On the algebraic characterization of invariant sets of switched linear systems

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Abstract

In this paper, a suitable LaSalle principle for continuous-time linear switched systems is used to characterize invariant sets and their associated switching laws. An algorithm to determine algebraically these invariants is proposed. The main novelty of our approach is that we require no dwell time conditions on the switching laws. By not focusing on restricted control classes we are able to describe the asymptotic properties of the considered switched systems. Observability analysis of a flying capacitor converter is proposed as an illustration.

Key words: Invariant set characterization, switched systems, invariance principle, observer design, power converter

1 Introduction

Switched systems have attracted a growing interest in recent years [18]. Such systems are common across a diverse range of application areas. As an example, switched systems play a major role in the field of power systems where interactions between continuous dynamics and discrete events are an intrinsic part of their behavior.

Stability of switched systems is a fundamental property for which many contributions have been proposed. Some important results go back to the works on the stability of systems driven by discontinuous feedback laws [25,23]. The importance of discontinuous feedback laws in control theory has been acknowledged since the late seventies, see [27,2]. More specific results for switched and hybrid systems have since then been developed (see [7,10] for multiple Lyapunov based approach, [18] for Lie algebra based results , [15] for the role of dwell time and [26] for a survey on stability criteria for switched and hybrid systems). One of the most fundamental results in the field of dynamical systems analysis and feedback control is the LaSalle principle. In the context of switched systems, recent investigations (see [9,16,5,20]) provide interesting contributions leading to extremely general results that require little structure on the family of solutions of the hybrid system [24,14]. We study in this paper the characterization of invariant sets and the associated control laws in the special case of linear switched systems and we apply it to the design of a switched observer. The invariance result on which our analysis is based uses essentially a result given in [19]. A finer characterization is possible in the framework of our paper, due to the linearity of the vector fields and the fact that the Lyapunov function is assumed to be quadratic.

The main novelty of our approach is that we require no dwell time conditions on the switching laws. By not focusing on restricted control classes we are able to describe the asymptotic properties of the considered switched systems. We can, in particular, characterize ω-limit sets of solutions of the switched system in terms of special invariant sets of the differential inclusion obtained by convexifying the set of admissible velocities. We show how to compute algebraically such invariant sets and the associated control laws. Such a characterization can be used to determine stabilizing switching laws. We illustrate this fact on a practical application from power systems, namely, the observation of a flying capacitor converter. No mode of this switched system is fully observable, in the sense that the observability matrix associated to each mode has not full rank. Our aim is to analyze which switching laws lead to unobservability.

The paper is organized as follows. The next section gives definitions of some important notions used in the paper. Section 3 presents the description of ω-limit sets and corresponding switching laws in terms of invariant sets of the associated differential inclusion. In Section 4 we propose an algorithm allowing to compute algebraically the invariant sets and their associated controls laws. The application to the observation problem of a flying capacitor converter is detailed in Section 5.

2 Preliminaries

Let us recall some standard notions (see, for instance, [13], section 3.12.4, p.129).

Definition 1 *A function $x : [0, +\infty) \to \mathbb{R}^n$ is said to approach a set $S \subset \mathbb{R}^n$ if $\inf_{s \in S} \|x(t) - s\| \to 0$ as $t \to \infty$.*

Definition 2 *A point $l \in \mathbb{R}^n$ is an ω-limit point of $x : [0, +\infty) \to \mathbb{R}^n$, if there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to \infty$ and $x(t_n) \to l$ as $n \to \infty$. The set $\Omega(x)$ of all such ω-limit points is the ω-limit set of x.*

Property 3 *The ω-limit set $\Omega(x)$ is always closed. If x is bounded and continuous, then $\Omega(x)$ is nonempty, compact, connected, is approached by x and it is the smallest closed set so approached.*

3 Problem statement and characterization of invariant sets

Consider the linear switched system

$$\dot{x} = A_{\sigma(t)}x, \tag{1}$$

where the state $x(t)$ belongs to $\mathbb{R}^n$, $A_1, \ldots, A_I$ are square matrices and the switching law σ belongs to $L^\infty([0, +\infty), \{1, \ldots, I\})$.

Notice that the initial-value problem associated with (1) has a unique global Carathéodory solution (see for instance [1] or [19]).

As it is well known, a sufficient condition guaranteeing the global asymptotic stability of the continuous-time switched system (1) is the existence of a common quadratic Lyapunov function [12]. Nevertheless, common quadratic Lyapunov functions are not necessary for asymptotic stability of switched systems (see [22,11] and also [21] for similar results on Lyapunov functions of higher degree). The main advantage in using quadratic Lyapunov functions is the fact that LMI methods provide powerful numerical tools to compute them [6].

In this paper, we consider *weak* common Lyapunov functions, which differ from standard Lyapunov ones in having just *nonpositive* derivative along system trajectories. The main difference with a LMI based approach is related to fact that asymptotic stability is analyzed using an extension of the LaSalle principle and we provide a characterization of the ω-limit sets and their associated switching laws. Using weak Lyapunov functions, global asymptotic stability can be guaranteed using additional conditions such as a dwell time assumption [16,5,20,24,14]. Here, no such assumptions are used and it is only supposed that the switching law is a Lebesgue measurable function.

Assume that there exists a symmetric positive definite matrix P ($P > 0, P^T = P$) such that for all $x \in \mathbb{R}^n \backslash \{0\}$,

$$\begin{aligned} x^T P x &> 0 \\ x^T(A_i^T P + PA_i)x &\le 0, \forall i \in \{1, \ldots, I\}\,. \end{aligned}$$

The function $V(x) = x^T P x$ is a weak common quadratic Lyapunov function. The asymptotic behavior of the trajectories is described by the following theorem.

Theorem 4 *Let $x_0 \in \mathbb{R}^n$ and $\sigma(\cdot)$ be a switching law defined on $[0, +\infty)$. Let x be the solution of the switched system corresponding to $\sigma(\cdot)$ with initial condition x_0. Then the ω-limit set $\Omega(x)$ is contained in a level set of V and, for all $y_0 \in \Omega(x)$, there exists an absolutely continuous function y which satisfies*

$$\begin{aligned} \dot{y} &= A_\alpha y \quad \text{for a.e. } t \ge 0 \\ y(0) &= y_0 \\ y(t) &\in \Omega(x) \quad \text{for every } t \ge 0, \end{aligned} \tag{2}$$

where the time-dependent matrix $A_\alpha = \sum_{i=1}^I \alpha_i A_i$ corresponds to a control law $\alpha(\cdot)$ taking its values in the set

$$\Delta = \left\{ \alpha = (\alpha_1, \ldots, \alpha_I) \in [0,1]^I \mid \textstyle\sum_{i=1}^I \alpha_i = 1 \right\}.$$

In particular, $\dot{V}(y(t)) = 0$ for every $t \ge 0$, where, with a standard abuse of notations, $\dot{V}(y(t))$ denotes the derivative of $t \mapsto V(y(t))$.

The theorem is a direct consequence of known facts. Indeed, the existence of a solution y of

$$\dot{y} \in \left\{ \sum_{i=1}^I \alpha_i A_i y \mid \alpha \in \Delta \right\} =: F(y)$$

on $\Omega(x)$ follows from [19]. Then $\alpha(\cdot)$ can be chosen measurable thanks to classical selection results (see [3,4]).

Define, for any $x \in \mathbb{R}^n$ and $i = 1, \ldots, N$, $\dot{V}_i(x) = x^T(A_i^T P + PA_i)x \le 0$. For any $v = \sum_{i=1}^I \alpha_i A_i x \in F(x)$, the directional derivative of V in the direction v satisfies:

$$\begin{aligned} \dot{V}(x; v) &= x^T(A_\alpha^T P + PA_\alpha)x \\ &= \textstyle\sum_{i=1}^I \alpha_i (x^T(A_i^T P + PA_i)x) \\ &= \textstyle\sum_{i=1}^I \alpha_i \dot{V}_i(x) \le 0 \end{aligned}$$

since the α_i's are positive. The solutions of (2) evolving in a level set of V satisfy the following control problem.

Problem 5 *Find the controls $\alpha^* \in L^\infty([0,+\infty),\Delta)$ and the associated trajectories x satisfying*

$$\dot{x} = A_{\alpha^*} x \text{ a.e} \qquad (3)$$
$$\text{subject to } \sum_{i=1}^{I} \alpha_i^* \dot{V}_i(x) = 0.$$

Notice that if α^* and x satisfy Problem 5, then $\alpha^* \in \arg\max_{\alpha\in\Delta} \alpha_i \dot{V}_i(x)$.

Denote by Inv the union of all supports of trajectories x corresponding to solutions of Problem 5, by Ω_{Inv} the union of all their ω-limit sets and by Ω the union of all ω-limit sets of trajectories of (1).

Proposition 6 *The following chain of inclusion holds*

$$\Omega_{\text{Inv}} \subset \Omega \subset \text{Inv}. \qquad (4)$$

Both inclusions can be strict.

Proof. The inclusion of Ω in Inv is a direct consequence of Theorem 4. In order to prove that $\Omega_{\text{Inv}} \subset \Omega$, recall the following result [17, Theorem 1]: if x is a global solution of (2) starting from x_0 and $\varepsilon : [0,+\infty) \to (0,+\infty)$ is continuous, then there exists a solution ξ of (1) starting from $\xi_0 \in B(x_0, \varepsilon(0))$ such that $\|\xi(t) - x(t)\| < \varepsilon(t)$ for all $t \in [0,+\infty)$. Therefore, the ω-limit set of any trajectory of (2) is the ω-limit set of a trajectory of (1). This concludes the proof of (4). □

An example where both inclusions are strict can be constructed as follows. Let $n = 3$ and P be the identity matrix. For every $a \in \mathbb{R}^3$, let $S(a)$ be the skew-symmetric matrix such that $S(a)v = a \times v$, where $\times$ denotes the cross product. Fix two orthogonal unit vectors v and w of $\mathbb{R}^3$. If $I = 2$, $A_1 = -ww^T + S(w + v)$ and $A_2 = -ww^T + S(-w + w \times v)$, then $V(x) = \|x\|^2$ is nondecreasing along the dynamics of A_1 and A_2 and the directional derivative of V at a point x along $A_1 x$ or $A_2 x$ is zero if and only if x is orthogonal to w. Notice that the only invariant subset of $\{x \mid x^T w = 0\}$ for the dynamics of A_1 or A_2 is $\{0\}$. It is easy to check that the solutions to Problem 5 are either constant trajectories on points of the line $L = \text{span}(v + w \times v)$ or arcs of circle centered at the origin and connecting λv to $(\lambda\sqrt{2}/2)(v + w \times v)$ or $\lambda v \times w$ to $(\lambda\sqrt{2}/2)(v + w \times v)$, for some $\lambda \in \mathbb{R}$. These arcs are parameterized with a velocity going to zero as the trajectory approaches L. Thus, $\Omega_{\text{Inv}} = \Omega = L$ and $\text{Inv} = \{x \mid x^T w = 0,\ (x^T v)(x^T(v \times w)) \geq 0\}$.
The previous argument can be refined in order to get that both inclusions in (4) are strict. In order to do so, denote by $\mathcal{A}(v, w)$ the pair of matrices $\{A_1, A_2\}$ introduced above, seen as a function of the vectors v and w. Similarly, let $\mathcal{I}(v, w) = \{x \mid x^T w = 0,\ (x^T v)(x^T(v \times w)) \geq 0\}$. Let T be a geodesic equilateral triangle on the unit sphere of $\mathbb{R}^3$ with edges of length $\pi/4$. Denote by v_1, v_2, v_3 its vertices and define $w_i = v_i \times v_{i+1}/\|v_i \times v_{i+1}\|$, $i = 1, 2, 3$, with the convention that $v_4 = v_1$. Let $I = 6$ and consider the set $\{A_1, \ldots, A_6\}$ obtained as union of $\mathcal{A}(v_1, w_1)$, $\mathcal{A}(v_2, w_2)$, $\mathcal{A}(v_3, w_3)$. Then $\Omega_{\text{Inv}} = \mathbb{R}\{v_1, v_2, v_3\}$, $\Omega = \mathbb{R}T$ and $\text{Inv} = \cup_{i=1}^{3} \mathcal{I}(v_i, w_i)$.

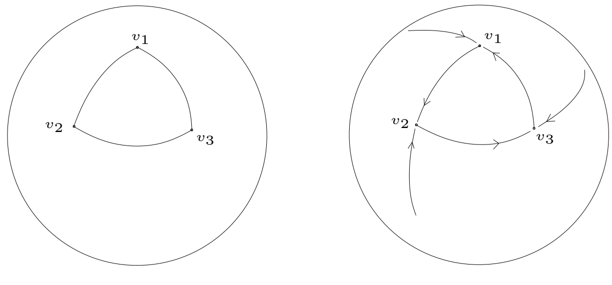

Fig. 1. The intersection of Ω (left) and Inv (right) with the unit sphere.

Proposition 6 states that if we are able to characterize completely the set of solutions of Problem 5 then we can bound from above and from below the set Ω containing all possible ω-limit sets of the original switched system. This motivates the results of the next section.

Another relation between Problem 5 and the convergence to the origin of a trajectory of (1) is the following.

Proposition 7 *Let x be a solution of (1) and $\alpha : [0,+\infty) \to \Delta$ be the piecewise constant function taking value on the vertices of Δ such that $\dot{x} = A_\alpha x$. If there exists a sequence of times $t_n \to +\infty$ such that the sequence $(\alpha(t_n + \cdot))_{n\in\mathbb{N}}$ weak-$\star$ converges in $L^\infty([0,+\infty),\Delta)$ to some $\alpha_\star$ that does not correspond to any nonzero solution of Problem 5, then $x(t) \to 0$ as $t \to +\infty$.*

Recall that a sequence $\alpha_n \in L^\infty([0,+\infty),\Delta)$ weak-$\star$ converges to $\alpha_\star$ if for every $\varphi \in L^1([0,+\infty),\Delta)$ the sequence of integrals $\int_0^\infty \alpha_n(t) \cdot \phi(t)dt$ converges to $\int_0^\infty \alpha_\star(t) \cdot \phi(t)dt$ as $n \to \infty$. Due to the Banach-Alaoglu theorem, $L^\infty([0,+\infty),\Delta)$ is sequentially weak-$\star$ compact.

Proposition 7 follows from the remark that $x(t_n)$, being bounded, converges, up to the extraction of a subsequence, to some $x_\star \in \mathbb{R}^n$ and that $x(t_n + \cdot)$ converges uniformly on compact sets to the trajectory of (3) corresponding to $\alpha_\star$ with $x_0 = x_\star$. (See, e.g, [8] for details.)

Proposition 7 can be used to guarantee the convergence to the origin of solutions of (1) corresponding to a wide class of switching laws, once the controls corresponding to solutions of Problem 5 are characterized.

4 Algebraic characterization

We discuss in this section a procedure to get algebraically the solutions of Problem 5. For every $i = 1, \ldots, I$, let C_i

be the matrix defined by

$$C_i^T C_i = -(A_i^T P + PA_i).$$

Then, along a solution x of Problem 5,

$$\begin{aligned}0 = \dot{V}(x) &= x^T(A_\alpha^T P + PA_\alpha)x \\ &= -x^T C_\alpha^T C_\alpha x = 0 \text{ a.e.}\end{aligned}$$

with $C_\alpha = \sum_{i=1}^{I} \sqrt{\alpha_i} C_i$. Hence, for a.a. t, $C_\alpha x = 0$. Let us assume that there exists a subset of indices $I_0 \subseteq \{1, \ldots, I\}$ of cardinality $|I_0| \geq 1$ on a time interval (a, b), $a < b$, such that for all $i_0 \in I_0$,

$$C_{i_0} x \equiv 0 \tag{5}$$

and $C_j x(t) \neq 0$ if $j \notin I_0$. In particular x, restricted to the interval (a, b), evolves in $\bigcap_{i_0 \in I_0} \mathrm{Ker}(C_{i_0})$ and the associated control law takes its values in the subset

$$\Delta_{I_0} = \left\{\alpha \in \Delta \mid \textstyle\sum_{i_0 \in I_0} \alpha_{i_0} = 1\right\}. \tag{6}$$

By differentiating (5) and replacing $\dot{x}$ by $\sum_{i \in I_0} \alpha_i A_i x$, we get for all $i_0 \in I_0$, $\sum_{i \in I_0} \alpha_i C_{i_0} A_i x = 0$. We are therefore justified to define, for every $i_0 \in I_0$,

$$p_{i_0}(x) = \min\left\{k \mid \exists i \in I_0, \frac{\partial}{\partial \alpha_i} \frac{d^k}{dt^k} C_{i_0} x \not\equiv 0 \text{ on } (a, b)\right\},$$

that is, $p_{i_0}(x) \in \mathbb{N} \cup \{+\infty\}$ is the minimal number of time derivatives of $C_{i_0} x$ guaranteeing the appearance of at least one component of the control with a nonzero coefficient. Notice that the time derivative appearing in the definition of $p_{i_0}(x)$ corresponds to the formal replacement of $\dot{x}$ by $A_\alpha x$ and that $p_{i_0}(x)$ depends on I_0. Let p_{i_0} be the minimum value of $p_{i_0}(x(t))$ as t varies in (a, b) and assume that p_{i_0} is finite. For each $i_0 \in I_0$ the following conditions are fulfilled on (a, b),

$$\begin{cases} C_{i_0} x & = 0 \\ C_{i_0} A_{i_1} x & = 0,\ i_1 \in I_0 \\ \quad\vdots & \\ C_{i_0} A_{i_1} A_{i_2} \cdots A_{i_{p_{i_0}-1}} x & = 0, \left(i_1, \cdots, i_{p_{i_0}-1}\right) \in I_0^{p_{i_0}-1} \end{cases} \tag{7}$$

and

$$\textstyle\sum_{i_k \in I_0} \alpha_{i_k} C_{i_0} A_{i_1} A_{i_2} \cdots A_{i_{p_{i_0}}} x = 0, \left(i_1, \cdots, i_{p_{i_0}}\right) \in I_0^{p_{i_0}}. \tag{8}$$

The system of equations (6), (7), (8) provides us, at every instant of time, with a set of algebraic relations between the control α and the point x. Let us rewrite system (7) in matrix form as $M_{i_0, I_0, p_{i_0}} x = 0$ and denote by $S_{i_0, I_0, p_{i_0}}$ the kernel of $M_{i_0, I_0, p_{i_0}}$.

Lemma 8 *For every set I_0 and every $i_0 \in I_0$, there exists a finite number $p_{i_0, I_0}^{\max}$ beyond which (that is for all $p > p_{i_0, I_0}^{\max}$) $S_{i_0, I_0, p}$ is constant. Moreover, $p_{i_0, I_0}^{\max}$ is the smallest p such that $S_{i_0, I_0, p} = S_{i_0, I_0, p+1}$.*

Proof. It is clear that $S_{i_0, I_0, p}$ is monotone non-increasing w.r.t. p. We prove by recurrence on k that if $S_{i_0, I_0, p} = S_{i_0, I_0, p+1}$ then $S_{i_0, I_0, p} = S_{i_0, I_0, p+k}$ for all $k \geq 1$. If $x \in S_{i_0, I_0, p+1}$ then, for every $i \in I_0$,

$$y_i = A_i x \in S_{i_0, I_0, p} = S_{i_0, I_0, p+1},$$

i.e. $M_{i_0, I_0, p+1} y_i = 0$. Replacing y_i by $A_i x$ for all $i \in I_0$, we get that $x \in S_{i_0, I_0, p+2}$. □

Notice that if $p_{i_0} \geq p_{i_0, I_0}^{\max}$ then (8) is verified. To avoid pathological cases we introduce the following definition.

Definition 9 *A control α solving Problem 5 is called regular if there exists a sequence of concatenated time intervals $[a_k, b_k)$ whose union is $[0, +\infty)$ and an associated sequence I_0^k of subsets of $\{1, \ldots, I\}$ such that for a.a. $t \in (a_k, b_k)$, $\alpha(t) \in \Delta_{I_0^k}$.*

We can now state the following proposition.

Proposition 10 *If (x, α) is a solution of Problem 5 with α regular then its support in $\mathbb{R}^n \times \Delta$ is included in the union of all subsets obtained through the following algorithm: take a subset $I_0 = \{s_1, s_2, \ldots, s_{|I_0|}\}$ of $\{1, \ldots, I\}$, take $p = (p_{s_1}, p_{s_2}, \ldots, p_{s_{|I_0|}})$ with $1 \leq p_{s_k} \leq p_{s_k, I_0}^{\max}$, $k = 1, 2, \ldots, |I_0|$, and solve the algebraic system of equations (6), (7), (8) associated to I_0 and p.*

The projection on $\mathbb{R}^n$ of the set obtained through the algorithm proposed in Proposition 10 usually contains strictly the set Inv. The following proposition gives a criterion guaranteeing that the output of the algorithm is contained in Inv.

Proposition 11 *Fix I_0 and p as in the statement of Proposition 10 and assume that the projection on $\mathbb{R}^n$ of the solutions of (6), (7), (8) is the same as the projection of the solutions of (7), (8). Then such projection is contained in* Inv.

The proposition follows from the fact that the projection of the solutions of (7),(8) is a linear space L and the equality of projections guarantees that there exists a velocity tangent to L that is admissible for Problem 5.

5 Illustrative application

We will show in this section how the characterization proposed in the previous sections can be used to study observability of a flying capacitor converter.

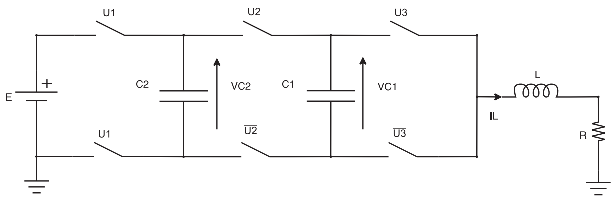

Fig. 2. Flying capacitor converter.

5.1 Problem statement

The state equations of the converter are given by

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{3} u_i(t)(A_i x(t) + B_i) \qquad (9)$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $u = (u_1, u_2, u_3) \in \{0, 1\}^3$. The matrices A_i, $i = 0, 1, 2, 3$, are defined by

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & -\frac{1}{C_1} \\ 0 & 0 & 0 \\ \frac{1}{L} & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} \\ 0 & 0 & -\frac{1}{C_2} \\ -\frac{1}{L} & \frac{1}{L} & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_2} \\ 0 & -\frac{1}{L} & 0 \end{bmatrix},$$

and $B_1 = B_2 = 0$, $B_3 = (0, 0, E/L)^T$. The states x_1, x_2 are the voltages in each capacitor and x_3 is the load current. The control u refers to the switches position and each among the possible 2^3 values of u corresponds a mode of (9). Here and in the following we take $C_1 = C_2 = 40\mu F$, $L = 0.01H$, $R = 30\Omega$. The aim is to estimate the voltage of the capacitors by measuring only the load current. The output y is thus defined by the equation $y = Cx$ with $C = (0, 0, 1)$.

Notice that the state components are only partially observable for every fixed mode, since the observability matrix has never full rank. Thus, no observer will converge to the actual state for arbitrary switching laws. A question of interest is then how to characterize the switching laws for which it is possible to observe the state.

Here such characterization is illustrated in the case of a Luenberger switched observer of the form

$$\dot{\hat{x}} = A_0 \hat{x} + L_0(y - \hat{y}) + \sum_{i=1}^{3} u_i (A_i \hat{x} + B_i + L_i(y - \hat{y})).$$

The dynamics of $e = x - \hat{x}$ are given by

$$\dot{e} = \tilde{A}_0 e + \sum_{i=1}^{3} u_i \tilde{A}_i e, \qquad (10)$$

with $\tilde{A}_i = A_i - L_i C$, $i = 0, 1, 2, 3$. It is easy to show that if a matrix $P > 0$ satisfies $\tilde{A}_i^T P + P\tilde{A}_i \leqslant 0$, for $i = 0, 1, 2, 3$, then P and the gains L_i must be of the type

$$P = \begin{bmatrix} \wp_1 & \wp_4 & 0 \\ \wp_4 & \wp_2 & 0 \\ 0 & 0 & \wp_3 \end{bmatrix},$$

$$\wp_1 \left(\frac{u_2 - u_1}{C_1} - L_1(u) \right) + \wp_4 \left(\frac{u_3 - u_2}{C_1} - L_2(u) \right) + \wp_3 \frac{u_1 - u_2}{L} = 0,$$
$$\wp_4 \left(\frac{u_2 - u_1}{C_1} - L_1(u) \right) + \wp_2 \left(\frac{u_3 - u_2}{C_1} - L_2(u) \right) + \wp_3 \frac{u_2 - u_3}{L} = 0,$$

where $L_1(u)$ and $L_2(u)$ are respectively the first and the second component of $L(u) = L_0 + \sum_{i=1}^{3} u_i L_i$. By pole placement, we fix $L_0 = 10^4(0, 0, 5.7)$, $L_1 = 10^6(8.975, 4.5, 0)$, $L_2 = 10^6(-4.475, 4.475, 0)$, $L_3 = 10^6(-4.5, -8.975, 0)$ and $\wp_1 = \wp_2 = 90$, $\wp_3 = 6.075 \times 10^6$, $\wp_4 = -45$.

5.2 Invariant set and control characterization

A simple computation shows that zero is the only admissible velocity for Problem 5 and that its trajectories are constrained on $\{e \mid e_3 = 0\}$. Therefore,

$$\mathrm{Inv} = \Omega_{\mathrm{Inv}} = \{e \mid e_3 = 0\}.$$

It follows from Proposition 6 that $\Omega = \{e \mid e_3 = 0\}$. Moreover, the control law $u = u(t) \in [0, 1]^3$ corresponding to a trajectory of Problem 5 must satisfy a.e. the relation

$$(u_3 - u_2)e_2 + (u_2 - u_1)e_1 = 0. \qquad (11)$$

Equation (11) and Proposition 7 guarantee the convergence to zero of the estimation error for a wide class of switching laws. In Figure 3 and 4, we have applied a switching law satisfying (11) to system (10). Once the error e_3 approaches 0 (that is, e approaches Inv) the estimation errors of the voltage values remain constant. Although the control applied is a singular one, taking values in Δ but not always in its vertices, the trajectory can be approximated arbitrarily well by a solution of the switched system (10), as it follows from [17, Theorem 1].

5.3 Observability at the operating point

Generally, the goal of the control is to regulate the load current and to maintain in average the voltage in each capacitor to a fixed value of $2E/3$ in capacitor C_2 and $E/3$ in capacitor C_1. The operating point of the flying capacitor converter may therefore be defined in average value by $x_{\mathrm{ref}} = (2E/3, E/3, i_{\mathrm{ref}})$. Notice that the control law which maintains the current at a prescribed nonzero value is a singular law. The operating points are defined as the equilibria of the average state model, that is, the elements of the set

$$\Big\{ x_{\mathrm{ref}} \in \mathbb{R}^3 \mid A_0 x_{\mathrm{ref}} + \sum_{i=1}^{3} u_{i,\mathrm{ref}}(A_i x_{\mathrm{ref}} + B_i) = 0 \text{ for some } u_{\mathrm{ref}} \in [0, 1]^3 \Big\}.$$

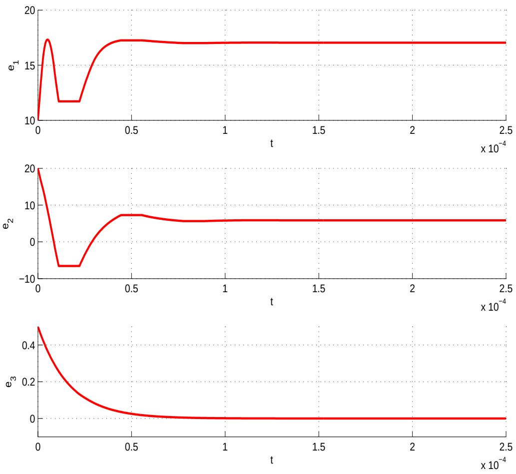

Fig. 3. Observation errors

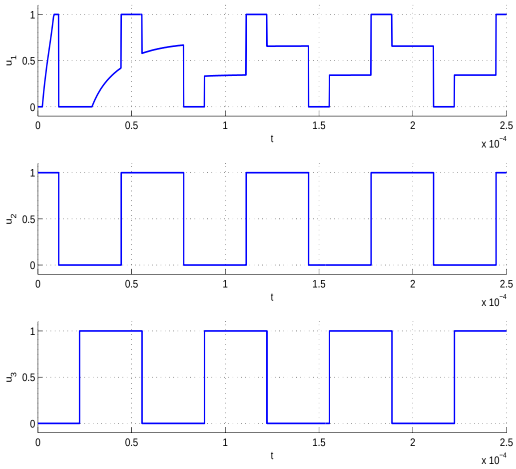

Fig. 4. A control making the converter unobservable

The equation $A_0x+\sum_{i=1}^{3}u_i(A_ix+B_i)=0$ has a unique solution $u_1=u_2=u_3$ leading to the equilibrium $i=\frac{E}{R}u_3\neq 0$. This law corresponds to a singular law of the type seen at the beginning of the section, rendering the system unobservable. Consequently, the better the switching law approximates the singular law $u_1=u_2=u_3$, the smaller is the convergence rate of the observer. For example, if we consider three periodic switching laws, one obtained from the other by a uniform time-rescaling, realizing $u_1=u_2=u_3$ in average, we can see in Figure 5 that the convergence rate of system (10) decreases when the switching frequency increases (the chosen switching frequencies are $f_s=5$, 15, 25kHz).

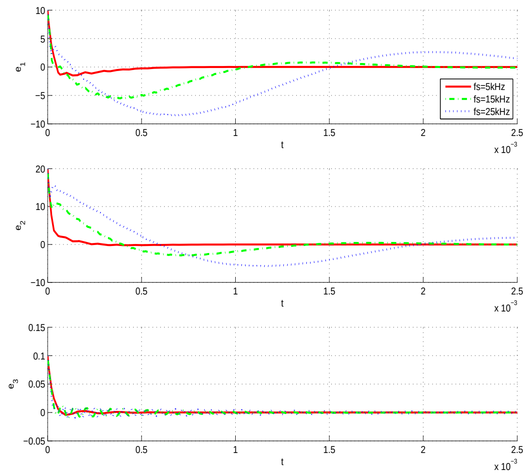

Fig. 5. Observation errors for time-rescaling switching laws

6 Conclusion

In this paper, a characterization of invariant sets and the associated switching laws for continuous-time switched systems is proposed. In particular, a description is given of the relations between the ω-limit sets of a switched system having a common weak Lyapunov function and the trajectories of the convexified system lying on a level set of the Lyapunov function. The result is used to analyse observability properties of a flying capacitor converter.

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