About solving hybrid optimal control problems

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Abstract - The main objective of this paper is to discuss numerical difficulties in solving hybrid optimal control problems and to propose a multiple phase-multiple shooting formulation for hybrid optimal control design. Such a formulation allows to solve directly the problem using nonlinear programming techniques. In the case of switched systems, it is shown that the switching rule can be obtained in a direct way avoiding combinatory explosion.

Keywords— **Hybrid systems, Optimal Control, Algorithms, NLP.**

I. INTRODUCTION

In hybrid systems context, the necessary conditions for optimal control are now well known [1], [2], [3]. These conditions mix discrete and continuous classical necessary conditions on the optimal control. The discrete dynamic involves dynamic programming methods whereas between the a priori unknown discrete values of time, optimization of the continuous dynamic is performed using the maximum principle (MP) or Hamilton Jacobi Bellmann equations (HJB). At the switching instants, a set of boundary-tranversality necessary conditions ensures a global optimization of the hybrid system.

From a practical point of view, it is particulary very hard to perform such an optimization. The major raison is that discrete dynamic requires to evaluate the optimal cost along all branches of the tree of all possible discrete trajectories. Dynamic programming is then used, but the duration between two switchings and the continuous optimization procedure make the task really hard. This makes the complexity increasing and only problems with a poor coupling between continuous and discrete parts can be reasonably solved.

Recent works have proposed to solve optimal switching problems by using a fixed switching schedule. By switched systems we mean a class of hybrid dynamical systems consisting of a family of continuous (or discrete) time subsystems and a rule that governs the switching between them (to be determined). The optimization consists then in determining the optimal switching instants and the optimal continuous control assuming the number of switchings and the order of active subsystems already given. In [4], [5], the authors develop a search algorithm based on dynamic programming to obtain derivatives of the value function with respect to the switching instants. Then a nonlinear search method is used to determine the optimal solution.

Recently, in the case of piecewise linear systems for which the different dynamics are associated to a fixed partition of the state space, B. Lincoln and A. Rantzer have proposed a Relaxed Dynamic programming methodology to obtain a suboptimal state feedback solution in the case of a quadratic criteria [6]. These works complete preceding approaches [7], [8], [9].

Shaikh and Caines have proposed algorithms based on the maximum principle for both multiple controlled and autonomous switchings with fixed schedule [1]. The algorithms use the transversality conditions at switching instants. Then, the authors develop a combinatoric search in order to determine the optimal switching schedule [10]. In [11], a *strong variations algorithms* based on the maximum principle are proposed and applied to a eight order hybrid system with seven discrete states.

In the context of discrete time systems, some interesting results on state or output feedback must be mentioned [12], [14] , [13] and [15]. Finally and more recently, always for a fixed schedule, M. Egerstedt et al. have proposed a gradient based algorithm for switched systems [16] and for switching systems [17].

This paper is organized as follows. In section II, we start by staying the optimal control problem in a hybrid framework. In section III, we discuss numerical methods based on the Maximum Principle and the Bellmann Principle. Then in section IV and in complement of all the methods resulting from the resolution of the necessary conditions of optimality, we propose to use a multiple-phase multiple-shooting formulation which enables the use of standard constraint nonlinear programming methods. This formulation is applied to hybrid systems with autonomous and controlled switchings and seems to be of interest in practice due to the simplicity of implementation.

In the case of controlled switching (section V), we show that the proposed approach gives directly the optimal switching schedule. A convex formulation of the initial problem can also be used to solve the problem and possibly to detect singular control [18]. This fact is illustrated in section (VI) with a numerical example.

II. PROBLEM FORMULATION

The class of hybrid systems under consideration in this paper is defined as follows :

For a given finite set of discrete states $\underline{Q} = \{1, ..., Q\}$, there is an associated collection of continuous dynamics defined by differential equations

$$\dot{x}(t) = f_q(x(t), u(t), t) \quad (1)$$

where $q \in \underline{Q}$, the continuous state $x(.) \in \mathbb{R}^n$ ($n \in \mathbb{N}$), the continuous control $u(.) \in \mathbb{R}^m$ ($m \in \mathbb{N}$), the vector fields f_q are supposed defined and continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^m \times [a, b]$, $\forall q \in \underline{Q}$. In addition, some algebraic constraints of the form

$$g_{L_q} \leq g_q(x, u, t) \leq g_{U_q} \quad (2)$$

and bounds on the state

$$x_{L_q} \leq x(t) \leq x_{U_q} \quad (3)$$

and control

$$u_{L_q} \leq u(t) \leq u_{U_q} \quad (4)$$

are considered. Actually, (3) and (4) are included in the condition (2) but usually we give them separately to exhibit state and control constraints.
The discrete dynamic is defined using a transition function ν of the form:

$$q(t^+) = \nu(x(t^-), q(t^-), d(t), t) \quad (5)$$

with $q(.)$ the discrete state ($q(t) \in \underline{Q}$) and $d(.)$ the discrete control ($d : [a, b] \rightarrow \underline{D}$ where $\underline{D} = \{1, ..., D\}$ is a finite set). ν is a map from $\mathbb{R}^n \times \underline{Q} \times \underline{D} \times [a, b]$ to $\underline{Q}$.
The discrete variable $q(.)$ is a piecewise constant function of the time. This is indicated by t^- and t^+ in (5) meaning just before and just after time t.
The value of the transition function ν depends on two kinds of discrete phenomena which can affect the evolution of $q(.)$: changes in the discrete control $d(.)$ and boundary conditions on (x, t) of the form $C_{(q,q')}(x, t) = 0$ which modify the set of attainable discrete states.
This hybrid model covers a very large class of hybrid phenomena. It takes into account autonomous and/or controlled events.
Let $[t_0 = a, t_1, ..., t_i, ..., t_m = b]$ and $[q_0, ..., q_i, ..., q_m]$ be the sequence of switching times and the associated mode sequence corresponding to the control $(u, d)(.)$ on the time interval $[a, b]$.
A hybrid criterion is introduced as:

$$J(u, d) = \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} L_{q_i}(x(t), u(t), t)dt + \phi(x(t_i), t_i) \quad (6)$$

where $q_i \in \underline{Q}$.
The optimal control problem consists in finding the hybrid control $(u, d)(.)$ that minimizes the cost function J over the time interval $[a, b]$.

III. DISCUSSION ON THE NUMERICAL METHODS BASED ON NECESSARY CONDITIONS FOR OPTIMALITY

In optimal control theory, there are two major ways to solve a continuous problem:

i Methods classified as variational methods (e.g. using the Maximum Principle)

ii Methods which use the Bellmann Principle with the dynamic programming and the Hamilton_Jacobi_Bellmann equation.

A. *The necessary conditions resulting from the Maximum principle*

Applying a hybrid version of the maximum principle to the problem formulation above yields a set of necessary conditions on the optimal control (under suitable assumptions). For simplicity we do not consider path constraints or inequality constraints. The reader may refer to [2] for a description of the suitable assumptions.
As in the continuous case, at any time t, the following maximum condition holds for $(p, p_0, x, q)(t)$:

$$\frac{\partial H_q}{\partial u} = 0 \quad (7)$$

where $H_q(p, p_0, x, u, t) = p^T f_q(x, u, t) + p_0 L_q(x, u, t)$ denotes the Hamiltonian function and p the costate or adjoint variable whose dynamic is given by

$$\dot{p} = -\frac{\partial H_q}{\partial x}. \quad (8)$$

At switching time t_i, $i = 0, \ldots, m$, the following transversality conditions are satisfied: there exists a vector π_i such that

$$p(t_i^-) = p(t_i^+) + \frac{\partial C_{q_{i-1},q_i}}{\partial x}^T \pi_i \quad (9)$$

$$H_{q_{i-1}}(t_i^-) = H_{q_i}(t_i^+) - \frac{\partial C_{q_{i-1},q_i}}{\partial t}^T \pi_i \quad (10)$$

The notations (9), (10) imply that π_i must be equal to zero if t_i is a controlled switching time without boundary conditions giving thus the continuity of p and H. Equations (9) and (10) must obviously be adapted according to the final and initial constraints under the state (x, q) at time $t = a$ and $t = b$ (not necessary specified).
The complete set of necessary conditions are the equation (1) to (8) and the boundary conditions at switching instants

t_i, (9) (10). This problem is known as *a multipoint boundary value problem.* It must be noticed that the above set of necessary conditions express only local optimality properties and they do not make it possible to determine the optimal switching sequence. A way of doing it is to use the dynamic programming jointly.

In the continuous case and in order to solve a boundary value problem, the user has to determine the good values for the state and adjoint variables at initial and /or final time. A shooting method [20] can be a way to solve numerically this problem. The procedure consists of determining an initial guess such that final boundary conditions are met. It can be described by the following algorithm:

1) choose an initial condition for the differential system (1) and (8),

2) integrate the system along the time interval

3) determine whether final boundary conditions are met or not

4) if not, use a procedure which adjusts the initial guess and repeat from step 2

Unfortunately this method is sensitive : a small change of the initial guess can produce a large change in the final condition. Moreover as the choice of the adjoint variable is not intuitive (non physical), it may be very difficult to determine a good guess.
Finally numerical integration of (1) and (8) can be very ill-conditioned : for example in case of linear systems, if the direct system has λ as an eigenvalue then the adjoint system has $-\lambda$ as an eigenvalue leading to opposite dynamic. In order to avoid this ill_contitioning, it is better to proceed in the following way:

1) choose a control u for the differential system (1),

2) integrate forward the system (1)

3) integrate backward the system (8) using the relevant optimal conditions at final time to determine $p(b)$

4) determine whether optimal conditions are met or not

5) if not, use a procedure (gradient with $\frac{\partial H}{\partial u}$) which adjusts the initial control u and repeat from step 2.

In the hybrid case, due to discrete dynamic, the switching instants as well as their number and the mode sequence, are a priori unknown. So it is difficult to impose the transversality conditions (9) (10) due to combinatory explosion [10], [11].

B. Necessary conditions using Bellmann Principle

To summarize for a given classical optimal control problem, the Bellmann Principle expresses that the value function on $[a, b]$ satisfies for a given initial state x_0 :

$$J^*(a, b, x_0) = \min_{u\big|_{[a\ c]}} \{J(a, c, x_0, u) + J^*(c, b, x_c)\} \quad (11)$$

where $x_c = x(c)$ is the resulting state at time $c \in [a, b]$ for the control $u\big|_{[a\ c]}$. In discrete time, this relation defines a recurrent relation whose solution can be obtained backward in time.
In continuous time, one gets (by differentiating (11)) the Hamilton Jacobi Bellmann equation (HJB). This leads in the hybrid case to :

$$\frac{\partial J^*(t, b, x, q)}{\partial t} = -\min_u \{L_q(x, u, t) + \frac{\partial J^*(t, b, x, q)^T}{\partial x} f_q(x, u, t)\} \quad (12)$$

$$J^*(t, b, x, q) \leq J^*(t, b, x', q') \quad (13)$$

where (x', q') is the resulting state after an authorized jump.
There is also some major difficulties to solve (12) and (13). As (12) is a partial differential equation, numerical solutions imply a discretization on the whole time-state space. In that case, we have to solve globally the problem and the dimension of the problem appears as a brake. In the hybrid case, a set of inequalities (13) on the value functions and defined for each locations are added(see [6]) which increases notably the difficulties.

These two ways to solve optimal control problem can be refered as indirect methods since they attempt to solve necessary conditions derived from the initial control problem. Another way to proceed can be to reformulate the optimal control problem as a classical optimization problem for a simple reason: multiple-phase multiple-shooting formulation can be a good formulation for hybrid optimization.

IV. MULTIPLE-PHASES MULTIPLE-SHOOTING FORMULATION

Hybrid trajectories can be viewed as made up of a collection of N phases. A phase is a portion of trajectory in which the system of differential equations remains unchanged. Within the phase k, the discrete state, q_k, is fixed and the continuous dynamic equations are given by constraints (2,3,4) and $\dot{x}(t) = f_{q_k}(x(t), u(t), t)$ for $t_{I_k} \leq t \leq t_{F_k}$.
At the switching instants, the phases are linked by boundary conditions of the general form:

$$\begin{aligned} \psi_L \leq \psi(&x(t_{I_1}), u(t_{I_1}), t_{I_1}, x(t_{F_1}), u(t_{F_1}), t_{F_1}, \\ &x(t_{I_2}), u(t_{I_2}), t_{I_2}, x(t_{F_2}), u(t_{F_2}), t_{F_2}, \\ &\ldots \\ &x(t_{I_N}), u(t_{I_N}), t_{I_N}, x(t_{F_N}), u(t_{F_N}), t_{F_N}) \leq \psi_U \end{aligned} \quad (14)$$

This boundary conditions are used to ensure the junction conditions for example:

$$t_{I_{k+1}} = t_{F_k} \quad (15)$$
$$x(t_{I_{k+1}}) = x(t_{F_k}) \quad (16)$$

or in the case of a *jump* of the state at t_{F_k}

$$x(t_{I_{k+1}}) = \Upsilon(x(t_{F_k}), u(t_{F_k}), t_{F_k}) \quad (17)$$

and in the case of an autonomous switching, an algebraic constraint of the form

$$C_{q_k,q_{k+1}}(x(t_{F_k}), t_{F_k}) = 0. \quad (18)$$

To sum up, we have a collection of phases corresponding to a sequence q_k, $k = 1...N$, a constraint dynamical system in each phase and a set of conditions evaluated at the phases boundaries. It can be noticed that the sequence q_k, $k = 1...N$, as well as the number N are a priori unknown. It depends on the controls u and d and on equations (1)(5).
Now, recall that a shooting method is a procedure which consists of determining an initial guess such that final boundary conditions are met. In order to reduce the sensitivity [22], [21] of a shooting method (a small change of the initial guess can produce a large change in the final condition), it is widely fruitful to split the time interval into M smaller segments $[t_m t_{m+1}]$. In this case, an initial value x^0_m for the shoot on each segment m must be guessed. As final boundary conditions, constraints to force continuity of the state between the segments must be added i.e. $x_m(t_{m+1}) = x^0_{m+1}$. This approach is called multiple-shooting [20]. As additional variables and constraints are introduced, the size of the problem increases but robustness is improved [22]. In our case, it is advantageous to divide all phases in a multiple-shooting formulation.
In order to solve numerically the multiple-phase multiple-shooting formulation of the above optimization problem, discretization schemes must be chosen for (1) and (6). For a fixed switching schedule q_k, $k = 1...N$, the resulting finite dimensional problem can be solved using standard nonlinear programming.

At this point, it is interesting to mention the fact that NLP applies to optimal hybrid control problems and can be useful to solve them. It can be noticed that the multiple-phase formulation is of importance in order to consider discontinuous vector fields. Concerning the convergence results, they are obviously related to NLP algorithms.
It is important in such programs to be able to exploit the sparsity of the corresponding matrices and call refinement procedures. Some standard code exist and make it properly [23]. Naturally, a better accuracy can be obtained by passing formally expression of the gradient of the cost and constraints.
Then, it is necessary to repeat the optimization with combinatoric algorithms (for example [1]) along all admissible switching schedules. This last point can be omitted in the case of switched systems as it is shown in the next section.

V. THE CASE OF SWITCHED SYSTEMS

Switched systems are the most simple hybrid systems : the only hybrid phenomena are controlled switchings. So, such systems can also be expressed using a single vector field meaning,

$$\dot{x}(t) = F(x, u, \alpha) = \sum_{q=1}^{Q} \alpha_q(t) f_q(x(t), u(t)) \quad (19)$$

with $x \in \mathbb{R}^n$, $\alpha(t)$ a Boolean vector ($\alpha(t) \in \{0,1\}^Q$) and $\alpha_q(t)$ refers to the q^{th}-component of $\alpha(t)$ so that there is one and only one component of $\alpha(t)$ equal to 1, i.e. $\alpha(t) \in D$ where $D = \{\alpha \in \{0,1\}^Q : \sum_{q=1}^{Q} \alpha_q = 1\}$.
By switching between the different values of D, the function α plays the role of the discrete control.
Two approaches can be used to solve the optimal control problem in such a situation.
The first one consists in the following :
Choose an arbitrary but reasonably large number of phases N multiple of Q ($N = mQ$). Choose the periodic schedule sequence $s = \{1, 2, ..., Q, 1, 2, ..., Q, ..., 1, 2, ..., Q\}$. Set the duration at each location q to $I_q = T/N$ ($T = b - a$). Guess an initial control u^0 and compute the resulting trajectory x^0.
Solve the corresponding multiple phase multiple shooting problem formulation with the above initial conditions.
The boundary phase constraints are :

$$x(t_{I_{k+1}}) = x(t_{F_k}), \;\; I_{q_k} \geq 0, \quad \sum_{k=1}^{N} I_{q_k} = T. \quad (20)$$

Remark : The schedule seems to be imposed. Actually it is not the case since the optimization procedure can produce $I_{q_k} = 0$ which means that the mode q_k must not be used locally. Hence, the optimization is made for all possible sequences of length less or equal to N.
The second approach consists in extending the set D to its convex hull

$$co\{D\} = \{\alpha \in [0\;1]^Q : \sum_{q=1}^{Q} \alpha_q = 1\} \quad (21)$$

and treating the problem as a continuous optimization problem using nonlinear programming. A bang bang optimal control α (at the vertices of the set $[0\;1]^Q$) corresponds to an optimal solution for the switched systems. The advantage of such a formulation is that it enables the detection of singular arcs (the case where H_{uu} is singular which leads to controls which are not bang-bang) which is a frequent situation into switched system optimization problems [18], [25].

The methods based on the maximum principle are not able to detect singular control and no solution can be produced without more mathematic analysis. On another hand direct methods take the advantage in such situation as explained in the next section.
In fact various constraints on the solution can be formulated and solved with this numerical approach: the sequence schedule can be fixed i.e. the number and/or the order of the switchings, as minimum delay between two switchings and so on...

VI. EXAMPLE

Let us illustrate our purpose on a time optimal control problem proposed in a preceding paper [18]. We consider a switched system formed by two linear systems

$$\dot{x}(t) = \alpha(t)A_0x(t) + (1 - \alpha(t))A_1x(t) \qquad (22)$$

with $A_0 = \begin{pmatrix} 0.4 & 0.3 \\ -1.3 & 1.1 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0.2 & -1.4 \\ 0.8 & -0.7 \end{pmatrix}$.

The values of the discrete control $\alpha(.)$ are 0 or 1. The problem is to find the optimal time transfer from the initial position x_0 to the final position x_F. Although there exist trajectories which connect $(x_0\ ,\, x_F)$, an optimal hybrid trajectory doesn't exist. Now if the set of control values is extended to its convex hull ($\alpha(t) \in [0\ 1]$), there exists an optimal solution (figure 1). As it has been mentioned in [18], when line D (fig 2) is reached from initial position by applying control $\alpha(t) = 1$, the control $\alpha(t) \cong 0.5127$ leads to a sliding motion on D until reaching the second intersect point where the control switch to $\alpha(t) = 0$. In fact, the value $\alpha(t) \cong 0.5127$ refers to a singular control. It corresponds to the case where $\frac{\partial H}{\partial u} = 0, \frac{\partial^2 H}{\partial u^2} = 0$. The presence of a singular control explains why the original problem has no solution. Nevertheless, sub-optimal trajectories for the switched system can be obtained by chattering throughout the line D.
On figure 2, we have drawn numerical solutions of two continuous problems :

1) a one-phase one-shooting formulation for the embedding (i.e. $\alpha(t) \in [0\ 1]$) continuous problem

2) a multiple-phase multiple-shooting formulation for the embedding continuous problem

On figure 2, we can see that due to the presence of a singular control, the numerical solution of the embedding problem ($\alpha(t) \in [0\ 1]$) gives rise to a bad convergency and oscillations in the neighborhood of line D. It is clear that the multiple-phase multiple-shooting formulation yields better results than the one-phase one-shooting formulation and gives results close to the analytic solution because no continuity w.r.t. the vector field is imposed.
Finally, a multiple-phase multiple-shooting formulation for the hybrid problem ($\alpha(t) \in \{0, 1\}$) with time constraint

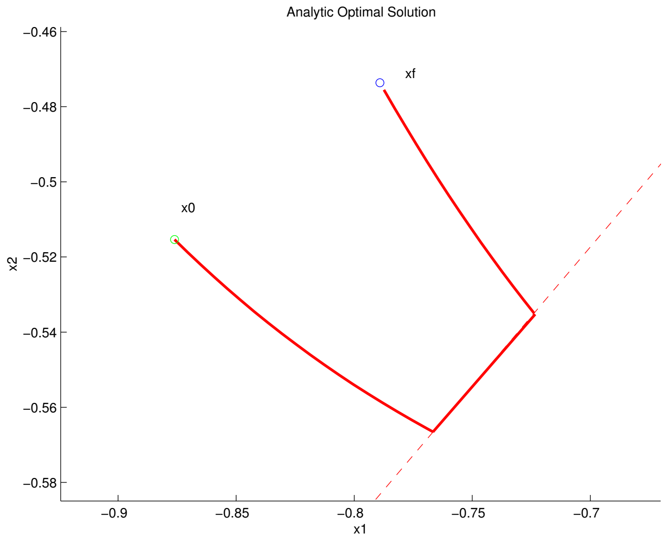

Fig. 1. Optimal trajectory for the embedding problem $J = 0.7869$

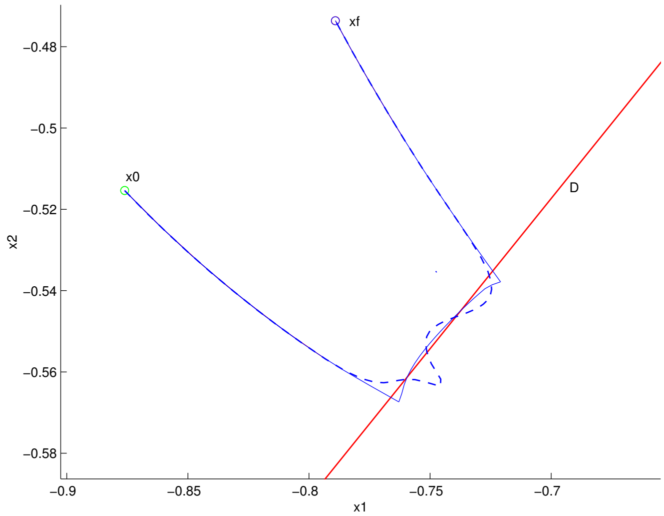

Fig. 2. Numerical solution for the embedding problem. Single (dash, $J = 0.7875$) versus multiple-phase multiple-shooting (dash dot $J = 0.7874$) formulation.

between two switchings and/or a fixed maximum number of switchings is considered. On figure 3, for 32 maximum switchings, we obtain the best result ($J = 0.7874$) by chattering in exactly 32 switchings near the line D (dash dot).
We can also impose a minimum duration (here $\Delta t = 0.02s$) between two switchings (fig. 4) without any constraint on the maximum number of switchings. This case may represent physical constraints on the actuator, for example. The best result is obtained for 24 switchings (fig. 4)($J = 0.7875$).
Finally, we have drawn the result for 32 switchings and a

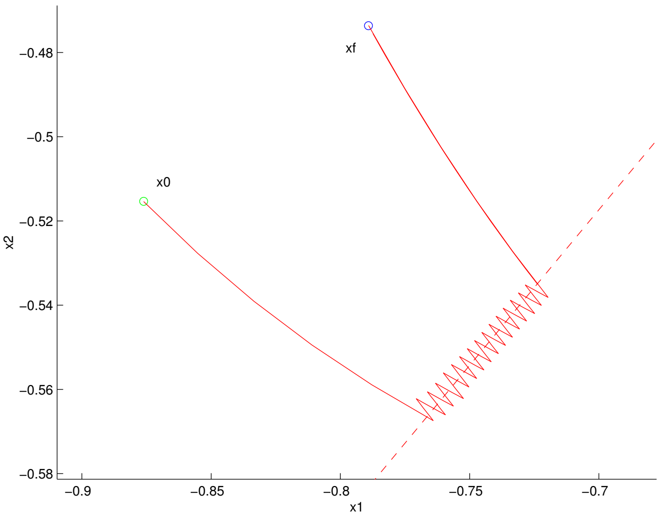

Fig. 3. Optimal trajectory for a maximum of 32 switchings. Cost $J = 0.7874$.

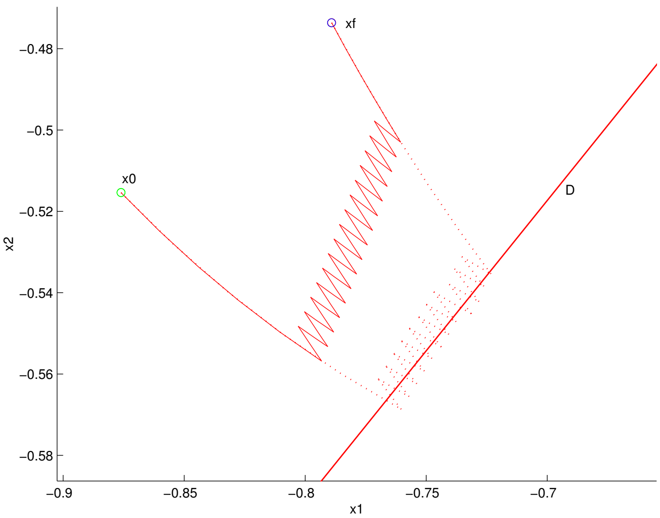

Fig. 4. Optimal Trajectories with time delay $\Delta t = 0.02$ between two switchings and no maximum number of switchings. Best result : 24 switchings and $J = 0.7875$.

minimum duration ($\Delta t = 0.02s$) between two switchings on figure 4. In this last case, we obtain $J = 0.8043$.

VII. CONCLUSION

In this paper, practical methods for optimal hybrid control are considered. After a short recall of existing methods, we have mentioned that a multiple-phase multiple-shooting formulation can be really useful to solve hybrid problems with the help of nonlinear programming. The proposed method presents the advantage to be particulary robust and easy to implement on a computer and must be mentioned as a complement to existing methods. As an example and in order to show the effectiveness of the method, we have chosen a singular time optimal control problem for which methods based on the Maximum Principle are unable to give any result. It has been shown that several constraints can be added such as time delay between two switchings, maximum number of switchings... Nevertheless in the general case, it remains the problem of discrete dynamics which imply to use combinatoric search and may lead to explosive computational effort (as in the discrete case with the dynamic programming).

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