Abstract—This paper address the continuous state estimation of a class of switched systems having modes in which the state is not or partially observable. Using a relevant LaSalle principle for switched systems, a characterization of invariant sets and associated control laws for which the state cannot be estimated, is given. This result is applied on a flying capacitor converter to prove convergency of the estimation error under suitable control laws. This work is also used to explain why rate of the estimate error decreases with high switching frequency scheme.

Index Terms—Observer design, invariance principle, power converter

I. INTRODUCTION

In the case of industrial applications with power of a few megawatts, the switching components voltage becomes very high (several kilovolts). Therefore, the switching frequency must be maintained to a low value and bulky filters are needed for obtaining an appropriated output [1]. To palliate this drawback, a new class of power electronic converters has appeared, called multilevel converters. These structures consist of a series connection of switching devices with passive storage elements, which are used to generate intermediate voltage levels [2]. The control law for these structures needs to maintain the intermediate voltage levels at some constant values and to regulate the voltage or the load current. The main advantages of the multilevel converter is that the spectral quality of the output signal is improved by a high switching frequency between the intermediate voltage levels [3]. The downside is that, excepting simpler DC-DC converters, the control of multilevel converters is more complex [4], [5].

Emerging control techniques such as stabilizing control [6], predictive control [7], [8], flat control [9] are based on observer technics. Moreover, the robustness of the proposed schemes is also improved and ensured with parameters estimations. Thus, there are a demanding for state or parameter estimations to improve algorithms and to reduce the number of sensors. The recursive filter such as Extended or Unscented Kalman Filter are efficient and the most frequently used techniques [10], [11]. But proof of the convergency are generally omitted or shown that that the state estimation of the flying capacitor relies in the stability of a linear switched systems having non positive definite matrices. The section III is devoted to the characterization of invariant set and associated control laws. A discuss is introduced at the beginning of section IV on how to determine the invariant set. Then an algorithm is proposed to solve this problem. Switched observer design with pole placement is illustrated in section V. Then the proposed characterization is used to explain why the rate of convergency of the observer is decreasing with the switching frequency.

II. PROBLEM STATEMENT

The state equations of the converter have an affine form given by

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{3} u_i(t)(A_i x(t) + B_i)$$

with a three dimensional state $x = [x_1 \ x_2 \ x_3]^T$, a three dimensional boolean control vector $u = [u_1 \ u_2 \ u_3] \in \{0, 1\}^3$, the matrices $A_i$, $i = 0, 1, 2, 3$, defined by

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & -\frac{1}{C_1} \\ 0 & 0 & 0 \\ \frac{1}{L} & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & -\frac{1}{C_2} \\ 0 & 0 & 0 \\ -\frac{1}{L} & \frac{1}{L} & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_3} \\ 0 & 0 & 0 \end{bmatrix},$$

and $B_1 = B_2 = 0$, $B_3 = [0, 0, E/L]^T$. The states $x_1$, $x_2$ are the voltages in each capacitor and $x_3$ is the load current.
The control $u$ refers to the switches position and we count $2^3$ modes following the values of $u$.

The aim is to estimate the voltage of the capacitors in the case where only the current in the load is measured. The output $y$ is defined by the equation $y = C\dot{x}$ with $C = [0\ 0\ 1]$.

Notice that the state components are only partially observable for every fixed configuration of the switches. For example, if the switch value is $u = [1\ 0\ 0]$, then the voltage in the capacitor $C_1$ cannot be estimated since it is disconnected or if the switch value is $u = [1\ 0\ 1]$ then only the sum of the voltages in capacitors $C_1$ and $C_2$ can be estimated. Thus, it is not possible to consider an observer for arbitrary switching law. A question of interest is then how to characterize the switching law for which it is impossible to observe the state?

Here such characterization will be illustrated in the case of a Luennberger switched observer. The purpose is to study the switching law for which it is impossible to observe the state $x$.

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Inv contains both \( \Omega_{\text{Inv}} \) and the set \( \Omega \), defined as the union of all \( \omega \)-limit sets associated to (3).

Therefore, Problem 2 may be useful to characterize the \( \omega \)-limit set and the associated control for the switched system (3). An important question is: “What is the relative size of \( \omega \) with respect to Inv and \( \Omega_{\text{Inv}} \)?” The answer can be obtained using the following density result.

**Theorem 3:** (See [19, Theorem 1.]) The Carathéodory solutions of (3) are dense among the solutions of \( \dot{x} \in F(x) \) in the following sense: Let \( x \) a global solution of (4) starting from \( x_0 \) and let \( \varepsilon : [0, +\infty) \to (0, +\infty) \) be continuous. Then there exists a solution \( \xi \) of (3) starting from \( \xi_0 \in B(x_0, \varepsilon(0)) \) such that \( ||\xi(t) - x(t)|| < \varepsilon(t) \) for all \( t \in [0, +\infty) \).

In other words, all global solution of (4) could be approached arbitrarily close by global solutions of (3).

In view of this density theorem the \( \omega \)-limit set of any trajectory of (4) is the \( \omega \)-limit set of a trajectory of (3). Therefore, \( \Omega_{\text{Inv}} \subset \Omega \subset \text{Inv} \).

Assume now that we get a trajectory solving Problem 2 which is not entirely an \( \omega \)-limit set. Since this solution is included in a level set of \( V \), it is clear, that its \( \omega \)-limit set remains in the same level set.

**IV. ALGORITHM TO COMPUTE INVARIANT SET**

The simplest situation occurs when on a time interval \((a, b)\), \( a < b \), there exists one index \( i \) such that

\[
\dot{V}_i(x) = x^T C_i^T C_i x = 0
\]

and \( \dot{V}_j(x) \neq 0 \) for \( j \neq i \).

The control is then given by \( \alpha_j(t) = \delta_{ij} \) for all \( t \in (a, b) \).

In this situation, since \( x^T C_i^T C_i x = 0 \) vanishes identically, by successive differentiations we get the set of equations:

\[
\begin{align*}
C_i x &= 0 \\
C_i A_i x &= 0 \\
&\vdots \\
C_i A_i^{n-1} x &= 0
\end{align*}
\]

and the trajectory \( x \) is therefore contained in the unobservable subspace associated to the pair \((C_i, A_i)\), denoted in the following by \( \text{Ker}(O(C_i, A_i)) \).

Assume now that there exists a subset of indices \( I_0 \subseteq \{1, \ldots, I\} \) of cardinality \(|I_0| > 1\) such that for all \( i_0 \in I_0 \),

\[
C_{i_0} x = 0
\]

on \((a, b)\) and \( C_{i_0} x(t) \neq 0 \) if \( j \not\in I_0 \) and \( t \in (a, b) \). In particular \( x \), restricted to the interval \((a, b)\), evolves in \( \bigcap_{i_0 \in I_0} \text{Ker}(C_{i_0}) \) and the associated control law (see Problem 2) takes its values in the subset

\[
\Delta_{I_0} = \{ \alpha \in \Delta \mid \sum_{i_0 \in I_0} \alpha_{i_0} = 1 \}.
\]

By differentiating (6) and replacing \( \dot{x} \) by \( \sum_{i \in I_0} \alpha_i A_i x \), we get

\[
\sum_{i \in I_0} \alpha_i C_{i_0} A_i x = 0
\]

for all \( i_0 \in I_0 \).

We see the former as a linear relation between control components, and we notice that either at least one coefficient \( C_{i_0} A_i x \) is a non-vanishing function (giving an algebraic condition on \( \alpha \)) or they all vanish, providing us with \(|I_0| \) extra algebraic conditions on \( x \) that can again be differentiated. In the case in which some coefficients are vanishing and some other are not, we are given an algebraic condition on \( \alpha \) and some algebraic conditions on \( x \) that can again be differentiated.

We obtain in this way a recurrent procedure. Notice that we do not need to require additional regularity assumptions on \( x \) in order to consider all these subsequent differentiations, since at each step only the differentiability of \( x \) is required.

We are therefore justified to define, for every \( i_0 \in I_0 \),

\[
p_{i_0} = \min \left\{ k \mid \exists \alpha_i \in I_0, \frac{\partial}{\partial \alpha_i} \frac{d^k}{d\alpha_i} C_{i_0} x \neq 0 \text{ on } (a, b) \right\},
\]

that is, \( p_{i_0} \) is the minimal number of time derivatives of \( C_{i_0} x \) guaranteeing the appearance of at least one component of the control with a nonzero coefficient.

By eventually restricting the interval \((a, b)\), we will assume that all such non-vanishing coefficients are nonzero everywhere in \((a, b)\).

For each \( i_0 \in I_0 \) the following conditions are fulfilled on \((a, b)\),

\[
\begin{align*}
C_{i_0} x &= 0 \\
C_{i_0} A_i x &= 0, \ i_1 \in I_0 \\
&\quad \quad \vdots \\
C_{i_0} A_i A_i \cdots A_i_{i_{p_{i_0}} - 1} x &= 0, \ (i_1, \ldots, i_{p_{i_0} - 1}) \in I_0^{p_{i_0} - 1}
\end{align*}
\]

and

\[
\sum_{i \in I_0} \alpha_{i_k} C_{i_0} A_i A_i \cdots A_i_{i_{p_{i_0}}} x = 0, \ (i_1, \ldots, i_{p_{i_0}}) \in I_0^{p_{i_0}}.
\]

Even if the the last equation can be differentiated once again (this depends on the regularity of \( \alpha \)), no additional information concerning the values of the control components can be obtained since their derivatives will appear.

The system of equations (7), (8), (9) provides us, at every instant of time, with a set of algebraic relations between the control \( \alpha \) and the point \( x \).

The set of solution of (8), denoted \( S_{p_{i_0}} \), is monotone non increasing w.r.t. \( p_{i_0} \). Then, assume that for every set \( I_0 \), there exists a finite maximum number \( p_{i_0} \) beyond which (that is for all \( p_{i_0} > p_{i_0} \)) the solution set of (8) is constant.

As the number \( p_{i_0} \) is a priori unknown, the following algorithm may be used to try to determine a solution.

**Algorithm 4:** for all subset \( I_0 = \{s_1, s_2, \ldots, s_{|I_0|}\} \) of \( \{1, \ldots, I\} \) for each \( i_0 \in I_0 \)

set \( p_{i_0} = 1 \) and \( S_{p_0} = \{\emptyset\} \)

step 1: check if the solution set of (8) is constant

\[
S_{p_{i_0}} = S_{p_{i_0} - 1}
\]

if not set \( p_{i_0} := p_{i_0} + 1 \) and return to step 1

end if

set \( p_{i_0} \) max := \( p_{i_0} - 1 \)

end for
for all $p = (p_{s_1}, p_{s_2}, \ldots, p_{s_{|I_0|}})$ with $1 \leq p_{s_k} \leq p_{s_k \text{ max}}$ for $k = 1, 2, \ldots, |I_0|$ compute the solution set of (8) taking $i_0 = s_k$ for $k = 1, \ldots, |I_0|$ and check if the intersection of all such solution sets reduces to the singleton \{0\} if not solve the algebraic system of equations (7), (8), (9)

end if

end for

end for

V. APPLICATION TO THE FLYING CAPACITOR CONVERTER

It is easy to show that if a matrix $P > 0$ satisfies $\tilde{A}_i^T P + PA_i \leq 0$, for $i = 0, 1, 2, 3$, then $P$ must be of the type

$$P = \begin{pmatrix} p_1 & p_4 & 0 \\ p_4 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}.$$ 

Moreover, in order to avoid a positive eigenvalue in $\tilde{A}_i^T P + PA_i$, the gains $L_i$ and $P$ must be such that

$$p_1 \left( \frac{u_2 - u_1}{c_1} - L_1(u) \right) + p_4 \left( \frac{u_3 - u_2}{c_1} - L_2(u) \right) + p_3 \frac{u_1 - u_2}{L} = 0$$

$$p_4 \left( \frac{u_2 - u_1}{c_1} - L_1(u) \right) + p_2 \left( \frac{u_3 - u_2}{c_1} - L_2(u) \right) + p_3 \frac{u_2 - u_3}{L} = 0$$

where $L_1(u)$ and $L_2(u)$ are respectively the first and the second component of $L(u) = L_0 + \sum_{i=1}^3 u_i L_i$.

As mention above, for a given mode, the subsystem is not or partially observable. The dynamic of such a system is fast and the smallest eigenvalues is $p = -3000$. Choosing a pole $p$ for all $\{u_1, u_2, e_3, e_2\}$ for which this is the case.

In the case where $e_2 = 0$, we get $u_1 = u_2$ and $e_3$ free.

For all control laws built in this way, system (2) is just stable and evolves on a level set of the matrix $P$. In the figure (2), for example, we have applied the singular control $u_3 = -\frac{u_1 e_1 + u_2 e_2 + u_3 C_2}{e_2}$. It can be observed that once the error $e_3$ is in or near 0 (that is $e$ in or near Inv) the errors in the estimation of the voltage values remain constant.

B. Observability of the operating point

Generally, the goal of the control applied on the flying capacitor converter is to regulate the load current and to maintain in average the voltage in each capacitor to a fixed value of $2E/3$ in capacitor $C_2$ and $E/3$ in capacitor $C_1$. The operating points of the flying capacitor converter may therefore be defined in average value by $x_{\text{ref}} = [2E/3 \ E/3 \ i_{\text{ref}}]^T$.

It is interesting to notice that the only control value that maintains the current in average around a nonzero value is a singular law. Indeed, for systems exhibiting a cyclic behavior, the desired operating point is the mean value of $x$ on the cycle.

To find the admissible operating point set, one can study the evolution of the average state $\bar{x}$ given by the convolution product

$$\bar{x}(t) = \mathbb{E} \{x(t)\} = \frac{1}{T_p} \int_{t-T_p}^t x(\tau)d\tau$$

(12)

where $T_p$ is the cycle period and $\mathbb{E}$ is a rectangular window function.

The dynamical model of $\bar{x}$ is obtained differentiating (12). However, the derivative is generally hard to use because of its nonlinear form.
A solution consists in defining the average state model
\[ \dot{x} = A_0\bar{x} + \sum_{i=1}^{3} \bar{u}_i(t)(A_i\bar{x}(t) + B_i), \quad \Pi_i \in [0 1] \] (13)
which gives an approximation of the dynamics of \( \bar{x} \). \( \bar{u} \) is the average value of \( u \) on the cycle. It has been proven that \( \bar{x} \) is close to \( x \) and \( x \) when \( T_n \) is small with respect to the system dynamics (see [20]).

The operating points are then defined as the equilibria of the average state model, that is, the elements of the set
\[ X_{ref} = \{x_{ref} \in \mathbb{R}^n \mid A_0x_{ref} + \sum_{i=1}^{3} u_{i,ref}(A_ix_{ref} + B_i) = 0, \quad u_{i,ref} \in [0, 1] \} \] (14)
Since no mode allows to hold the given reference \( x_{ref} \) as \( u_{ref} \in (0, 1) \), the only possibility for the switched system is to enter into a cyclic behavior around the reference \( x_{ref} \).

Here, equation \( A_0x + \sum_{i=1}^{3} u_i(A_ix + B_i) = 0 \) has a unique solution \( u_1 = u_2 = u_3 \) leading to the equilibrium \( \bar{x} = \frac{1}{2}u_3 \neq 0 \). This law corresponds to a singular law of the type seen at the beginning of the section, rendering the system unobservable.

Consequently, a better approximation of the singular law \( u_1 = u_2 = u_3 \) by the switching law leads to a smaller convergence rate of the observer. For example, if we consider two periodic switching laws, one obtained from the other by a uniform time-rescaling, which realize \( u_1 = u_2 = u_3 \) in average, we can see in Figures 3 and 4 that the convergence rate of system (2) decreases when the switching frequency increases. For this example the observer convergence is achieved at time \( t = 2 \times 10^{-3}\) s for a switching frequency of \( f_s = 10 \) kHz while it only happens at time \( t = 1.0 \times 10^{-2}\) s for a switching frequency of \( f_s = 30 \) kHz.

The method presented in this article has been validated in simulation with the nominal parameter values \( C_1 = C_2 = 40\mu F, L = 0.01H, R = 30\Omega \).
VI. CONCLUSION

In this paper, a pole placement design of a switched observer is proposed and applied on a flying capacitor converter. A complete characterization of invariant set and associated control laws for the estimate error dynamic systems explains in which case the system state cannot be estimated. A contrary, the result proves the asymptotic convergency of the observer under suitable control laws (i.e. for switching law not associated to an invariant set). An amazing consequence is that the operating points of the flying capacitor are associated to an invariant set. An amazing consequence is that the points are defined by mean values remaining observable the system in a neighborhood. In view of these results, it is clear that asymptotic stability property are not sufficient and an important question concerns the estimation of the observer rate convergency.

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REFERENCES


