Observer-based switched control design with pole placement for discrete-time switched systems

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ABSTRACT: In this paper, switched linear discrete-time systems are considered. We propose a method to design an observer-based switched control which guarantees that the switched system is asymptotically stable. The contribution of this paper is twofold. We propose a method to design a switched control which allows to place the poles of each subsystem in desired locations and which guarantee that the switched system is asymptotically stable using Lyapunov arguments. In case of output feedback, we prove a separation principle for this class of hybrid systems. Hence, the design of the switched state feedback control and the switched observer can be carried out independently.

KEYWORDS: Switched Discrete Time Systems, Switched Lyapunov functions, Observer based control, Switched observers, Separation principle, LMI.

1 INTRODUCTION

In recent years, the study of hybrid dynamical systems has received a growing attention in control theory and practice [1], [2]. Switched systems are an important class of hybrid systems consisting of a family of continuous (or discrete) time subsystems and a rule that orchestrates the switching between them. A survey of basic problems in stability and design of switched systems is given in [3] where some contributions are summarized. Recent contributions dealing with stability analysis or design of state feedback based control laws are proposed in [4], [5], [6]. To our knowledge, there is no result in the literature which combines stability and pole placement for hybrid systems. The motivations of this problem are numerous. It is well known that it is possible to design a switched control which stabilizes a set of unstable systems. In the opposite, a set of stable systems can give an unstable hybrid system by an inappropriate switching sequence. By state feedback, you can imagine to place the poles of each subsystem and get an unconditionally stable switched system. The advantages are obvious : the dynamic of each subsystem is fixed so as to get a correct behavior when a model keeps constant for any long time. When the switching is activated because of variation in the parameters, or in the operating point, or by a

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supervisor which selects the best control strategy, the stability is ensured whatever the switching control can be.

In this paper, we propose a $\mathcal{LMI}$ based approach to address this problem. The goal is to ensure, in addition of stability of the closed loop switched system, that the closed loop subsystems have their poles in desired location. The result can be used for switched state feedback design problems or for the more realistic switched output feedback control design problem. In the case of output feedback, we design an observer-based switched control using a separation principle proved in this paper for linear discrete-time switched systems. Hence, one may perform a separate design of the switched state feedback control and the switched observer. The advantage of the proposed control, in addition to be a stabilizing output feedback based control with constraints on the linear dynamics, is that the observer allows a direct access to all the components of the state vector. This may be useful for people interested by fault detection problems in the switched systems framework.

The paper is organized as follows. In the next section, the design of a switched state feedback with pole placement is proposed. In Section 3 a separation principle is proved. This shows that an observer-based switched control can be designed. Observer based switched control with pole placement constraints is also derived. We end the paper by some illustrative examples and a conclusion.

Notations: We use standard notations throughout the paper. M^T is the transpose of the matrix M. $M > \mathbf{0}$ ($M < \mathbf{0}$) means that M is positive definite (negative definite). $\mathbf{0}$ and $\mathbf{I}$ denote the null matrix and the identity matrix with appropriate dimensions. $B(0, R)$ denotes the ball with center 0 and radius R and :

$\|x\|$ denotes the euclidian norm.
$\|A\| = \sup \frac{\|Ax\|}{\|x\|}$ hence $\|AB\| < \|A\| \, \|B\|$
$\prod_{k=1}^{p} A_{i_k} = A_{i_p} A_{i_{p-1}} \cdots A_{i_1}$.

2 SWITCHED STATE FEEDBACK WITH POLE PLACEMENT

2.1 Complete pole placement

Consider the switched system defined by:

$$x_{k+1} \quad = \quad A_\alpha x_k + B_\alpha u_k \qquad (1)$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input. $\{(A_i, B_i) : i \in \mathcal{E}\}$ are a family of matrices parameterized by an index set $\mathcal{E} = \{1, 2, ..., N\}$ and $\alpha : \mathbb{R}^n \times \mathbb{N} \rightarrow \mathcal{E}$ is a switching signal ($i = \alpha(x_k, k)$). The results proposed in this paper applies also when the switching sequence is generated by any hybrid strategy or supervisor.

We assume that the switching signal is unknown a priori but real time available. The classical stabilizing switched state feedback control problem is to find

$$u_k = K_\alpha x_k, \tag{2}$$

ensuring stability of the corresponding closed loop switched system

$$x_{k+1} = \big(A_\alpha + B_\alpha K_\alpha\big) x_k \tag{3}$$

This problem reduces to the computation of the gain matrices K_i, $i \in \mathcal{E}$, ensuring the asymptotic stability for the switched system (3) under arbitrary switching signal. A solution has been proposed to this problem in [7]. It is based on the use of switched quadratic Lyapunov functions and gives less conservative results than the search of a common quadratic Lyapunov function [3], [6]. This solution is recalled in the following theorem.

Theorem 1 *If there exist symmetric matrices S_i, matrices G_i, and R_i, $\forall i \in \mathcal{E}$, such that $\forall (i,j) \in \mathcal{E} \times \mathcal{E}$*

$$\begin{pmatrix} G_i + G_i^T - S_i & (A_i G_i + B_i R_i)^T \\ A_i G_i + B_i R_i & S_j \end{pmatrix} > \mathbf{0} \tag{4}$$

then the state feedback control given by (2) with

$$K_i = R_i G_i^{-1} \qquad \forall i \in \mathcal{E} \tag{5}$$

stabilizes asymptotically the system (3).

Proof: see [7].

On the other hand, computing a state feedback matrix gain K which ensures that the closed loop dynamical matrix $(A + BK)$ for the LTI system (A, B) has $\{\lambda_1, \ldots, \lambda_n\}$ as its eigenvalues is well known if (A, B) is controllable. The existence of the various K solutions of the problem has already been considered in [8]. The proposed procedure has been used recently in [9] for continuous time LTI systems and leads to the computation of the matrix gain K in two steps :

1. For $q = 1, \ldots, n$ compute bases

$$\begin{bmatrix} M_q \\ \\ N_q \end{bmatrix} \quad \text{of the null-space of} \quad \begin{bmatrix} A - \lambda_q \mathbf{I} & & B \end{bmatrix} \tag{6}$$

2. For arbitrary column vectors v_q of dimension m compute the gains

$$K \triangleq R G^{-1} \tag{7}$$

where

$$R \triangleq \begin{bmatrix} N_1, & \ldots, & N_n \end{bmatrix} diag_{q=1}^{n} v_q,$$
$$G \triangleq \begin{bmatrix} M_1, & \ldots, & M_n \end{bmatrix} diag_{q=1}^{n} v_q$$

The only condition on the vectors v_q is that the matrix G must be invertible. We have so the degree of freefom in the choice of K. This procedure has to be modified for complex eigenvalues. As complex eigenvalues occur in conjugate pairs $(\lambda_r, \lambda_{(r+1)})$ with $\lambda_{(r+1)} = \lambda_r^*$, the null-space computation must be replaced with

$$\begin{bmatrix} M_r \\ N_r \\ M_{(r+1)} \\ N_{(r+1)} \end{bmatrix} \qquad (8)$$

null-space of

$$\begin{bmatrix} A - Re(\lambda_r)\mathbf{I} & B & Im(\lambda_r)\mathbf{I} & \mathbf{0} \\ -Im(\lambda_r)\mathbf{I} & \mathbf{0} & A - Re(\lambda_r)\mathbf{I} & B \end{bmatrix}$$

The matrix gain K is given by (7) with:

$$R \triangleq N\Psi, \quad G \triangleq M\Psi \qquad (9)$$

where

$$\begin{array}{rcl} M & \triangleq & \left[\ldots, M_q, \ldots, M_r, M_{(r+1)}, \ldots\right] \\ N & \triangleq & \left[\ldots, N_q, \ldots, N_r, N_{(r+1)}, \ldots\right] \\ \Psi & \triangleq & diag\big(\ldots, v_q, \ldots, v_r, v_{(r+1)}, \ldots\big), \\ & & \quad v_q \in \mathbb{R}^{p\times 1}, \quad v_r = v_{(r+1)} \in \mathbb{R}^{2p\times 1} \end{array}$$

with the subscripts q for real eigenvalues and r, $r+1$ for complex eigenvalues.

Here, for switched linear systems, we are interested in assigning the eigenvalues of the closed loop matrices $(A_i + B_i K_i)$, $\forall i \in \mathcal{E}$, to prescribed locations of the complex plane while ensuring stability of the closed loop switched system (3). Hence, we are looking for a switched state feedback such that

- the closed loop dynamic spectra satisfies

$$spec(A_i + B_i K_i) = \{\lambda_{i1}, \ldots, \lambda_{in}\} \qquad (10)$$

  where the λ_{iq} , $q = 1, ..., n$, denote desired eigenvalues

- the closed loop switched system is unconditionally globally asymptotically stable.

To answer this problem, we propose to use the previous procedure and Theorem 1. Our problem is to look for each linear subsystem for gain matrices K_i, $i \in \mathcal{E}$ such that (10) is satisfied while ensuring global asymptotic stability of the switched closed loop system by enforcing the Lyapunov-type constraints given in Theorem 1.

As in [9] for the linear time invariant case, we notice that in the framework of switched linear discrete systems, the stabilizing switched control structure (5) is similar to the matrix gains formulas (7). We can then select among the stabilizing switched controls the one ensuring that the closed loop matrices $(A_i + B_iK_i)$, $i \in \mathcal{E}$ have $\{\lambda_{i1}, \ldots, \lambda_{in}\}$ as eigenvalues. This can be done by replacing in the LMIs of Theorem 1 the unknowns R_i and G_i by those of (9). The matrix Ψ_i, defined for each subsystem i following the previous procedure, gathers all degrees of freedom and is used to satisfy the Lyapunov constraints of Theorem 1. This is summarized in the following Theorem.

Theorem 2 *[10] Assume that M_i and N_i have been computed for each liner subsystem i as indicated previously. If there exist symmetric matrices S_i and matrices Ψ_i solutions of:*

$$\begin{bmatrix} M_i\Psi_i + \Psi_i^T M_i^T - S_i & \Psi_i^T M_i^T A_i^T + \Psi_i^T N_i^T B_i^T \\ A_iM_i\Psi_i + B_iN_i\Psi_i & S_j \end{bmatrix} > 0, \tag{11}$$

$\forall (i,j) \in \mathcal{E} \times \mathcal{E}$, then a stabilizing switched state feedback exists and the resulting gains K_i are given by

$$K_i = (N_i\Psi_i)(M_i\Psi_i)^{-1}$$

Moreover, such gains ensure that the closed loop dynamic is characterized by spec$(A_i + B_iK_i) = \{\lambda_{i1}, \ldots, \lambda_{in}\}$.

Proof : Assume that the LMIs (11) are feasible. This is equivalent to the feasibility of the LMIs (4) with

$$R_i = N_i\Psi_i, \qquad G_i = M_i\Psi_i$$

and the closed loop switched system obtained with $K_i = R_iG_i^{-1} = (N_i\Psi_i)(M_i\Psi_i)^{-1}$ is asymptotically stable. To complete the proof, one has to follow similar reasoning as in [8] to check that $\{\lambda_{i1}, \ldots, \lambda_{in}\}$ is a set of closed loop eigenvalues of the matrices $(A_i + B_iK_i)$, $i \in \mathcal{E}$. $\square$

2.2 Partial pole placement

The success of the proposed procedure is related to the availability of degrees of freedom in excess with respect to a pure pole placement. It it well known that for single input systems the matrix gain ensuring a complete pole placement is uniquely defined and one has no excess of degree of freedom in this case. The approach given in the previous section can be used for multi-input systems only. However, partial pole placement is a case of practical interest and can be addressed for single inuput systems. In this case only a subset of column vectors G_i and R_i have to satisfy the subspace inclusions defined in (6) and (8). The remaining freedom can then be used to meet additional constraints. In this case, for each linear subsystem i, $i \in \mathcal{E}$, the procedure becomes:

1. Compute for $q = 1, \ldots, l$ with $l < n$ bases

$$\begin{bmatrix} M_{iq} \\ \\ N_{iq} \end{bmatrix} \text{ of the null-space of } [A_i - \lambda_{iq}\mathbf{I} \qquad\qquad B_i]$$

2. The matrix gains are given by

$$K_i \triangleq R_i G_i^{-1}, \quad R_i \triangleq N_i \Psi_{N_i}, \quad G_i \triangleq M_i \Psi_{M_i}$$

with

$$M_i \triangleq \big[M_{i1}, \ldots, M_{il}, \underbrace{\mathbf{I}, \mathbf{I}, \ldots, \mathbf{I}}_{n-l \quad times}\big],$$

$$N_i \triangleq \big[N_{i1}, \ldots, N_{il}, \underbrace{\mathbf{I}, \mathbf{I}, \ldots, \mathbf{I}}_{n-l \quad times}\big]$$

$$\begin{array}{l} \Psi_{M_i} \triangleq diag(v_{i1}, \ldots, v_{il}, s_{i1}, \ldots, s_{i(n-l)}), \\ \quad v_{iq} \in \mathbb{R}^p, \quad v_{ir} = v_{i(r+1)} \in \mathbb{R}^{2p}, \quad s_{ik} \in \mathbb{R}^n \\ \\ \Psi_{N_i} \triangleq diag(v_{i1}, \ldots, v_{il}, t_{i1}, \ldots, t_{i(n-l)}), \\ \quad v_{iq} \in \mathbb{R}^p, \quad v_{ir} = v_{i(r+1)} \in \mathbb{R}^{2p}, \quad t_{ik} \in \mathbb{R}^p \end{array} \tag{12}$$

Theorem 3 *Let Ψ_{M_i}, Ψ_{N_i} matrices given by (12). If there exist symmetric matrices S_i, matrices Ψ_{M_i}, Ψ_{N_i} solutions of:*

$$\begin{bmatrix} M_i\Psi_{M_i} + (M_i\Psi_{M_i})^T - S_i & (A_iM_i\Psi_{M_i} + B_iN_i\Psi_{N_i})^T \\ A_iM_i\Psi_{M_i} + B_iN_i\Psi_{N_i} & S_j \end{bmatrix} > 0, \tag{13}$$

$\forall (i,j) \in \mathcal{E} \times \mathcal{E}$, then a stabilizing switched state feedback exists and the resulting gains K_i are given by

$$K_i = (N_i\Psi_{N_i})(M_i\Psi_{M_i})^{-1}$$

Moreover, such gains ensure that the closed loop dynamic is characterized by $\{\lambda_{i1}, \ldots, \lambda_{il}\} \subset spec(A_i + B_iK_i)$.

Proof: The proof is similar to the proof of Theorem 2.

3 OBSERVER BASED OUTPUT FEEDBACK

Consider the switched system defined by:

$$\begin{array}{rcl} x_{k+1} & = & A_\alpha x_k + B_\alpha u_k \\ y_k & = & C_\alpha x_k \end{array} \tag{14}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input and $y_k \in \mathbb{R}^p$ is the output vector. $\{(A_i, B_i, C_i) : i \in \mathcal{E}\}$ are a family of matrices of appropriate dimensions parameterized by an index set $\mathcal{E} = \{1, 2, ..., N\}$ and $\alpha : \mathbb{R}^n \times \mathbb{N} \to \mathcal{E}$ is a switching signal defined as previously.

The problem now is the design of an observer-based switched control law of the following form :

$$\begin{aligned} \hat{x}_{k+1} &= A_\alpha \hat{x}_k + B_\alpha u_k + L_\alpha(y_k - \hat{y}_k) && (15)\\ \hat{y}_k &= C_\alpha \hat{x}_k \\ u_k &= K_\alpha \hat{x}_k, && (16) \end{aligned}$$

such that the corresponding closed loop switched system

$$\begin{pmatrix} x_{k+1} \\ \epsilon_{k+1} \end{pmatrix} = \begin{bmatrix} \tilde{A}_\alpha & \tilde{B}_\alpha \\ \mathbf{0} & \hat{A}_\alpha \end{bmatrix} \begin{pmatrix} x_k \\ \epsilon_k \end{pmatrix} \quad (17)$$

is asymptotically stable. $\epsilon_k = x_k - \hat{x}_k$ denotes the observation error and

$$\begin{aligned} \tilde{A}_\alpha &= A_\alpha + B_\alpha K_\alpha \\ \hat{A}_\alpha &= A_\alpha - L_\alpha C_\alpha \\ \tilde{B}_\alpha &= -B_\alpha K_\alpha \end{aligned} \quad (18)$$

Such a switched control law is more realistic than the classical switched state feedback which requires the availability of all the state vector components.

3.1 A separate design of the switched control and the switched observer

The switched state feedback design reduces to the computation of

$$u_k = K_\alpha x_k, \quad (19)$$

ensuring stability of the corresponding closed loop switched system

$$x_{k+1} = \underbrace{(A_\alpha + B_\alpha K_\alpha)}_{\tilde{A}_\alpha} x_k \quad (20)$$

under arbitrary switching signal. A solution has been proposed to this problem in Theorem 1.

The design of a switched observer :

$$\begin{aligned} \hat{x}_{k+1} &= A_\alpha \hat{x}_k + B_\alpha u_k + L_\alpha(y_k - \hat{y}_k) \\ \hat{y}_k &= C_\alpha \hat{x} \end{aligned} \quad (21)$$

consists in computing the gain matrices L_i, $i \in \mathcal{E}$ such that the observation error between the state x_k of the switched system (14) and the state $\hat{x}_k$ of the observer (21) is asymptotically stable. The convergence to the origin of the observation error has to be independent of the initial conditions x_0 and $\hat{x}_0$, the input u_k and the switching signal α.

Define the observation error by $\epsilon_k = x_k - \hat{x}_k$, the error dynamic is given by:

$$\epsilon_{k+1} = \underbrace{(A_\alpha - L_\alpha C_\alpha)}_{\hat{A}_\alpha} \epsilon_k. \tag{22}$$

The following Theorem, gives sufficient conditions to build such a switched observer.

Theorem 4 *If there exist symmetric matrices S_i, matrices F_i and G_i solutions of:*

$$\begin{bmatrix} G_i + G_i^T - S_i & G_i^T A_i - F_i^T C_i \\ A_i^T G_i - C_i^T F_i & S_j \end{bmatrix} > 0, \quad \forall (i,j) \in \mathcal{E} \times \mathcal{E}, \tag{23}$$

then a switched observer (21) *exists and the resulting gains L_i are given by*

$$L_i = G_i^{-T} F_i^T \qquad \forall i \in \mathcal{E}$$

Proof: See theorem 4 in [11]. $\square$

3.2 A separation principle for discrete-time switched systems

In this section, we show that the switched output feedback control obtained by combining the switched state feedback and the switched observer computed independently in the previous section guarantees that the closed loop switched system (3) is asymptotically stable.

Theorem 5 *[12] Assume that the matrix gains K_i and L_i, $\forall i \in \mathcal{E}$, have been computed as indicated in Theorems 1 and 4. Then the observer-based switched control (15)-(16) stabilizes asymptotically the closed switched system (17).*

Proof : Assume that the matrix gains K_i and L_i, $\forall i \in \mathcal{E}$, have been computed as indicated in Theorems 1 and 4. The closed loop system resulting from the combination of the switched state feedback and the switched observer is given by (17). As the error ϵ_k is asymptotically stable, it remains to show that

$$x_{k+1} = \tilde{A}_\alpha x_k + \tilde{B}_\alpha \epsilon_k \tag{24}$$

is also asymptotically stable which is done in the Appendix. $\square$

It is well known that the separation principle does not hold in general and one has to check carefully the validity of this principle in other cases than the classical linear time invariant case. Hence, it is not obvious that the switched system (24) is asymptotically stable even if the error ϵ_k converges asymptotically to 0. Theorem 5 gives a rigorous justification for a separate design of the switched control and the switched observer.

3.3 output feedback stabilization with pole placement

The separation principle stated previously can also be used to design observer based switched control with pole placement constraints. First, we reformulate the observer design procedure to include pole placement constraints. The following Theorem allows to design a switched observer such that

- the error dynamic spectra satisfies

$$spec(A_i - L_iC_i) = \{\tilde{\lambda}_{i1}, \ldots, \tilde{\lambda}_{in}\} \tag{25}$$

  where the $\tilde{\lambda}_{iq}$, $q = 1, ..., n$, denote desired eigenvalues

- the error dynamic is unconditionally globally asymptotically stable.

Theorem 6 *Assume that M_i and N_i have been computed using the procedure of section 2.1 with A_i and B_i replaced by A_i^T and C_i^T respectively. If there exist symmetric matrices S_i and matrices Ψ_i solutions of:*

$$\begin{bmatrix} M_i\Psi_i + \Psi_i^T M_i^T - S_i & \Psi_i^T M_i^T A_i - \Psi_i^T N_i^T C_i \\ A_i^T M_i \Psi_i - C_i^T N_i \Psi_i & S_j \end{bmatrix} > 0, \quad \forall (i,j) \in \mathcal{E} \times \mathcal{E}, \tag{26}$$

then a switched observer (21) *exists and the resulting gains L_i are given by*

$$L_i = (M_i\Psi_i)^{-T}(N_i\Psi_i)^T \qquad \forall i \in \mathcal{E}$$

Moreover, the dynamical matrices involved in the observation error are characterized by $spec(A_i - L_iC_i) = \{\tilde{\lambda}_{i1}, \ldots, \tilde{\lambda}_{in}\}$.

Proof : The proof can be immediately deduced from the proof of Theorem 2.

Now an observer based switched control with pole placement constraints can be designed using the previous separation principle. This is stated in the following Theorem.

Theorem 7 *Assume that the matrix gains K_i and L_i, $\forall i \in \mathcal{E}$, have been computed as indicated in Theorems 2 and 6. Then the observer-based switched control (15)-(16) stabilizes globally asymptotically the closed switched system (17). Moreover, for each subsystem, the eigenvalues of the closed loop dynamical matrix are $\{\lambda_{i1}, \ldots, \lambda_{in}, \tilde{\lambda}_{i1}, \ldots, \tilde{\lambda}_{in}\}$.*

Proof : for the stability part, the proof is the same as the proof of Theorem 5. For pole placement, one can notice that the dynamical matrices of the closed loop switched system (17) are triangular. Hence, the eigenvalues of each closed loop subsystem are the eigenvalues of the closed loop state feedback matrix and the eigenvalues of the error dynamic matrix. □

This Theorem can be used only for multi-input multi-output systems. From the discussion provided before Theorem 3 in the previous section, one can easily reformulate this result for the following cases

- single-input/single-output systems : asymptotic stability of the closed switched system (17) with partial pole placement contraints in the control and the observer design.
- single-input/multi-output systems : asymptotic stability of the closed switched system (17) with partial pole placement contraints in the control design and complete pole placement in the observer design.
- multi-input/single-output systems : asymptotic stability of the closed switched system (17) with complete pole placement contraints in the control design and partial pole placement contraints in the observer design.

4 ILLUSTRATIVE EXAMPLES

4.1 Example 1

This example illustrates the steps to follow when one is faced with the implementation of the procedure proposed in section 2.

Consider a switched system given by [11]

$$x_{k+1} = A_\alpha x_k + B_\alpha u_k \tag{27}$$

where $\{A_i : i \in \mathcal{E}\}$ and $\{B_i : i \in \mathcal{E}\}$ are a family of matrices parameterized by an index set $\mathcal{E} = \{1, 2\}$ and

$$A_1 = \begin{bmatrix} 0.0094 & 0.3010 \\ -3.0098 & 0.0094 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.0094 & 3.0098 \\ -0.3010 & 0.0094 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In the autonomous case ($u_k = 0$), the discrete time invariant subsystems characterized by A_1 and A_2 are stable and the matrices A_1 and A_2 have the same eigenvalues: $0.0094 \pm 0.9518i$. However, if the switching signal is characterized by

$$\alpha = \begin{cases} 1 & \text{if } x_k^1 x_k^2 \geq 0, \quad \text{with} \quad x_k = [x_k^1 \;\; x_k^2]' \\ 2 & \text{otherwise} \end{cases} \tag{28}$$

then the result is unstable. This can be easily checked by depicting the trajectories of such a switched system (Figure 1).

One may be interested in designing a switched state feedback that ensures stability of the closed loop system under arbitrary switching signals. In addition, one may be interested in specifying some constraints on the dynamic of each closed loop subsystem. This example is single input and only a partial pole assignment may be achieved. A stabilizing switched state feedback control with the property that each closed loop subsystem has $\lambda = 0$ as one of its eigenvalues can be computed using the procedure proposed in this paper.

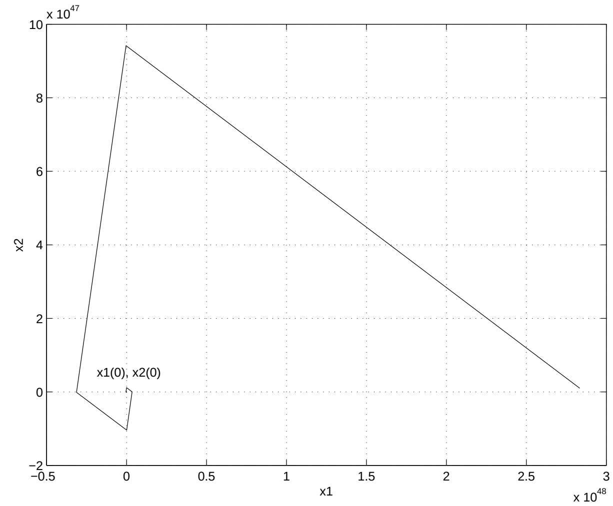

Figure 1: Trajectory of the open loop switched system

Let $i = 1$, the first subsystem, compute for $q = 1, ..., l$ with $l < n$ the null-space of $[A_i - \lambda_{iq}\mathbf{I} \quad B_i]$. Here $n = 2$ and $l = 1$ since only one eigenvalue may be fixed for each subsystem. The result is :

$$\begin{bmatrix} 0.0030 \\ 0.9576 \\ -0.2883 \end{bmatrix}$$

which means that

$$M_{11} = \begin{bmatrix} 0.0030 \\ 0.9576 \end{bmatrix}, \qquad N_{11} = -0.2883$$

Build the matrices

$$M_i \triangleq \left[M_{i1}, \ldots, M_{il}, \underbrace{\mathbf{I}, \mathbf{I}, \ldots, \mathbf{I}}_{n-l \quad times}\right],$$

$$N_i \triangleq \left[N_{i1}, \ldots, N_{il}, \underbrace{\mathbf{I}, \mathbf{I}, \ldots, \mathbf{I}}_{n-l \quad times}\right]$$

that is

$$M_1 = \begin{bmatrix} M_{11} & \mathbf{I}_{2\times 2} \end{bmatrix} = \begin{bmatrix} 0.0030 & 1 & 0 \\ 0.9576 & 0 & 1 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} N_{11} & \mathbf{I}_{2\times 2} \end{bmatrix} = \begin{bmatrix} -0.2883 & 1 \end{bmatrix}$$

One has to do similar computations for $i = 2$ the second subsystem and obtain

$$M_2 = \begin{bmatrix} 0.0098 & 1 & 0 \\ 0.3153 & 0 & 1 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} -0.9490 & 1 \end{bmatrix}$$

The next step is to check the feasibility of the Linear Matrix Inequalities of Theorem 3 where the unknowns are the Lyapunov matrices S_i and the matrices

$$\Psi_{M_i} = \begin{bmatrix} v_{i1} & 0 \\ \mathbf{0}_{2\times 1} & s_{i1} \end{bmatrix}, \qquad \Psi_{N_i} = \begin{bmatrix} v_{i1} & 0 \\ 0 & t_{i1} \end{bmatrix}$$

$$\text{with} \quad v_{11} \in \mathbb{R}, \quad s_{11} \in \mathbb{R}^2 \quad \text{and} \quad t_{i1} \in \mathbb{R}$$

for $i = 1$ and $i = 2$; Such LMIs are found feasible and

$$S_1 = 1.0e-03 \begin{bmatrix} 0.0100 & 0.0005 \\ 0.0005 & 0.1912 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 0.0026 & 0.0462 \\ 0.0462 & 1.4822 \end{bmatrix}$$

$$\Psi_{M_1} = \begin{bmatrix} 0.0066 & 0 \\ 0 & -0.0001 \\ 0 & 2.0158 \end{bmatrix}, \qquad \Psi_{N_1} = \begin{bmatrix} 0.0066 & 0 \\ 0 & -0.6067 \end{bmatrix}$$

$$\Psi_{M_2} = \begin{bmatrix} 0.2459 & 0 \\ 0 & 0.0230 \\ 0 & 2.1505 \end{bmatrix}, \qquad \Psi_{N_1} = \begin{bmatrix} 0.2459 & 0 \\ 0 & -6.4726 \end{bmatrix}$$

This leads to the following control gains

$$K_1 = (N_1\Psi_{N_1})(M_1\Psi_{M_1})^{-1} = \begin{bmatrix} -0.01786 & -0.30097 \end{bmatrix}$$

$$K_2 = (N_2\Psi_{N_2})(M_2\Psi_{M_2})^{-1} = \begin{bmatrix} -0.0102 & -3.0098 \end{bmatrix}$$

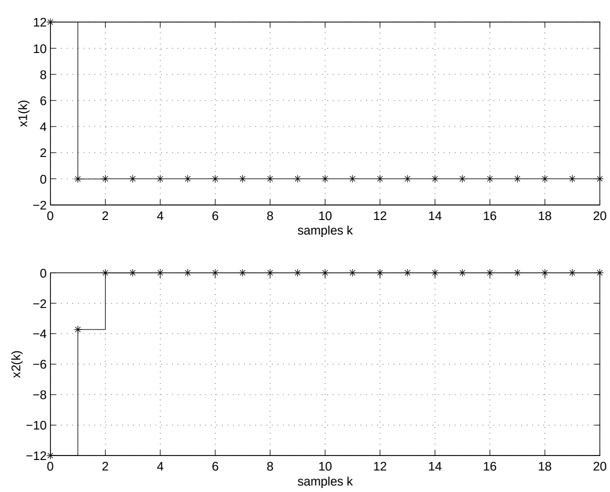

Figure 2: The closed loop behavior

Using the switched control characterized by these values for K_1 and K_2 the closed loop switched system is globally asymptotically stable and the dynamic of each subsystem in the closed loop configuration is characterized by $spec(A_1 + B_1K_1) = \{1.7e-13, \;\; 0.93e-3\}$ and $spec(A_2 + B_2K_2) = \{-7e-14, \;\; 0.0085\}$. The common specified eigenvalue is achieved within the computer accuracy. The closed loop behavior is depicted in Figure 2 with α the switching signal given by (28).

4.2 Example 2

Consider the switched system used in [11] and given by (14) where $\{A_i : i \in \mathcal{E}\}$, $\{B_i : i \in \mathcal{E}\}$ and $\{C_i : i \in \mathcal{E}\}$ are a family of matrices parameterized by an index set $\mathcal{E} = \{1, 2\}$ and

$$A_i = \begin{bmatrix} 0 & 0.89 & 0.5 \\ h_i & 0.89 & 0 \\ -0.1 & 0 & 0.9 \end{bmatrix}$$

with $i = 1, 2$ and $h_1 = -a\lambda_1$, $h_2 = \lambda_2$

$$B_1 = [0\ 0\ 1]^T, \qquad B_2 = [0\ \ -6(\lambda_1 + \lambda_2)\ 0]^T$$

$$C_1 = [-1\ \ 1\ \ -2] \quad \text{and} \quad C_2 = [-2\ \ 0.35\ \ 1]$$

If the switching signal is characterized by

$$\alpha = \begin{cases} 1 & \text{if } x_k^1 < 6, \quad \text{with} \quad x_k = [x_k^1\ \ x_k^2\ \ x_k^3]^T \\ 2 & \text{otherwise} \end{cases} \tag{29}$$

and the parameters $\lambda_1 = 1.12$, $\lambda_2 = 2$ then the open loop switched system may exhibit a chaotic motion as shown in figure 3 by using an open loop control $u_k = 0$ if $x_k^1 < 6$ and $u_k = 1$ otherwise.

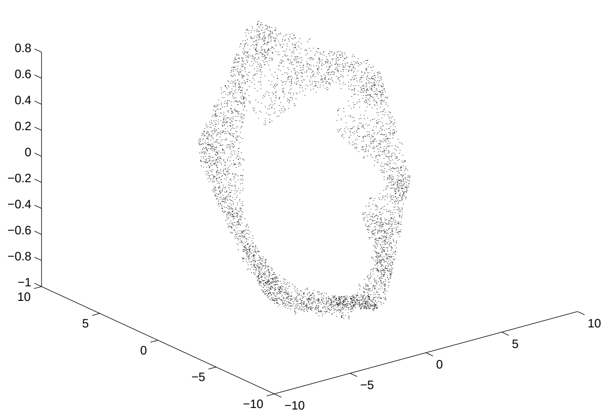

Figure 3: Chaotic attractor generated by the switched system in the three dimensional state space

First, we design a switched state feedback using Theorem 1. The corresponding LMIs are found to be feasible and the obtained matrix gains are :

$$K_1 = \begin{bmatrix} 1.3724 & -0.7652 & -0.7618 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 0.1021 & 0.0360 & 0.0358 \end{bmatrix}$$

We design separately a switched observer using Theorem 4. The corresponding LMIs are found to be feasible and the obtained matrix gains are :

$$L_1 = \begin{bmatrix} -0.3450 \\ 0.5650 \\ -0.1756 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.3033 \\ -1.3599 \\ -0.0463 \end{bmatrix}$$

The observer-based switched control obtained by combining the previous results leads to a closed behavior corresponding to the one depicted in figure 4.

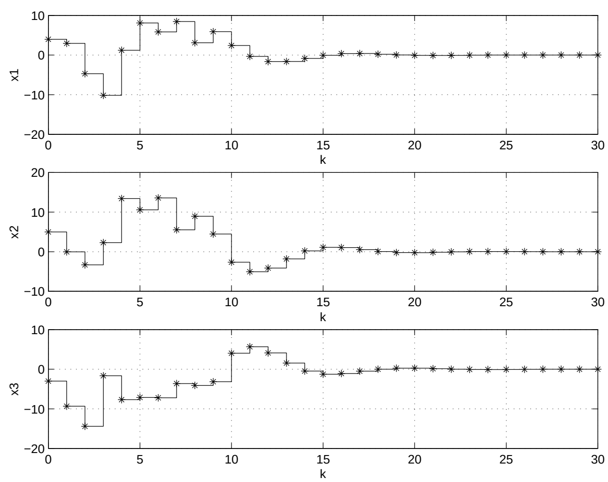

Figure 4: The state components $x_k = [x_k^1 \quad x_k^2 \quad x_k^3]^T$ of the closed loop switched system

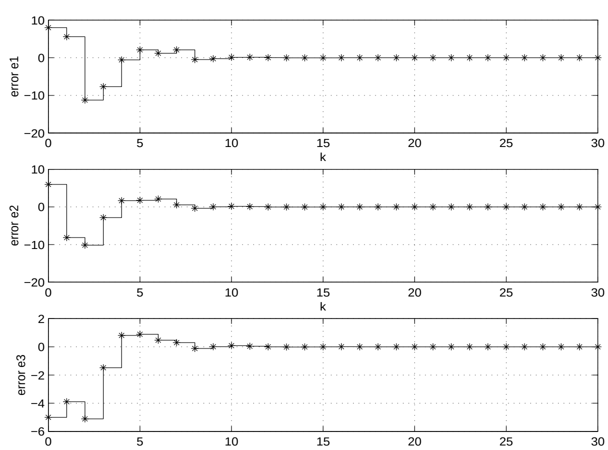

Figure 5: The error behavior $\epsilon_k = x_k - \hat{x}_k$

The error behavior is shown in figure 5. The simulation is performed for the same switching rule as in (29). The designed observer-based control guarantees that the closed loop system in unconditionally stable that is under arbitrary switching signal α. A simulation has been performed with $\alpha = 1$ for even samples k and $\alpha = 2$ for odd samples k and the results are depicted in figures 6 and 7.

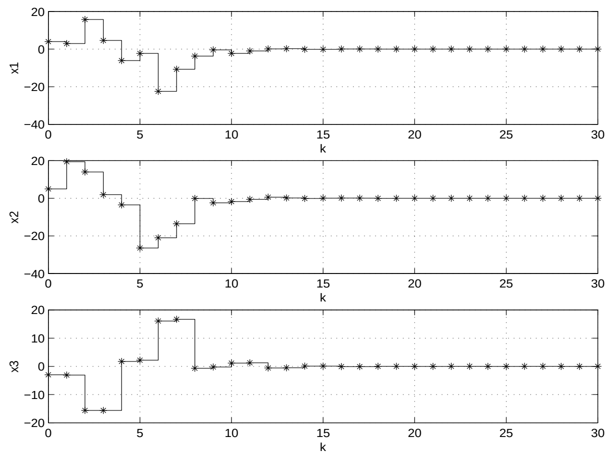

Figure 6: The state components $x_k = [x_k^1 \quad x_k^2 \quad x_k^3]^T$ of the closed loop switched system

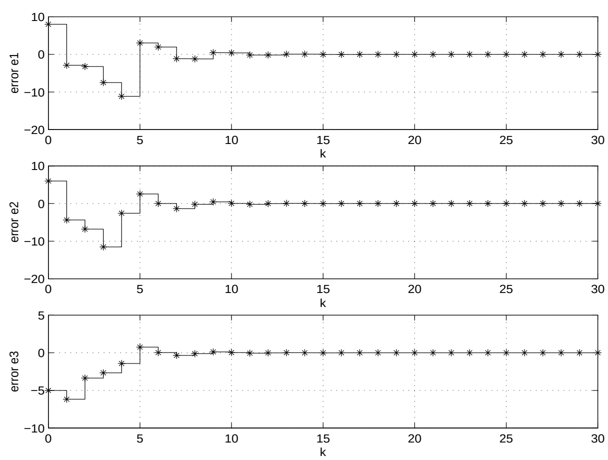

Figure 7: The error behavior $\epsilon_k = x_k - \hat{x}_k$

5 CONCLUSION

Designing a switched state feedback control which allows to place the poles of each subsystem in desired locations and which leads to an unconditionally stable switched system is proposed. To get this control law, one has to check the feasibility of a number of LMI conditions. This reduces to solve a convex optimization problem for which many tools are available (LMI toolbox by MATALB, Sedumi solver...). The case of switched output feedback control is also addressed. A separation principle for linear discrete-time switched systems is proved. It allows to perform the control and the observer designs independently. Finally, these results are also applicable to the classical discrete time varying case. This case corresponds to a fixed switching sequence and can be solved by checking only the LMIs corresponding to the allowable sequence.

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APPENDIX A

Before stating the proof of Theorem 5, we notice that computing a switched state feedback computed as indicated in Theorem 1 ensures that the closed loop switched system (20) is asymptotically stable under arbitrary switching signal α and this is equivalent to

$$\begin{aligned} &\forall x_0 \in \mathbb{R}^n, \\ &\forall s \in \{(i_0, i_1, i_2, \ldots, i_k, \ldots) : \forall k \geq 0,\ i_k \in \{1, 2, ..., N\}\}, \end{aligned}$$

$$\lim_{p \to \infty} \prod_{k=0}^{p} \tilde{A}_{i_k} x_0 = 0$$

$\Leftrightarrow$

$$\begin{aligned} &\forall R > 0, \forall x_0 \in B(0, R), \exists n_{\tilde{A}}, \forall p \geq n_{\tilde{A}}, \\ &\forall s_p \in \{(i_0, i_1, i_2, \ldots, i_p) : \\ &\forall k = 0, 1, ..., p, i_k \in \{1, 2, ..., N\}\} \end{aligned}$$

$$x_p = \prod_{k=0}^{p} \tilde{A}_{i_k} x_0 \in B(0, R/2)$$

$\Leftrightarrow$

$$\begin{aligned} &\forall \mu > 1, \exists n_{\tilde{A}}(\mu), \forall p \geq n_{\tilde{A}}, \\ &\forall s_p \in \{(i_0, i_1, i_2, \ldots, i_p) : \forall k = 0, 1, ..., p, \\ &\ i_k \in \{1, 2, ..., N\}\}, \end{aligned}$$

$$\left\| \prod_{k=0}^{p} \tilde{A}_{i_k} \right\| < \frac{1}{\mu}$$

This states the fact that stability under arbitrary switching rule can be rewritten to exhibit contraction properties. In other words, after a certain number of iterations a contraction of the state trajectory is achieved whatever the switching sequence be. Moreover, a switched observer computed as indicated in Theorem 4 ensures that the error dynamic (22) is asymptotically stable under arbitrary switching signal α. Hence, the previous contraction properties are verified with $n_{\tilde{A}}$ and $\tilde{A}_{i_k}$ replaced by $n_{\hat{A}_{i_k}}$ and $\hat{A}$ respectively.

Proof of Theorem 5 : Assume that the matrix gains K_i and L_i, $\forall i \in \mathcal{E}$, have been computed as indicated in Theorems 1 and 4. The closed loop system resulting from the combination of the switched state feedback and the switched observer is given by (17). As the error ϵ_k is asymptotically stable, it remains to show that

$$x_{k+1} = \tilde{A}_\alpha x_k + \tilde{B}_\alpha \epsilon_k \tag{30}$$

is also asymptotically stable. The later equation writes :

$$
\begin{aligned}
x_1 &= \tilde{A}_{i_0} x_0 + \tilde{B}_{i_0} e_0 \\
x_2 &= \tilde{A}_{i_1} \tilde{A}_{i_0} x_0 + \tilde{A}_{i_1} \tilde{B}_{i_0} e_0 + \tilde{B}_{i_1} \hat{A}_{i_0} e_0 \\
&\dots \\
&\dots \\
x_{p+1} &= \prod_{k=0}^{p} \tilde{A}_{i_k} x_0 + \left[\prod_{k=1}^{p} \tilde{A}_{i_k} \tilde{B}_{i_0} + \prod_{k=2}^{p} \tilde{A}_{i_k} \tilde{B}_{i1} \hat{A}_{i_0} \right. \\
&+ \dots + \prod_{k=j}^{p} \tilde{A}_{i_k} \tilde{B}_{i_{j-1}} \prod_{k=0}^{j-2} \hat{A}_{i_k} + \dots \\
&\left. + \tilde{A}_{i_p} \tilde{B}_{i_{p-1}} \prod_{k=0}^{p-2} \hat{A}_{i_k} + \tilde{B}_{i_p} \prod_{k=0}^{p-1} \hat{A}_{i_k} \right] e_0
\end{aligned}
\tag{31}
$$

According to contraction properties stated before, let $\mu = 2$ and

$$m = \max\left(n_{\tilde{A}}, n_{\hat{A}}\right)$$

we have,

$$\forall s_n \in \{(i_0, i_1, i_2, \dots, i_n) : \forall k = 0, 1, ..., n,\ i_k \in \{1, 2, ..., N\}\},$$

$$\left\| \prod_{k=0}^{n} \tilde{A}_{i_k} \right\| < \left(\frac{1}{2}\right)^l \sigma^r$$

where $n = lm + r$ with $r < m$ and $\sigma = \max_{i \in \{1,2,\dots,N\}} \left\| \tilde{A}_i \right\|$

Hence,

$$\left\| \prod_{k=j}^{pm} \tilde{A}_{i_k} \tilde{B}_{i_{j-1}} \prod_{k=0}^{j-2} \hat{A}_{i_k} \right\| \le \left\| \prod_{k=j}^{pm} \tilde{A}_{i_k} \right\| \left\| \tilde{B}_{i_{j-1}} \right\| \left\| \prod_{k=0}^{j-2} \hat{A}_{i_k} \right\| \tag{32}$$

$$\left\| \prod_{k=j}^{pm} \tilde{A}_{i_k} \tilde{B}_{i_{j-1}} \prod_{k=0}^{j-2} \hat{A}_{i_k} \right\| \le \hat{\sigma} \beta \hat{\gamma} \frac{1}{2^{p-1}} \tag{33}$$

with

$$\sigma = \max_{i \in \{1,2,\dots,N\}} \left\| \tilde{A}_i \right\|, \quad \beta = \max_{i \in \{1,2,\dots,N\}} \left\| \tilde{B}_i \right\|$$

and

$$\gamma = \max_{i \in \{1,2,\dots,N\}} \left\| \hat{A}_i \right\|, \qquad \hat{\sigma} = \max(\sigma, \sigma^m), \qquad \hat{\gamma} = \max(\gamma, \gamma^m)$$

As :

$$\begin{aligned}
\|x_{pm+1}\| \quad \leq \quad & \left\| \prod_{k=0}^{pm} \tilde{A}_{i_k} x_0 \right\| + \left[\left\| \prod_{k=1}^{pm} \tilde{A}_{i_k} \tilde{B}_{i_0} \right\| \right. \\
& + \left\| \prod_{k=2}^{pm} \tilde{A}_{i_k} \tilde{B}_{i1} \hat{A}_{i_0} \right\| + \cdots \\
& + \left\| \prod_{k=j}^{pm} \tilde{A}_{i_k} \tilde{B}_{i_{j-1}} \prod_{k=0}^{j-2} \hat{A}_{i_k} \right\| + \cdots \\
& + \left\| \tilde{A}_{i_{pm}} \tilde{B}_{i_{pm-1}} \prod_{k=0}^{pm-2} \hat{A}_{i_k} \right\| \\
& + \left. \left\| \tilde{B}_{i_{pm}} \prod_{k=0}^{pm-1} \hat{A}_{i_k} \right\| \right] \|e_0\|
\end{aligned}$$

we have :

$$\|x_{pm+1}\| \leq \left\| \prod_{k=0}^{pm} \tilde{A}_{i_k} x_0 \right\| + \hat{\sigma} \beta \hat{\gamma} \sum_{k=1}^{pm} \frac{1}{2^{p-1}} \|e_0\| \tag{34}$$

$$\|x_{pm+1}\| \leq \left(\frac{1}{2} \right)^p \|x_0\| + \hat{\sigma} \beta \hat{\gamma} \frac{pm}{2^{p-1}} \|e_0\| \tag{35}$$

$$\|x_{pm+1}\| \to 0 \text{ if } p \to \infty \qquad \square$$