i}) = 0 \ . \quad (20)$$

Thus the first equality in (19a) is now defined by the two possibilities:

$$\lambda_i(\tau^-) = -\lambda_{-i}(\tau^+) \quad (21a)$$

$$\lambda_i(\tau^-) = \lambda_{-i}(\tau^+) - 2ie^{-t}. \quad (21b)$$

So, (15) at the switching time is:

$$u_i(\tau^-) = -u_{-i}(\tau^+) \quad (22a)$$

$$u_i(\tau^-) = u_{-i}(\tau^+) - 2i. \quad (22b)$$

In other respect, as $u = e^t\lambda$ (16) can be rewritten as a LTI system:

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ q & 1 \end{pmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} i \\ 0 \end{bmatrix}. \quad (23)$$

Hence the system can be entirely determined. Let $[0, \tau_1, ..., \tau_k, ...], k = 0, 1, 2, ...$ be the sequence of switching time, then the analytical solution of (23) takes the form on each time interval $[\tau_k, \tau_{k+1}]$ :

$$x_i(t) = \frac{i}{q} + A_k e^{r_1(t-\tau_k)} + B_k e^{r_2(t-\tau_k)} \quad (24a)$$

$$u_i(t) = -i + r_1 A_k e^{r_1(t-\tau_k)} + r_2 B_k e^{r_2(t-\tau_k)} \quad (24b)$$

with

$$r_1 = \frac{1-\sqrt{1+4q}}{2} \quad (24c)$$

$$r_2 = \frac{1+\sqrt{1+4q}}{2} \quad (24d)$$

$$A_k = \frac{1}{r_1\sqrt{1+4q}}\left(ir_2 - qx_i(\tau_k) - r_1u_i(\tau_k)\right) \quad (24e)$$

$$B_k = \frac{1}{r_2\sqrt{1+4q}}\left(-ir_1 + qx_i(\tau_k) + r_2u_i(\tau_k)\right) \quad (24f)$$

The constants $A_k, B_k$ are determined using the transversality conditions (22). Notice that in (24e) and (24f), $x_i(\tau_k) = -i\Delta$, for $k \geqslant 1$.

Suppose for example we start from a position $x(0) = x_0 < \Delta$ and a control $u(0) = u_0$ in the mode 1, then as $r_1 < 0$ and $r_2 > 1, \forall q$, three cases can be distinguished:

$$x(t) \to -\infty \quad (25)$$

$$x(t) \to \frac{1}{q} \text{ if } \frac{1}{q} \leqslant \Delta \quad (26)$$

$$\exists \tau_1, x(\tau_1) = \Delta \quad (27)$$

The first case implies infinite cost function since $(qx^2 + u^2)e^{-t} \approx (q + r_2^2)B_k^2 e^{(2r_2-1)t}$ and must be rejected. The second can be validated if $\frac{1}{q} \leqslant \Delta$.

The last leads to the following:

If the condition (22b) is used then as $\dot{x}(\tau_1^-) = 1 + u(\tau_1^-) \geqslant 0$, $u(\tau_{-1}^+) \geqslant 1$. So, a variation analysis of $x(t)$ and $u(t)$ with respect to the initial condition $u(\tau_{-1}^+)$ shows that if $u(\tau_{-1}^+) \geqslant -\frac{r_1+q\Delta}{r_2}$ ( which is true in this case since $u(\tau_{-1}^+) \geqslant 1 \geqslant -\frac{r_1+q\Delta}{r_2}$), the control $u(t)$ increases each time. In fact, $B_1 > 0$ and with the same preceding argument the cost function explodes.

Hence the only possibility is (22a).

With this choice, by symmetry, we have the following behavior:

$$x(t) \to \infty \quad (28)$$

$$x(t) \to -\frac{1}{q} \text{ if } \frac{1}{q} \leqslant \Delta \quad (29)$$

$$\exists \tau_2, x(\tau_2) = -\Delta \quad (30)$$

With an analysis with respect to the values of $u$ at switching time, it is shown that the only possible solutions are:

1. there is one infinite sequence obtained with the following conditions

$$u(\tau_{k+1}^-) = u(\tau_k^+) \tag{31}$$

$$q < q_{\lim} = \frac{2+\sqrt{4+\Delta^2}}{\Delta^2} \tag{32}$$

which leads to a periodic behavior,

2. if $\frac{1}{q} < \Delta$, there is finite switching sequences (one with respect to the number of switches) and the asymtotic behavior of the associated trajectory is

$$x(t) \to -\frac{1}{q} \quad \text{if the last mode is } -1 \tag{33}$$

$$x(t) \to \frac{1}{q} \quad \text{if the last mode is } 1 \tag{34}$$

At the last switching time, $B_k$ must be zero. Hence, initial conditions are determined by integrating the systems in reverse time.

**Remark 4** *In the case 1, the value of $u(\tau_k)$ is determined by solving the equation whose the unknowns are $t$ and $u(0)$*

$$u(t) = u(0) \tag{35a}$$

$$x(t) = \Delta \tag{35b}$$

*with initial conditions $x(0) = -\Delta$.*

Now we can observe that the optimal solution depends on the value of $q$.

**Case 1:** $\frac{1}{q} > \Delta$

The only solution is the infinite sequence defined above.

**Case 2:** $q \geqslant q_{\lim}$

There is only one trajectory with no switch which converges to $\frac{1}{q}$ with initial condition obtained with the condition $B = 0$.

**Case 3:** $\frac{1}{q} \leq \Delta$ and $q < q_{\lim}$

There is one trajectory with respect to the number of switches and one for the infinite case. Hence cost function must be computed in order to decide which one is optimal.

Figure 3 shows the cost function with respect to $q$ and the number of switches. The dotted line represents the cost obtained for the infinite sequence, the upper bound solid line the trajectory which converges to $\frac{1}{q}$ without switching and intermediate solid lines the cost obtained with different finite switching numbers (here, k=1,...,20). It follows that the optimal control for $q < q_{\lim}$ is clearly obtained from the infinite sequence and for $q \geqslant q_{\lim}$ from the only one trajectory with no switch which converges to $\frac{1}{q}$.

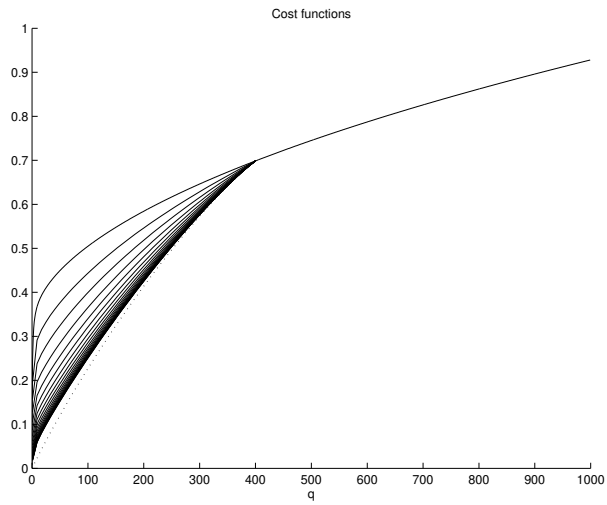


**Figure 3:** Cost functions versus $q$ and switching number ($k = 1, 2, ..., \infty$)

Below illustrative examples have been drawn for $\Delta = 0.1, x_0 = -0.2$ and for different values of $q$ (the three case mentioned above). The limite value of $q$ is $q_{\lim} \simeq 400.25$.

Thus, it can be noted that when $q \to_{q<q_{\lim}} q_{\lim}$, the value of $B_k$ converges to zero and the time beetwen two switches tends to infinity. The shape of $x(t)$ clearly shows this (Figure 5). Indeed, beetwen two switching times, in a first time $x$ converges to $\frac{1}{q}$ and in a second time the influence of the $B_k$ term leads to a new switch.

## 4 Conclusion

The use of a numerical algorithm in [19] does not exhibit the mandatory condition $B_k = 0$ when $q > q_{\lim}$, i.e. the divergent part of the Hamiltonian system disappears in the extremal trajectories. We can suppose that the algorithm ends when the criteria get a sufficient small decrease corresponding to a solution of (23).

The power of the maximum principle adapted to hybrid formulation is clearly shown in this example: When the optimal solution is periodic, we find the same results that Branicky et al. but the analytical solution is given. When no periodic solution verifies the necessary conditions of optimality, the analytical solution is given too but the numerical algorithm fails.

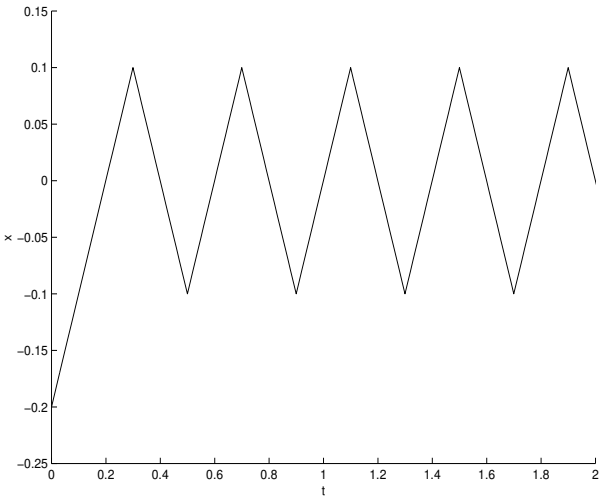


**Figure 4:** $x$ versus $t$, $q = \frac{1}{2}, x_0 = -0.2, \Delta = 0.1$

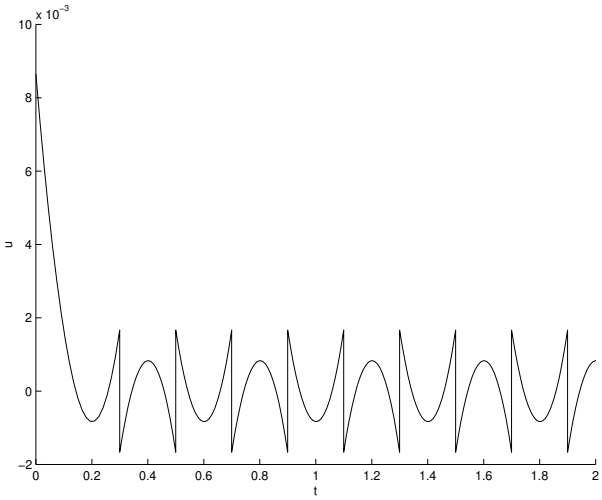


**Figure 7:** $u$ versus $t$, $q = \frac{1}{2}, x_0 = -0.2, \Delta = 0.1$

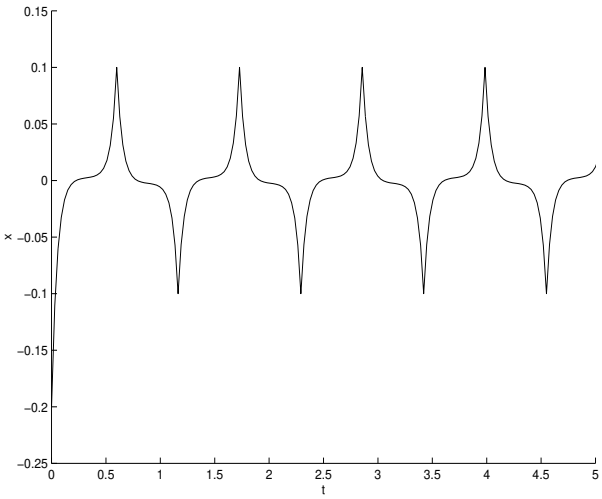


**Figure 5:** $x$ versus $t$, $q = 400, x_0 = -0.2, \Delta = 0.1$

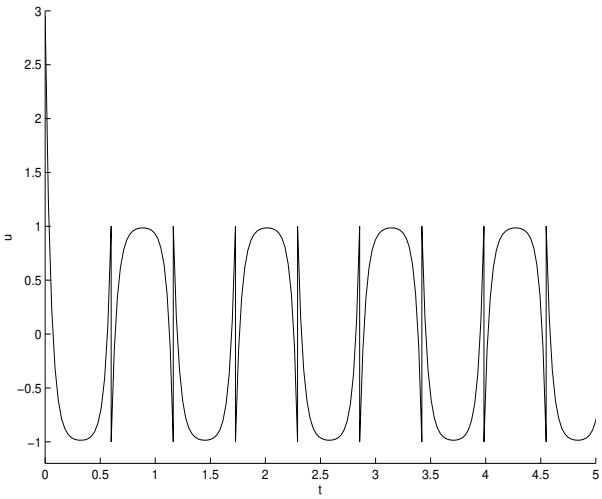


**Figure 8:** $u$ versus $t$, $q = 400, x_0 = -0.2, \Delta = 0.1$

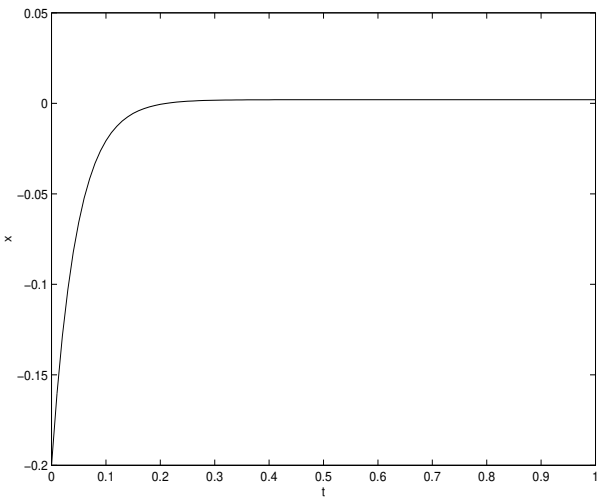


**Figure 6:** $x$ versus $t$, $q = 500, x_0 = -0.2, \Delta = 0.1$

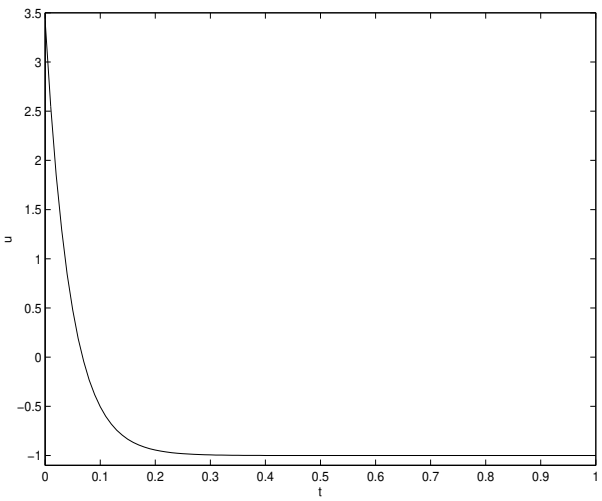


**Figure 9:** $u$ versus $t$, $q = 500, x_0 = -0.2, \Delta = 0.1$

This example is quite relevant. In spite of its relative simplicity, it exhibits some difficulties encountered in the optimal control of HDS.

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