Practical optimal state feedback control law for continuous-time switched affine systems with cyclic steady state

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Abstract

In this article, a method for computing an optimal state feedback control law for continuous-time switched affine systems exhibiting cyclic behavior in steady state is presented. The hybrid solutions are deduced from the Fillipov solutions. It is shown that the optimal trajectory synthesis implies to determine singular arcs. Algebraic conditions are given to obtain these particular arcs of the trajectory. A numerical procedure is then proposed to generate optimal trajectories on a given state space area avoiding the classical two-point boundary value problem occurring in optimal control synthesis. The interpolation of the solutions set, through a neural network, yields a state feedback control law. Several examples in the power converters field show the feasibility and the efficiency of the method.

1 Introduction

Hybrid systems are dynamical systems characterized by interactions between a continuous and a discrete dynamics. The meaning of the term "dynamical systems" was given by Kalman et al. in [Kalman et al.(1969)]

A particular and important class in terms of hybrid system applications is depicted by systems with a cyclic behavior in steady state. From the designer point of view, one of the most important control target is the average value of the state instead of its instantaneous value. Therefore, a required performance of the close-loop system is a state balance around a desired average value. For example, combustion engines belong to this category . Another application which has received a big theoretical contribution is the power converters class. Indeed, power supplies are currently embedded in computers, electric drives, cellular phones and generally in all electric devices. Their aim is to convert an electrical energy shape (voltage / current / frequency) to another one. This modulation process is carried out with semiconductor-based electronic switches which make them belong to a hybrid system class. This explains why most of the references in this field are related to power converters.

Usually, the analysis and the control design in these types of systems rely either on average models or on small signal approximations of the average model [Middlebrook and Wester(1973)], [Sanders et al.(1991)], [Iung et al.(1978)]. The averaging technique is convenient when the average is computed over a small time interval with respect to the system's dynamics. However, these are only low frequency approximations of the true dynamics. Moreover, the discontinuous effect of switchings is ignored and the resulting waveform may generate undesirable subharmonics or interharmonics of the cutting frequency [Möllerstedt and Bernhardsson(2000)]. On the other hand, the sampled linear methods use tangent approximations to the non-linear sampled model [Mahabir et al.(1990)], [Huliehel and Ben-Yaakov(1991)]. The recurrence is not easily obtained and it does not prevent high ripples between switching instants.

Nowadays, the constant components' integration into systems with a cycle as steady state leads to very specific performances in the transitory and also in stability. Oftentimes, the classical methods to control this kind of systems give insufficient results. Thus, control laws which address all performance requirements remain a challenging problem. To be able to consider the whole system dynamics, hybrid methods should be used. The idea is to operate the different system modes directly. In addition, the stability for the hybrid systems has been widely studied [Rantzer et al.(2000)], [Daafouz et al.(2002)], [Decarlo et al.(2000)], [Hespanha(2004)], as well as limit cycles [Flieller et al.(2006)], [Rubensson(2000)] [Goncalves(2004)].

Emerging approaches based on sliding modes [Perry et al.(2005)], [Richard et al.(2006)], [Sira-Ramirez et al.(2002)], [Ahmed et al.(2003)] optimal control [Bemporad et al.(2002)], [Mehta and Egerstedt(2006)], [Shaikh and Caines(2003)], [Joh(2007)] or predictive control [Béthoux and Barbot(2006)], [Silva et al.(2007)], [Beccuti et al.(2007)], [Donzel and Bornard(2000)], [Almer et al.(2007)] are promising.

The Model Predictive Control (MPC) [Geyer et al.(2005)], [Lazar and Keyser(2004)] is a well-known and useful tool for the switched systems. However, the receding horizon implies to solve an optimization program at each sampled time and the control signal may not always be available in real time. One of the solutions could be to interpolate off-line the optimal trajectories with a neural network [Baja et al.(2007)], [Patino et al.(2007)]. Other methods use a discrete time formulation with the Mixed Logical Dynamics (MLD) programming framework. To avoid time consumption, a MPC controller is designed in [Bemporad et al.(2000)], [Borelli et al.(2003)] and a look-up table is obtained off-line and applied on-line.

A switching table procedure is proposed in [Bemporad et al.(2002)], [Seatzu et al.(2006)]. This method is based on dynamic programming. Nevertheless, it presents computation problems due to a combinatory explosion. We can also mention passivity based control methods [Morvan et al.(2004)], [Sira-Ramirez and Ortega(1995)], [Zainea et al.(2007)] leading to control the system by sliding modes [Richard et al.(2006)], [Ahmed et al.(2003)]. Based on the use of the Lyapunov function, system stability is guaranteed. Flatness based control in hybrid systems also seems to be a promising approach because it prevents the energy loss in the switches [Payman et al.(2006)], [Sira-Ramírez and Silva-Ortigoza(2002)].

In our work, optimal techniques are investigated for high performance design in controlling affine switched systems depicting a limit cycle in steady state. To the authors' knowledge, the optimal control without approximation assumptions applied to this particular class of systems has not been sufficiently considered until recently.

Although the problem is tedious, it is possible to synthesize an optimal state feedback control law for low order systems. The goal of our article is to show how to compute this control law.

We will use the following notation and terminology. An arbitrary function f is C^1 when its derivative exists and is continuous. All vectors are column vectors. If x is a vector, x^T is its transpose. The abbreviation f_x is used for the partial derivative of an arbitrary function f with respect to x. If A is a matrix, $A > 0$ and $A \geq 0$ indicate respectively that A is positive definite and positive semidefinite. $\lambda \perp h$ denotes that a vector λ is orthogonal to another vector h. If U and V are subsets of E, the complementary subset V^c in E is such that $E = V^c \cup V$ and $V^c \cap V = \varnothing$. The subset U private V denoted $U \backslash V = U \cap V^c$. The convex hull of a set U is denoted by $co(U)$. The cardinality of a set U is denoted by $|U|$.

For two real numbers a and b, $[a\ b]$ is the closed interval while $\{a, b\}$ is the set composed by singletons $\{a\}$ and $\{b\}$.

Let us start by considering the following optimal control problem over a control-affine system:

$$\begin{aligned} &\min_{u(.)} \int_0^T L(z(t) - z_{ref})dt \qquad (1)\\ s.t.\ \dot{z}(t) &= r(z(t)) + s(z(t))u(t) \\ &= r(z(t)) + \sum_{i=1}^{m} u_i(t) s^i(z(t)) \\ z(0) &= z_0 \quad u(t) \in U = \{0,1\}^m \end{aligned}$$

where $z \in \mathbb{R}^\nu$ ($\nu \in \mathbb{N}$) is the state, and the functions $r, s^i \colon \mathbb{R}^\nu \to \mathbb{R}^\nu, i = 1, ..., m$ ($m \in \mathbb{N}$), and $L(.) : \mathbb{R}^\nu \to \mathbb{R}$ (L is the performance function). All these functions are supposed to be C^1. The term z_{ref} is a given equilibrium point in the sense defined later by equation (11), T is the final time ($T \in \mathbb{R}^+ \cup \{\infty\}$), $u(t)$ is an m-dimensional Boolean control vector, m is the number of admissible switches (number of switches that can be freely positioned). Notice that there exist 2^m modes or 2^m configurations.

The following assumption is considered:

Assumption 1 *There is no restriction related to the control set values U (Neither state nor time constraints). We assume that $u : [0, T] \to U$ is a Lebesgue measurable function.*

This assumption implies that only control switchings are taken into account and no autonomous switchings will be studied (threshold).

To solve the problem (1), necessary conditions given by the Minimum Principle (MP) can be used.

In this article, for this class of systems with a low order, we explain how to synthesize the appropriate control law among those satisfying the MP (extremal trajectories). In the case when the MP is not sufficient to compute the candidate control law, a singular arc appears. Precisely, the outcome presented here depends on the singular arcs. Algebraic conditions and second order necessary conditions will be shown. They are used to determine singular surfaces.

Once the potential singular surfaces are algebraically solved, a backward integration procedure from the equilibrium point and the associated singular arcs is used to numerically generate “optimal candidates”. The main advantage of this method is to avoid the classical two-point boundary value problem which occurs everytime the MP is used to find optimal trajectories.

Then, a neural network interpolates all the solutions on the whole working area. The result is a simple state feedback control law which can efficiently be implemented on-line.

The present article is organized as follows: Section 2 shows the problem statement and first order necessary conditions for the switched hybrid optimal control problem (1). The main contribution lays in Section 3. Standard results about singular control theory are firstly recalled. Then, an algebraic conditions set is deduced to obtain singular arcs (in Propositions 11, 12, 21 and 22). These conditions are easier to solve than the classical ones. A systematic procedure is proposed to get all singular arc candidates for low-order systems. Section 4 is devoted to the optimal trajectory synthesis using backward integration and a neural network interpolation. Indeed, we propose a methodology to obtain an optimal switching feedback control law from the open loop optimal control synthesis. In Section 5, two DC-DC converters are considered as examples: the step down and multilevel converters. Finally, we address some conclusions.

2 Problem formulation and necessary conditions

For clarity and simplicity, (1) is rewritten in a Mayer form. Adding a variable

$$x_n(t) = \int_0^t L(z(\tau) - z_{ref})d\tau$$

and defining

$$\begin{aligned} x(t) &= [z^T(t), x_n(t)]^T \\ x_0 &= [z_0^T, 0]^T \\ f(x(t)) &= [r^T(z(t)), L(z(t) - z_{ref})]^T \\ g(x(t)) &= [s^T(z(t)), 0]^T \end{aligned}$$

we get the equivalent problem (2):

$$\min_{u(.)} x_n(T) \tag{2}$$
$$s.t.\ \dot{x}(t) = f(x(t)) + g(x(t))u(t)$$
$$x(0) = x_0 \quad u(t) \in U = \{0,1\}^m$$

where $x \in \mathbb{R}^n$ ($n = \nu + 1$).

To simplify, time dependency of $x(t)$, $\lambda(t)$ and $u(t)$ is not mentioned. $f(x)$ is the drift term and $g(x)$ the control affine term.

The Hamiltonian H can be deduced from (2) :

$$\begin{aligned} H &= H(\lambda, x, u) \\ &= \lambda^T f(x) + \lambda^T g(x) u \end{aligned} \tag{3}$$

where $\lambda(t) \in \mathbb{R}^n$ is the adjoint variable. The dynamics of λ and x are given by:

$$\dot{\lambda} = -H_x \qquad \text{and} \qquad \dot{x} = H_\lambda \tag{4}$$

The MP to control the affine switched system (2) follows.

Theorem 2 *(Minimum principle for switched systems, [Pontryaguin et al.(1964)]) Let (x^*, u^*) solve (2). There exists an absolutely continuous function $\lambda^* : [0,T] \rightarrow \mathbb{R}^n$ non identically equivalent to 0, which verifies (4) almost everywhere, such that the following conditions are satisfied:*

1. *The control u^* minimizes the Hamiltonian function:*

$$H^* = H(\lambda^*, x^*, u^*) = \min_{u \in U} H(\lambda^*, x^*, u) \ a.e. \tag{5}$$

2. *For all $t \in [0,T]$*

$$H^*(t) = cst \tag{6}$$

where cst is a constant and $cst = 0$ if T is not specified.

3. *Initial and final conditions:*

$$\lambda^*(0) \ free \qquad and \qquad \lambda^*(T) = [0, ..., 0, 1]^T \tag{7}$$

This is a direct application of the Pontryagin Minimum Principle. The abnormal case when $\lambda_n^*(T) = 0$ is not considered here.

Remark 3 *The optimal switching law is one among those minimizing the Hamiltonian (5). This necessary condition does not hold when Assumption 1 is not true. When the available discrete control set U has a (discrete or continuous) state dependency, minimum condition (5) with respect to the discrete control must be reduced to the set of its connex components. Two control values are connex if no constraint concerning the switches between them exists. Therefore, in order to determine the extremals set, it is necessary to proceed with a dynamic programming procedure through the connex graph. Moreover, at switching time λ and H are discontinuous. This is one of the difficulties encountered in optimal hybrid control problems. For a complete explanation concerning the general case with a continuous control u and state jumps, we invite the reader to see [Riedinger et al.(2003)] or [Dmitruk and Kaganovich(2008)].*

Necessary conditions from Theorem 2 assume that a solution exists. This assumption is closely related to bang-bang solution existence for the relaxed problem as it has been shown in [Riedinger et al.(2003)] . The relaxed problem consists in extending the control set values U to its convex hull:

$$co(U) = [0\ 1]^m \tag{8}$$

Then, if the relaxed problem takes the form of a convex optimal control problem, minimizers' existence can be certified. Classical theorems state sufficient conditions for an optimal solution's existence under convexity assumptions [Berkowitz(1974)], [Young(1969)], [Roubicek(1997)], [Meziat et al.(2007)].

Hence, when the relaxed problem has a bang-bang solution (i.e. $\forall t$, $u(t) \in \{0,1\}^m$), the original problem (2) is also solved. Otherwise, it means that the optimal solution takes values $u(t) \in co(U)\backslash U$, $t \in \mathcal{T}$ where $\mathcal{T}$ is a non zero measure set of time (in the Lebesgue sense). Although this solution does not satisfy (2), one can obtain an average approximation by switchings between the different system modes. This leads to a sliding motion on the optimal trajectory [Utkin(1992)]. Proof of this statement can be found using relaxation theory and density results [Ingalls et al.(2003)]. Thus, studying problem (2) with its control set extended to its convex hull (8) is more convenient than studying the original problem with $u(t) \in U$. In other words, we try to determine the Fillipov's solution of (2) [Cortes(2008)].

Let us now consider the necessary conditions. Since the Hamiltonian function is affine with respect to the control, a switching function $\phi(t)$ can be defined by:

$$\phi(t) = \lambda^T(t) g(x(t))$$

and from minimum condition (5), the control is determined by the following the rule,

$$\text{For } i = 1, ..., m \qquad u_i(t) = \begin{cases} 0 \text{ if } \phi_i(t) > 0 \\ 1 \text{ if } \phi_i(t) < 0 \\ ? \text{ if } \phi_i(t) = 0 \end{cases}$$

When $u_i(t) = 0$ or 1, the control is called regular.

A difficulty arises when a component of $\phi(t)$ vanishes identically on a time interval $[a, b]$, $b > a$. In this case, the MP is inconclusive concerning the control value $u(.)$ on $[a, b]$. This situation is referred to singular control [Robbins(1967)] and it corresponds here to the case where $u(t)$ takes values in $co(U)\backslash U$, $t \in [a, b]$. We can conclude that solution segments for the relaxed problem, which are not admissible for the original problem, involve a singular control. This case will be defined and studied in detail in the next section.

Another reason to find optimal control values out of U comes from the nature of the operating points x_{ref}. Indeed, for systems exhibiting a cyclic behavior in steady state (e.g. torque in a combustion engine), the desired operating point is the average value of x over the cycle. To find the operating points set, the model dynamics of the average state $\overline{x}$ can be employed. $\overline{x}$ is given by the convolution product:

$$\overline{x}(t) = \sqcap_{T_p} * x(t) = \frac{1}{T_p} \int_{t-T_p}^{t} x(\tau) d\tau \tag{9}$$

where T_p is the cycle period and $\sqcap_{T_p}$ is a rectangular window function.

The dynamical model of $\overline{x}$ is obtained differentiating (9). However, the derivative is generally intractable or unusable because of its nonlinear form.

A solution consists in defining the average state model:

$$\dot{\tilde{x}} = f(\tilde{x}) + g(\tilde{x})\overline{u},\ \overline{u} \in co(U) \tag{10}$$

which gives an approximation of the dynamics of $\overline{x}$. $\overline{u}$ is the average value of u on the cycle. It has been proven that $\widetilde{x}$ is close to $\overline{x}$ and x when T_p is small with respect to the system dynamics [Sanders and Verhulst(1985)] ($\widetilde{x} \longrightarrow \overline{x}$ and x when $T_p \longrightarrow 0$).

The operating points are then defined as the equilibrium points of the average state model:

$$X_{ref} = \{x_{ref} \in \mathbb{R}^n : f(x_{ref}) + g(x_{ref})u_{ref} = 0, u_{ref} \in co(U)\} \tag{11}$$

Therefore, there exist equilibrium points x_{ref} whose control is $u_{ref} \in co(U)\backslash U$. Finally, it can be noticed that the model used in the relaxed problem is exactly the average state model.

Since no mode allows holding the given reference x_{ref} as $u_{ref} \in co(U)\backslash U$, the switched system enters into a cyclic behavior around the reference x_{ref}. Consequently, u_{ref} is nearly the average value of u over the cycle. Pulse Width Modulation (PWM) and sliding mode are practical methods to obtain the result in average.

3 Singular trajectories

This section describes a particular case for solving the problem (2) using necessary conditions from Theorem 2. Some literature concerning this difficult case can be found in [Robbins(1967)], [Krener(1977)], [Powers(1980)], [Michel(1996)], [Bonnard et al.(2003)], [Chitour et al.(2006)] and references therein.

The aim of this section is to summarize the standard results and to show how they can be used to compute the singular arcs. Some well known definitions and second order necessary conditions in singular control theory are recalled. This material is then used to deduce algebraic conditions set. This set involves only (x, u) (λ disappears) and is easier to solve. The result 11, 12, 15, 21, 22 and 24 have a practical and great interest for low-order systems, when the procedure ends with unique value for the control u as function of x.

To make the proposals clear, we will distinguish between the single control case and multiple controls case.

3.1 One-single control case $m = 1$

3.1.1 Standard results and second order necessary condition

Definition 4 *(Singular arc, [Krener(1977)], [Powers(1980)], [Michel(1996)]) If (x, λ, u) is an extremal for (2) such that*

$$\phi(t) = H_u \equiv 0$$

on a time interval $[a\ b] \subset T$, $a < b$, then (x, λ, u) is called a singular arc on time interval $[a\ b]$.

In this situation, the Hamiltonian function is independent of the control variable u and minimum condition (5) does not determine directly the control u. Nevertheless, additional conditions can be deduced with the total time derivatives of the switching function $\phi(t)$, *i.e.*,

$$\frac{d}{dt}\phi(t) = 0, \frac{d^2}{dt^2}\phi(t) = 0, ...$$

The following definition must be also given:

Definition 5 *(Problem order, [Robbins(1967)], [Michel(1996)]) The problem order is the smallest integer q such that u explicitly appears in the $(2q)$ derivative $\phi^{(2q)}$ where in each derivative $\dot{x}$ and $\dot{\lambda}$ are replaced by their expression as given by (4) ($\phi^{(k)} = \frac{d^k}{dt^k}\phi(t)$ is the total time derivative of order k).*

For an affine system, the control enters linearly in the $(2q)$ time derivative of ϕ and we get an expression of the form:

$$\phi^{(2q)} = A(x, \lambda) + B(x, \lambda)u. \tag{12}$$

Definition 6 *(Arc order, [Robbins(1967)], [Michel(1996)]) Let (x, λ, u) be a singular arc defined on $[a\ b] \subset T$ for (2). The arc order of (x, λ, u) is the smallest integer $p = \min\{0 \leq k < \infty : (\phi^{(2k)})_u = ((H_u)^{(2k)})_u,\ (x, \lambda, u) \neq 0$, for any $t \in [a\ b]\}$. When $k = \infty$, the arc order does not exist.*

As it has been mentioned in the above definition, the arc order not only depends on the extremal (x, λ, u) but also on the time interval $[a\ b]$. It may occur that a singular arc does not have the same order in different locations. This is why the following assumption is taken:

Assumption 7 *For every arc, there exists a neighborhood of trajectories where the arc order is constant.*

Exceptional cases where this assumption does not hold can be found in [Robbins(1967)].

It is clear that the arc order may be different from the problem order p if $B(x, \lambda) = 0$. Thus, a control problem may have different arc orders for the same problem order.

Definitions 4, 5 and 6 are consistent with the following necessary condition:

Proposition 8 *[Robbins(1967)], [Michel(1996)] The arc and problem order are obtained from an even number of derivative of ϕ.*

When the arc order exists on $[a, b]$, the following equality constraints along a given extremal (x, λ, u) can be deduced:

$$\phi^{(i)} = 0, i = 0, ..., 2p - 1 \tag{13}$$

$$\frac{\partial}{\partial u}\phi^{(i)} = 0, i = 0, ..., 2p - 1 \tag{14}$$

$$\phi^{(2p)} = A(x, \lambda) + B(x, \lambda)u = 0 \tag{15}$$

$$\text{with } B(x, \lambda) \neq 0 \text{ for all } t \in [a, b] \tag{16}$$

¿From the last equality, the control is determined by

$$u = -A(x, \lambda)/B(x, \lambda) \tag{17}$$

and higher order derivatives of ϕ give the relationship between the control and its derivatives because they must vanish identically in $[a\ b]$. No further relations are required.

¿From the solutions set (13), (14) and (15), a second order necessary condition is established. This condition excludes some of the non optimal candidates:

Theorem 9 *(Legendre-Clebsch Generalized Conditions, [Robbins(1967)], [Krener(1977)]). Let the problem (2) be given. If (x^*, λ^*, u^*) is an optimal arc, then the following properties hold*

a) If the problem order is q on $[a\ b]$ then

$$(-1)^q((H_u)^{(2q)})_u \geq 0$$

for all t on $[a\ b]$.

b) If the arc order is $p < +\infty$ on $[a\ b]$ then

$$(-1)^p((H_u)^{(2p)})_u > 0 \tag{18}$$

for all t on $[a\ b]$.

3.1.2 A practical method to determine singular arcs when $m = 1$

In general, nonlinear equations (13), (14) and (15) are not easy to solve. For the class of problems considered here, a solution can be found as we will show at the end of the subsection. It is based on algebraic conditions which are directly deduced from the material presented above and some additional assumptions related to the problem order and to the state space dimension.

The problem order can be determined by successive differentiation of the switching function until the explicit appearance of u. For an affine system and a problem order equal to q, the derivatives take the form:

$$\begin{aligned} \lambda^T ad_f^k g(x) = 0,\ k = 0, ..., 2q-1 \\ \lambda^T ad_f^{2q} g(x) + \lambda^T [g, ad_f^{2q-1} g](x) u = 0 \end{aligned} \tag{19}$$

where $[f, g](x) := g_x(x) f(x) - f_x(x) g(x)$ denotes the Lie bracket and $ad_f^k g(x) = [f(x), ad_f^{k-1} g(x)]$, $ad_f^0 g(x) = g(x)$ the iterated Lie bracket of order k.

The following proposition has been proven:

Proposition 10 *[Michel(1996), Fraser-Andrews(1989)] For optimal control problem (2) if $p = q$ then $2q \leq n$ and $ad_f^k g(x), k = 0, ..., 2q-1$, are linearly independent along the extremal (x, λ, u).*

Although this proposition is useful when $p = q$, there is no upper bound concerning the value of p. Then, the number of time derivatives which are necessary to determine the control is not *a priori* known.

¿From the order $2p$, it is observed that higher derivatives of ϕ i.e. $\phi^{(k)}$, $k > 2p$, involve successive derivatives of the control u. Therefore, these additional equations $\phi^{(k)}$, $k > 2p$, are not needed to find (x, λ). Note that time derivatives of u are justified by the analytic expression (17). The singular control dynamics of u is given by the differential equation $\phi^{(2p+1)} = 0$. Only the equations $\phi^{(k)} = 0$, $k = 0, ..., 2p$, are necessary to determine (x, λ, u) and the additional derivatives yield the terms $u^{(i)}$, $i = 1, 2, ...$

Usually, without more assumptions, the singular arc candidates cannot be explicitly determined. Let us assume that $p = q$ holds along the arc. From (19) and for a given x, we count $(n + 1)$ unknowns (λ and u) and also $2q + 1$ equations. From MP, λ is not trivially zero and when $2q + 1$ is equal to n, the following equation is obtained:

$$det([g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x), ad_f^{2q} g(x) + u[g, ad_f^{2q-1} g](x)]) = 0.$$

Using the independency property of $ad_f^k g(x), k = 0, ..., 2q-1$ along the singular extremal (proposition 10) and multi-linearity property of $det(.)$, the control is uniquely determined by:

$$u = -\frac{det([g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x), ad_f^{2q} g(x)])}{det([g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x), [g, ad_f^{2q-1} g](x)])}.$$

Unfortunately most of control problems have a problem order $q = 1$, then unique candidates w.r.t. x can only be guaranteed when n is limited to 3. Nevertheless, additional conditions can be taken into account if problems in which the Hamiltonian function vanishes are considered. This is the case for the most used in practice criteria which are time optimal control or quadratic criteria in infinite time. The next condition can be added:

$$\lambda^T f(x) = 0 \tag{20}$$

and the following proposition is deduced:

Proposition 11 *(Algebraic necessary conditions for singular arcs) Assume $p = q$, $n = 2(q+1)$ and $H = 0$ (required by the the MP). If $f(x)$ is linearly independent of $ad_f^k g(x), k = 0, ..., 2q-1$ along the singular extremal (x, λ, u), then the singular control u is uniquely determined by the state feedback*

$$u = -\frac{det([f(x), g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x), ad_f^{2q} g(x)])}{det([f(x), g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x), [g, ad_f^{2q-1} g](x)])} \tag{21}$$

and λ is deduced from

$$\lambda^T [f(x), g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x)] = 0. \tag{22}$$

Proposition 12 *(Algebraic necessary conditions for singular arcs) Assume that $p = q$ and $n = 2q+1$. The singular control u is uniquely determined by the state feedback*

$$u = -\frac{det([g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x), ad_f^{2q} g(x)])}{det([g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x), [g, ad_f^{2q-1} g](x)])} \tag{23}$$

and λ is deduced from

$$\lambda^T [g(x), ad_f^1 g(x), \ldots, ad_f^{2q-1} g(x)] = 0. \tag{24}$$

Remark 13 *By homogeneity, λ may be chosen such that $\|\lambda\| = 1$*

The cases where $n < 2q$, do not exist following proposition 10. As we have already mentioned, $q = 1$ is the most frequent case: it means that the two above propositions 11 and 12 can be used for $n = 3$ or 4. $n = 2$ is trivially solved from (19).

Remark 14 *In a practical case ($n = 2(q+1)$ or $n = 2q+1$), the above propositions require $p = q$. In order to ensure this, we must verify that $p \geq q+1$ cannot occur. A way to do it, is showing that the space $E = span\{g(x), ad_f^1 g(x), \ldots, ad_f^{2q+1} g(x)\}$ has a full rank n. Following the equation (13) and if $p = q+1$, then $\lambda^T [g(x), ad_f^1 g(x), \ldots, ad_f^{2q+1} g(x)] = 0$. Therefore, in the case that the dimension of space E is equal to n, the non triviality of λ is contradicted and $p > q$ cannot occur.*

Our proposal to determine admissible arcs can be now stated by the following proposition:

Proposition 15 *(Algorithm to obtain singular arcs) If the assumptions of proposition 11 or 12 are satisfied, the candidate singular arcs can be obtained through the steps:*

1. *Determine $u(x)$ and $\lambda(x)$ following related expressions in proposition 11 or 12*
2. *Store only solutions which satisfy $u(x) \in co(U)$.*
3. *Store only solutions which satisfy the second order condition from Theorem 9.*

3.2 Multi-input case $m > 1$

3.2.1 Standard results and second order necessary condition

The situation when $m > 1$ becomes more complex because of new conditions' appearance. Two methods are generally employed to find singular arcs:

- A term $\int_{t_0}^{t_f} (\epsilon/2)\alpha^2 dt$ is added to $J(x)$ in (2) with $\epsilon \to 0$. This is a penalization technique [Moylan and Moore(1971)], [Jacobson et al.(1970)], [Popescu(2005)]. Although such method can be indistinctively used when either $m = 1$ or $m > 1$, it becomes ill-conditioned when ϵ is very small.
- The solution can be found with a shooting algorithm for a given x_0. The method assumes a known solution structure; for example, the number of singular arcs in the trajectory. It requires a good initial estimation of λ [Fraser-Andrew(1989)], [Maurer(1976)].

In this section, some algebraic conditions are derived exclusively for $m > 1$. The proposed method has the advantage that neither approximations nor a particular form of the solutions are needed.

Firstly, some definitions from the previous subsection must be generalized. Once again, they are standard definitions on singular control. As in the case when $m = 1$, they are necessary for the comprehension of the proposed method.

The switching function ϕ is an m-dimensional vector

$$\begin{aligned}\phi(x, \lambda) &= [\phi_1(x, \lambda), ..., \phi_m(x, \lambda)]^T \\ &= H_u(x, \lambda, u) \\ &= [g^1(x), ..., g^m(x)]^T \lambda\end{aligned}$$

A singular trajectory may exist when at least one component of ϕ identically vanishes on a non zero measure time interval. Suppose that there exists a non empty fixed subset M of $\{1, 2, ..., m\}$ such that

$$\phi_i(t) \equiv 0, \forall i \in M \text{ on } [a, b] \quad (25)$$

$$\phi_i(t) \neq 0, \forall i \notin M \text{ on } [a, b] \quad (26)$$

(26) represents those controls which are regular and from the optimality conditions, $u_i(t) = 0$ or 1 following the sign of $\phi_i, \forall i \notin M$, on $[a, b]$. In what follows, these controls have a fixed value (0 or 1) on $[a, b]$. Since the fixed control u_i, $i \notin M$, does not enter in the singular control problem, the terms $g^i u_i$, $i \notin M$, are added to the drift term f. To highlight it, f is replaced by f_{drift} in sequel. The subscripts of M can be ordered as the set $\{1, 2, ..., m\}$ without loss of generality. Therefore, the definitions of singular arcs and orders become:

Definition 16 *(Problem order matrix, [Michel(1996)]) Let the optimal control problem (2) be given. The problem order matrix Q is a matrix with elements (q_{ij}), $i, j = 1, \ldots, m$. (q_{ij}) corresponds to the number where the control u_j appears explicitly for the first time in the $2q_{ij}$-th derivative with respect to t of the switching function ϕ_i.*

Definition 17 *(Arc order matrix, [Michel(1996)]) Let a singular extremal (u^*, x^*, λ^*) for the optimal control problem (2) be given. The arc order matrix P is a matrix whose elements $p_{i,j}$, $i, j = 1, \ldots, m$, are*

$$p_{ij} := \min_k \{k : k \geq 0, (\phi_i^{(2k)})_{u_j}(u^*, x^*, \lambda^*) \neq 0\}$$

Definition 18 *(Problem order and arc order [Robbins(1967)], [Michel(1996)]) The problem order q is*

$$q = min\{q_{ij}\}, \qquad i, j = 1, \ldots m$$

and the arc order:

$$p = min\{p_{ij}\}, \qquad i, j = 1, \ldots m$$

¿From Definition 18, [Robbins(1967)], [Krener(1977)], [Powers(1980)], [Melikyan(1994)] have developed some higher-order necessary conditions for optimality to distinguish the optimal controls from non optimal controls which satisfy the first order conditions (25).

A second-order necessary condition is described by the following theorem:

Theorem 19 *(Generalized Legendre-Clebsch Condition GLC, [Robbins(1967)]) Let an optimal singular extremal (x^*, λ^*, u^*) be given with an arc order $p < +\infty$ on $[a\ b]$. Therefore, along all the arc (x^*, λ^*, u^*), the matrix $(-1)^p((H_u)^{(2p)})_u$ is positive semidefinite for every t on $[a\ b]$.*

[Krener(1977)], [Michel(1996)], [Jacobson et al.(1970)], [Dmitruk(2007)] show third-order conditions for optimality of the singular problems.

Different from the one-single control case ($m = 1$), the problem order and the arc order might not be an integer number. However, the second order condition from Theorem 19 claims the following proposition:

Proposition 20 *[Robbins(1967)], [Vapnyarskii(1967)] The arc order of the optimal singular arcs is an integer number.*

[Vapnyarskii(1967)] also proved this proposition for the case where $2q = 1$.

3.2.2 A practical method to determine singular arcs

As in the case when $m = 1$, in this subsection, some algebraic conditions are deduced to find singular arcs using additional assumptions.

Let us summarize the definitions and conditions presented:

- There exists a components subset M of the switching function which identically vanishes on a time interval $[a\ b] \subset T$, $a < b$.
- For the components for which the switching function does not vanish, the value is fixed to 0 or 1.
- Considering the switching function $\phi(x, \lambda)$ without the fixed controls, we must solve $\phi(x, \lambda) \equiv 0$ on a time interval $[a\ b] \subset T$ with the help of second order necessary conditions.

Since $g(x) = \sum_{i \in M} g^i(x)u_i$, derivatives of the switching function components can be formally computed until the order $2q$, following the problem order definition

$$\begin{aligned} \phi_i^{(k)}(x, \lambda) &= \lambda^T ad^k_{f_{drift}} g^i(x) \quad (27) \\ &= 0, k = 0, ..., 2q - 1 \text{ and } i \in M \\ \phi_i^{(2q)}(x, \lambda) &= \lambda^T ad^{2q}_{f_{drift}} g^i(x) + \lambda^T \sum_{j \in M} [g^j, ad^{2q-1}_{f_{drift}} g^i](x) u_j \quad (28) \\ &= 0,\ i \in M \end{aligned}$$

For an extremal (x, λ, u) with an arc order p, if $2p > 2q$, we get:

$$\begin{aligned} \frac{\partial}{\partial u_j} \phi_i^{(2q+k)}(x, \lambda) &= \lambda^T [g^j, ad^{2q-1+k}_{f_{drift}} g^i](x) \quad (29) \\ &= 0, i \in M,\ j \in M, k = 0, ..., 2p - 2q - 1 \\ \phi_i^{(k)}(x, \lambda) &= \lambda^T ad^k_{f_{drift}} g^i(x) \quad (30) \\ &= 0, i \in M, k = 2q + 1, ..., 2p - 1 \\ \phi_i^{(2p)}(x, \lambda) &= \lambda^T (ad^{2p}_{f_{drift}} g^i(x) + \sum_{j \in M} [g^j, ad^{2p-1}_{f_{drift}} g^i](x) u_j) \quad (31) \\ &= 0, i \in M \end{aligned}$$

At this point, we note that:

1. All derivatives take the form $\lambda^T h_k(x, u)$ until order $2p$ is reached. For $k > 2p$, successive time derivatives of u enter in the expressions of h_k.
2. The adjoint λ is orthogonal to the space spanned by the vector field family $\{h_k(x, u), k = 0, 1, ...\}$.
3. Since the adjoint λ is not trivially zero, singular solutions are obtained when the space dimension is less than n.

4. The number $2p$ of derivatives to take into account is *a priori* unknown and is directly related to the number of linear independent vector fields h_k for a given (x, u).

5. If for a given derivation order all components of u are determined (as a function of x), then additional derivatives give the complete set of nonlinear differential equations involving u. No further derivatives are required to determine the values of the x candidates.

Assume now that $p = q$, from equations (27) and (28), and for a given x, we count $(n + |M|)$ unknowns (λ and u) and $(2q + 1) * |M|$ equations. From MP, λ is not trivially zero and if $2q + 1$ is equal to n, we get:

$$det([g^i(x), ad^1_{f_{drift}} g^i(x), \ldots, ad^{2q-1}_{f_{drift}} g^i(x), ad^{2q}_{f_{drift}} g^i(x) + \sum_{j \in M} [g^j, ad^{2q-1}_{f_{drift}} g^i](x) u_j]) = 0, \; i \in M.$$

Multi-linearity property of $det(.)$ leads to the following equation,

$$\begin{aligned} & det([g^i(x), ad^1_{f_{drift}} g^i(x), \ldots, ad^{2q-1}_{f_{drift}} g^i(x), ad^{2q}_{f_{drift}} g^i(x)]) \\ + & \sum_{j \in M} det([g^i(x), ad^1_{f_{drift}} g^i(x), \ldots, ad^{2q-1}_{f_{drift}} g^i(x), [g^j, ad^{2q-1}_{f_{drift}} g^i](x)]) u_j = 0, \; i \in M. \end{aligned} \tag{32}$$

Let us define $\left[\det_{(i,j)}(x)\right]$ the $|M| \times |M|$ matrix whose $(i, j) (\in M^2)$ entries are

$$\det_{(i,j)}(x) = det([g^i(x), ad^1_{f_{drift}} g^i(x), \ldots, ad^{2q-1}_{f_{drift}} g^i(x), [g^j, ad^{2q-1}_{f_{drift}} g^i](x)])$$

If $\left[\det_{(i,j)}(x)\right]$ is non singular, the control is uniquely determined by:

$$u = \left[\det_{(i,j)}(x)\right]^{-1} W(x)$$

where $i - th$ entry of vector $W(x)$ is given by

$$W_i(x) = det([g^i(x), ad^1_{f_{drift}} g^i(x), \ldots, ad^{2q-1}_{f_{drift}} g^i(x), ad^{2q}_{f_{drift}} g^i(x)]).$$

Similarly to the case where $m = 1$, the following proposition is deduced:

Proposition 21 *(Algebraic necessary conditions for singular arcs) Assume* $p = q$, $n = 2(q + 1)$ *and* $H = 0$ *(required by MP). If* $f_{drift}(x)$ *is linearly independent of* $ad^k_{f_{drift}} g(x)$, $k = 0, ..., 2q - 1$ *along the singular extremal* (x, λ, u) *and* $[\det_{(i,j)}(x)]$ *is non singular, then the singular control vector* u *is uniquely determined by the state feedback*

$$u = \left[\det_{(i,j)}(x)\right]^{-1} W(x) \tag{33}$$

where $\left[\det_{(i,j)}(x)\right]$ *is the* $|M|\times|M|$ *matrix whose* $(i,j)(\in M^2)$ *entries are defined by*

$$\det_{(i,j)}(x) = det([f_{drift}, g^i(x), ad^1_{f_{drift}} g^i(x), \ldots, ad^{2q-1}_{f_{drift}} g^i(x), [g^j, ad^{2q-1}_{f_{drift}} g^i](x)])$$

and where the i-th *entry of vector* $W(x)$ *is given by*

$$W_i(x) = det([f_{drift}, g^i(x), ad^1_{f_{drift}} g^i(x), \ldots, ad^{2q-1}_{f_{drift}} g^i(x), ad^{2q}_{f_{drift}} g^i(x)]). \quad (34)$$

Moreover λ *is deduced from*

$$\lambda^T [f_{drift}(x), g^i(x), ad^1_{f_{drift}} g^i(x), \ldots, ad^{2q-1}_{f_{drift}} g^i(x)] = 0. \quad (35)$$

Opposite to the case of a single control variable $m = 1$, the assumption $p = q$ does not usually hold. The most often case is to find $q = 1/2$ and $p = 1$ because the cross terms $[g^i, g^j](x)$, for $i \neq j$ do not disappear. Then the following proposition may be used to try to find a solution:

Proposition 22 *(Algebraic necessary conditions for singular arcs) Let* (x, u) *be a singular optimal solution to (2) with an arc order* p *such that* $2p \geqslant 2q$. *Then, the locus described by* (x, u) *verifies the algebraic equation:*

$$\bigcap_l \{(x,u) : S_l(x,u) = 0\} \quad (36)$$

where S_l *are all the minors of rang* n *i.e. an indexed sequence of determinants* $S_l(x,u) = det([h_{i_1}...h_{i_n}])$ *whose columns* $h_{i_k}(x,u)$, $1 \leq i_1 < \cdots < i_k < \cdots < i_n \leq i_{max} = 2(p-q)|M|^2 + (2p+1)|M|$ *are selected from the sets* $\alpha(x)$, $\beta(x)$ *and* $\gamma(x,u)$ *with*

$$\alpha(x) = \{ad^k_{f_{drift}} g^i(x), i \in M, k = 0, ..., 2p-1\}$$
$$\beta(x) = \{[g^j, ad^{(2q+k-1)}_{f_{drift}} g^i](x), i \in M, j \in M, k = 0, ..., 2p-2q-1\}$$
$$\gamma(x,u) = \{ad^{2p}_{f_{drift}} g^i(x) + \sum_{l\in M} [g^l, ad^{2p-1}_{f_{drift}} g^i](x) u_l, i \in M\}.$$

If $H = 0$ *is required by the MP, add* $f(x)$ *in the above list of vectors* h_k.

Remark 23 *The above proposition may fail if all the components of* u *are not entirely determined. It means that there exist at least one index* l *such that* $[g^l, ad^{2p-1}_{f_{drift}} g^i] \equiv 0$, $i \in M$.

Our proposal to determine admissible arcs can now be established as follows:

Proposition 24 *(Algorithm to obtain singular arcs) The candidate singular arcs can be obtained through the steps:*

1. *For all subsets* M *of control index* $\{1, 2, ..., m\}$, *and fixing to 0 or 1 the controls which are not in* M, *determine* $u(x)$ *following related expressions in proposition 22 (or 21 if it applies) for all admissible integer values of* p, $p \geq 2q$.

2. *Store only solutions which satisfy $u(x) \in co(U)$.*

3. *Determine $\lambda \perp h_i$, $i = 1, ..., i_{max}$. Check* a-posteriori *if the minimum condition of the MP is satisfied (due to the fixed value in the control variables).*

4. *Store only solutions which satisfy the second order condition given in Theorem 19.*

4 Optimal trajectory synthesis

It is well known that the necessary conditions given by the MP lead to solve a two-point boundary value problem.

Let us take the following assumption:

Assumption 25 *All optimal solutions reach the equilibrium in finite time or even in infinite time in case of infinite time criteria.*

Remark 26 *The assumption is obviously satisfied both for time optimal criteria or for quadratic criteria in infinite time. However, this is not the case for quadratic criteria in finite time.*

Here, due to the equilibrium point's nature and thanks to Propositions 15 and 24, we will show that final conditions are entirely known. Consequently, the problem appears as a simple initial condition problem.

The aim is to generate a dense set of trajectories ending at a given equilibrium point.

As we mentioned above, the equilibrium point x_{ref} belongs to a singular arc. Indeed, the control u_{ref} which holds the state on the equilibrium x_{ref} ($f(x_{ref}) + g(x_{ref})u_{ref} = 0$), is not a vertex of the control set. It means that if a trajectory reaches in finite time this equilibrium with a regular ($u_i = 0$ or 1, $i = 1, ..., m$) control. u_i, must switch to u_{ref} once $x(t_r) = x_{ref}$ at a switching time t_r. Then, from Propositions 15 and 24, admissible values λ_{ref} for $\lambda(t_r)$ are deduced. Consequently, it is not necessary to use a shooting algorithm to solve the two-point boundary value problem since final conditions are known.

Using a time backward integration of the Hamiltonian system (4) from the final state (x_{ref}, λ_{ref}) and control values $u_i, i = 1, ..., m$, equal to $0, 1$ or $u_{i,ref}$, all the trajectories ending at x_{ref} can be generated. $u_i = 0$, 1 or u_{ref} since a singular point is potentially a bifurcation point (the switching function vanishes).

This is not the best way to proceed for at least two reasons:

- Nonlinear differential equations which determine the singular control values are ill-conditioned and highly nonlinear.

- Singular arcs may converge asymptotically to the equilibrium point. This would make the backward arc computations numerically intractable.

It is really useful to find an algebraic solution of singular arcs and to start backward integration directly from points on the singular surface (avoiding the two difficulties pointed out above said above).

Backward integration starts with *ad hoc* conditions on x and λ, i.e., singular values obtained in the previous section from Propositions 15 and 24. The control is chosen such that if the i-th component of u is singular at the given point, two bifurcations are considered by switching the control u_i to 0 or 1. The rest of the trajectories are generated according to necessary conditions given by the MP. Hence, using a time backward integration of the Hamiltonian system (4) from points on singular arcs (ending in the final state x_{ref}), a dense set of optimal trajectories can be generated (see Figure 1).

It may arise that two trajectories intersect each other in the state space. In that case, unicity of the extremal does not hold. However, the conflict can be solved by considering the performance function value. The trajectory branch which has the higher cost is cut and lost (see Figure 2).

Remark 27 *For the following examples, the singular surface is unique and the equilibrium point lies within it. However, it may occur that an optimal trajectory has several disconnected singular arcs. This case is more difficult and conditions concerning the junction between singular and regular arcs are not completely known. See [Ruxton and Bell(1995)] and [McDanell and Powers(1971)].*

For a sufficiently dense set of optimal trajectories computed on a pavement of $\mathbb{R}^n$, a neural network is then used in order to interpolate optimal solutions. The resulting state feedback $u(x)$ whose evaluation is a simple limited number of sums and products is available in real time.

The following proposition summarizes the method presented in this article.

Proposition 28 *(Algorithm to compute a state feedback control law). A state feedback control law $u(x)$ can be obtained through the following steps:*

1. *Compute all admissible singular arcs following Propositions 15 or 24.*

2. *Integrate backward in time the Hamiltonian system (4) according the necessary conditions given by the MP. The integrations start with $x(0)$ and $\lambda(0)$ given by the points which belong to the singular arcs and changing u_i to $u_i = 0$ or $u_i = 1$, if u_i is singular. The end time is chosen sufficiently large to cover entirely a given state space area.*

3. *Store all the obtained trajectories.*

4. *Verify if there exist trajectories which intersect each other with a different control value.*

5. *Solve the conflict considering the performance function value. The trajectory with the higher cost is cut and lost from the conflict point.*

6. *Interpolate all the optimal trajectories with a neural network. The neural network has as inputs all the optimal trajectories x and as output the optimal control u.*

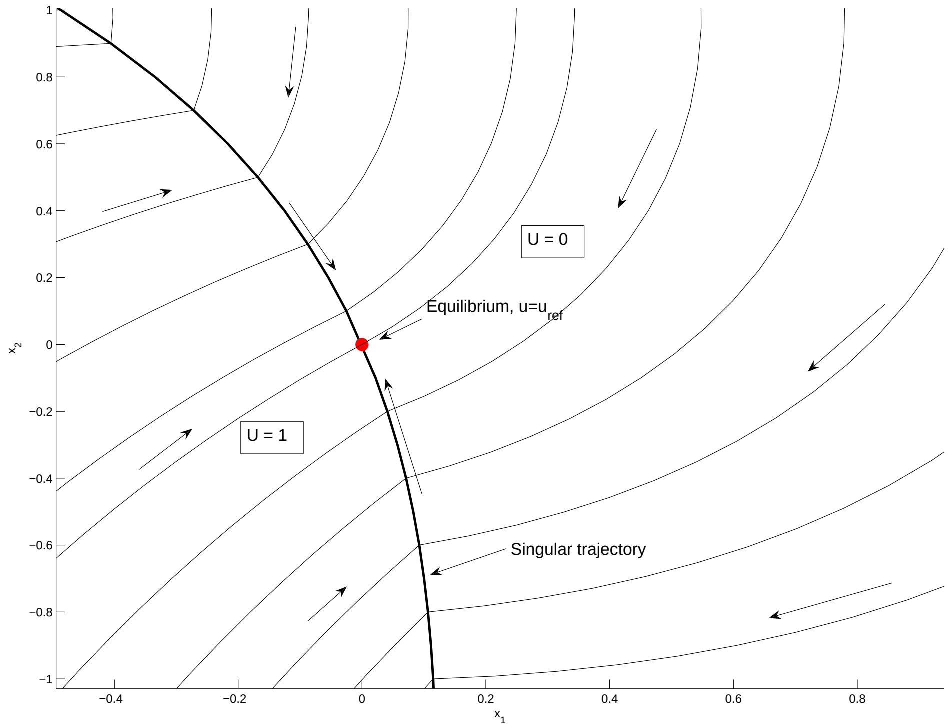

Figure 1: Generation of all trajectories: Backward integration from points on singular arcs.

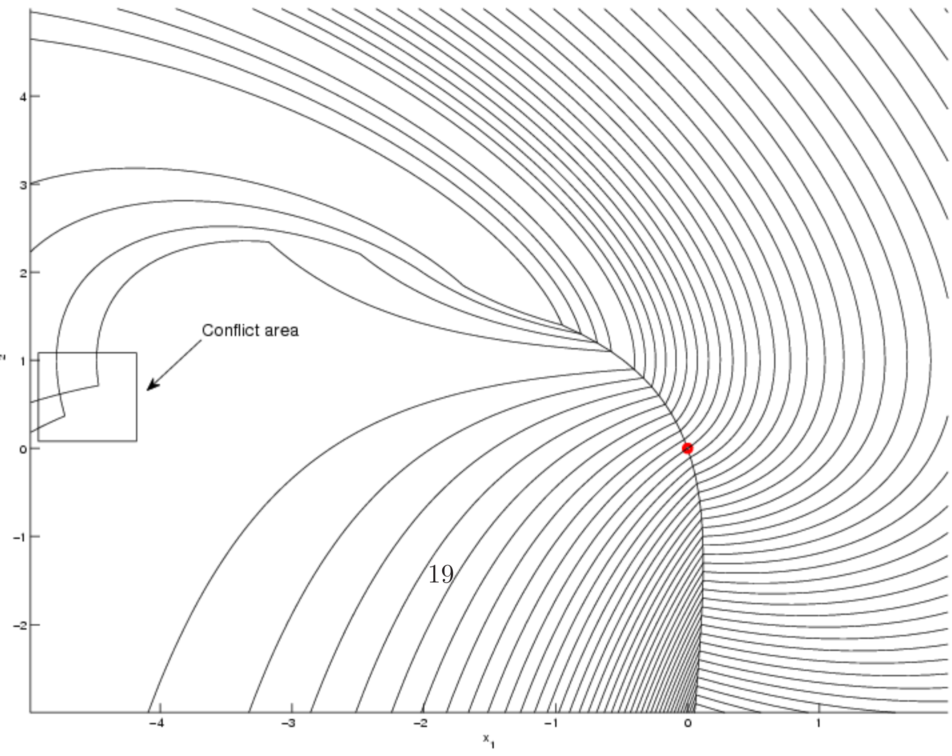

Figure 2: Conflict case. The square shows a possible conflict in the trajectories.

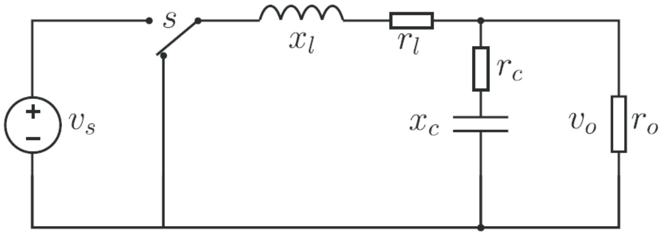

Figure 3: Step-down converter in continuous conduction mode. State variables are the inductance current and the output voltage. Control is the switch position.

Remark 29 *The success of the method practically implies:*

- *Low dimensional problems*
- *Unicity of the singular candidate control u and adjoint λ as function of the state x.*
- *Only one singular arc belongs to a trajectory.*

5 Examples

In this section, the control method is applied to two power converters.

5.1 Step-down converter

5.1.1 Problem statement

Consider a DC-DC step-down converter in Continuous Conduction Mode (CCM) whose topology is given in Figure 3 where r_0 is a resistive load.

The state variables are the inductance current i_l and the output voltage v_0. The switch position s gives two different dynamics:

1. When the switch is on, the current goes from the voltage source, v_s, to the load circuit.
2. When the switch is off, the source is not connected to the load.

The state equation is determined by: $\dot{z} = Az + Bu$, $u \in \{0, 1\}$, $z = [i_l, v_0]$, A and B take the form

$$A = \begin{bmatrix} -\frac{r_l}{x_l} & -\frac{1}{x_l} \\ -\frac{r_l r_c x_c r_o - r_o x_l}{x_c x_l (r_o + r_c)} & -\frac{r_o x_c r_c + x_l}{x_l x_c (r_o + r_c)} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \frac{1}{x_l} v_s \\ \frac{r_c r_o}{x_l(r_o+r_c)} v_s \end{bmatrix}$$

with $x_c := 100\mu F$, $x_l := 2mH$, $r_c := 0.15\ \Omega$, $r_l := 1\Omega$, $v_s := 50\ V$ and $r_0 := 50\Omega$.

The purpose is to regulate the output voltage v_0 around 25 volts despite load and input step variations. Inductance current is limited at 2.5 A.

For a given reference $z_{ref} = [i_{l_{ref}}, v_{0_{ref}}]^T = [0.5, 25]^T$, the criterion to optimize is:

$$J = \frac{1}{2} \int_0^{+\infty} (z - z_{ref})^T \mathcal{Q}(z - z_{ref}) dt$$

with the weighting matrix $\mathcal{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1000 \end{bmatrix}$.

5.1.2 Determination of the singular arcs

For this example, f and g can be written as

$$f(x) = [(Az)^T, \frac{1}{2}(z - z_{ref})^T \mathcal{Q}(z - z_{ref})]^T,$$
$$g(x) = [B^T, 0]^T.$$

$x = [z, x_n]^T$. Since the problem is in infinite time, the Hamiltonian is identically zero, $H = \lambda^T f + \lambda^T gu = 0$, $\forall t$. Therefore, the equation $\lambda^T f = 0$ can be added to the equation given by the switching function $\phi(t) = \lambda^T g = 0$. This leads to the following condition on x (the problem order is $q = 1$)

$$S = \det(f(x), g(x), [f, g](x)) = 0.$$

The adjoint variable must satisfy

$$\lambda \perp \{f(x), g(x), [f, g](x)\}$$

and u is deduced from the last equation of (19) (here $p = q$):

$$u = -\lambda^T ad_f^2 g(x)/\lambda^T [g, ad_f^1 g](x).$$

¿From the admissible candidates, optimal arcs are determined applying the second order condition (18).

Backward time integration gives all the trajectories ending at the equilibrium point (Figure 4). The control values which belong to a singular arc match the values in the interval $[0, 1]$. The dynamics of u is given by the nonlinear differential equation

$$\dot{u} = -\frac{\lambda^T (ad_f^3 g + u([g, [f, [f, g]]] + [f, [g, [f, g]]]) + u^2\, [g, [g, [f, g]]])}{\lambda^T\, [g, [f, g]]}$$

Therefore, singular arcs define an optimal sliding surface.

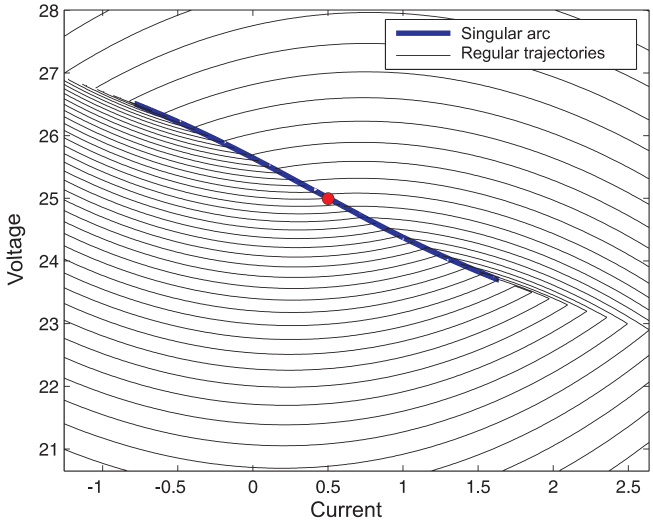

Figure 4: Singular arc and regular trajectories for the step-down converter. All the trajectories converge towards the singular arc and slide towards the equilibrium point.

5.1.3 State feedback

In order to obtain a feedback control law, a neural network is used with the optimal state x^* as input and the optimal control u^* as output. Once the learning phase in the neural network is completed, the state space is fractionated into two control regions. Each region gives the control value.

The upper bound in the current ($i_l < 2.5A$) is represented as a border line. This constraint is introduced after the feedback control law synthesis as an extra additional constraint. A switch is induced if the constraint is not verified. This strategy leads to a sliding motion. Nevertheless, the solutions could not be necessarily optimal in this configuration.

This is also a sliding trajectory at $i_l = 2.5$ A. See Figure 5.

The system response from zero initial conditions is shown in Figure 6. The learning phase of the control is carried out for different load values r_0 changing the reference (i_{ref} is a function of r_0). Load variations are observed in simulation from state and input measures through a simple gradient based estimator.

For control law robustness test, a step change of the input voltage is applied from 50 to 35 volts at time $t = 3$ ms, and at $t = 4$ ms from 35 to 50 V. Again, a load step change from 50 to 100 and from 100 to 50 Ω at times $t = 5$ ms and $t = 6.5$ ms is applied. The voltage output is regulated to 25 V.

Remark 30 *For this example, the off-line computation time necessary to establish the control is* 10.45 *seconds on a PC with* 1 *GB of RAM memory. The*

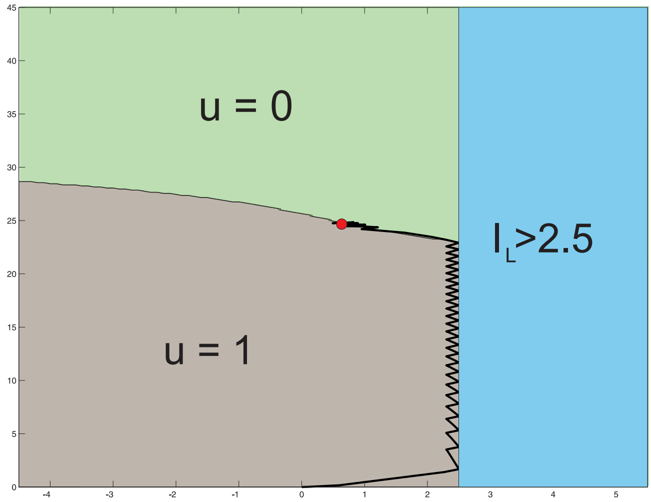

Figure 5: State Feedback control map. $u(x)$ is deduced from this map. A trajectory from start-up condition to the equilibrium point is also shown.

singular arcs computation, the trajectories generation and the solutions learning with a neural network are included in this time.

Remark 31 *The neural network used for this example has two layers with 10 neurons for the hidden layer.*

Remark 32 *The sliding motion on the singular arcs is obtained with a hysteresis around the surface which leads to an asynchronous control. For a numerical implementation, one can use a synchronous control $u(kT_e)$ with respect to a sampled time T_e. See Figure 7.*

Finally, the method is applied for an optimal time criterion. Figure 8 shows the system's response.

5.2 Multilevel converter

5.2.1 Problem statement

In the case of a few megawatt industrial power applications, the classical power converters have a very high voltage in the switching components (several kilovolts). To compensate this, a new class of power converters called multicellular converters appeared. A multicellular converter reduces the voltage throughout the switches. Its structure makes it possible to split the voltage constraints and to distribute them among several switches. It is composed of serial connections between semiconductor switching devices and passive storage elements to achieve the target operating voltage. See Figure 9.

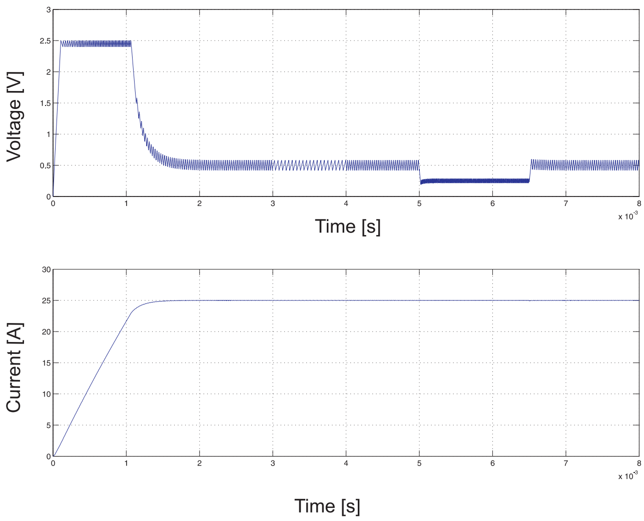

Figure 6: Transient response and robustness test for charge and source variation. Supply step from $50 \rightarrow 35$ V at $t = 3$ ms, at $t = 4$ ms, $35 \rightarrow 50$ V. Load step from $50 \rightarrow 100\Omega$ at $t = 5$ ms and $100 \rightarrow 50\Omega$ at $t = 6.5$ ms is applied. The voltage output is regulated to 25 V.

Three switching cells can be isolated, each one containing two switches that operate dually. The behavior of each cell can be described using only one boolean control variable $u_i \in \{0, 1\}$ with $i = 1, 2, 3$. $u_i = 1$ means that the upper switch is closed and the lower switch is open whereas $u_i = 0$ means that the upper switch is open and the lower switch is closed (Figure 9). There exist a lot of references concerning this converter design for medium and high voltage applications. Some of them are [Béthoux and Barbot(2006)], [Meynard et al.(2002)], [Lai and Peng(1996)], [Chiasson(2003)].

The state equations of the converter have an affine form given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{R}{L} z_3 \end{bmatrix} + \begin{bmatrix} -\frac{z_3}{C_1} & \frac{z_3}{C_1} & 0 \\ 0 & -\frac{z_3}{C_2} & \frac{z_3}{C_2} \\ \frac{z_1}{L} & \frac{z_2 - z_1}{L} & \frac{E - z_2}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{37}$$

$$= r(z) + [s^1(z), s^2(z), s^3(z)]u \tag{38}$$

where z_1, z_2 are the voltage on each capacitor and z_3 the load current.

The purpose is to regulate the state around an equilibrium point of the average state model. In order to improve the wave form, the capacitor voltages must be maintained in $2E/3$ and $E/3$, while the demanding load current is fixed to 0.6 A. Then, $z_{ref} = \begin{bmatrix} 2E/3 & E/3 & 0.6 \end{bmatrix}^T$.

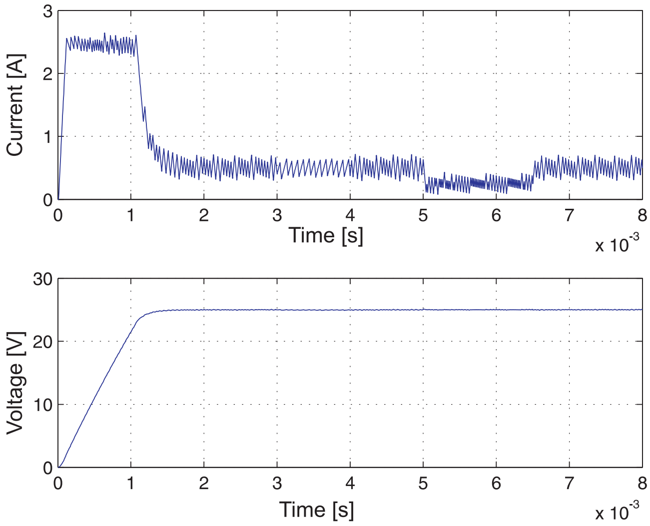

Figure 7: Transient response and robustness test for charge and source variation when the controller is sampled every 100kHz.

The criterion is again a quadratic one:

$$J = \frac{1}{2}\int_0^\infty (z - z_{ref})^T \mathcal{Q}(z - z_{ref})dt$$

with a weight matrix fixed to $\mathcal{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1000 \end{bmatrix}$.

5.2.2 Determination of the singular arcs

Here,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \frac{1}{2}\int_0^t (z - z_{ref})^T \mathcal{Q}(z - z_{ref})d\tau \end{bmatrix}$$

and

$$f(x) = \begin{bmatrix} r(z) \\ \frac{1}{2}(z - z_{ref})^T \mathcal{Q}(z - z_{ref}) \end{bmatrix} \quad g(x) = \left[g^1(x), g^2(x), g^3(x)\right] = \begin{bmatrix} s^1(z) & s^2(z) & s^3(z) \\ 0 & 0 & 0 \end{bmatrix}$$

As in the previous example, the Hamiltonian must be zero, $H = 0$, following the transversality conditions and $J < \infty$.

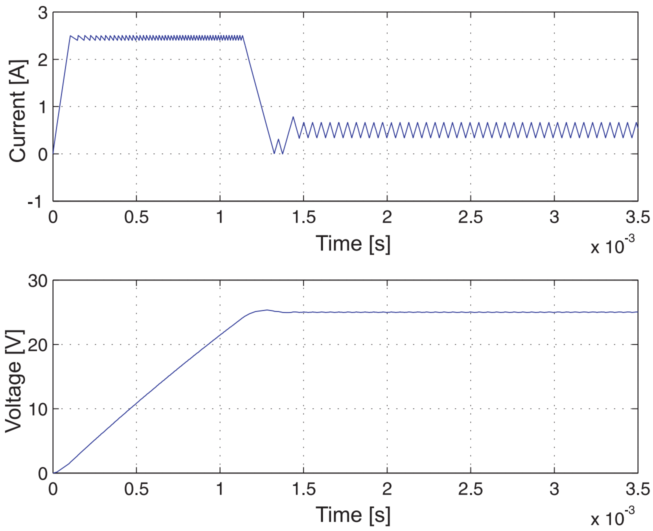

Figure 8: Optimal time start-up response. A minimum time criterion is considered.

The switching function is

$$\phi(x,\lambda) = \begin{bmatrix} \phi_1(x,\lambda) \\ \phi_2(x,\lambda) \\ \phi_3(x,\lambda) \end{bmatrix} = \begin{bmatrix} \frac{-x_3}{C_1}\lambda_1 + \frac{x_1}{L}\lambda_3 \\ \frac{x_3}{C_1}\lambda_1 - \frac{x_3}{C_2}\lambda_2 + \frac{x_2 - x_1}{L}\lambda_3 \\ \frac{x_3}{C_2}\lambda_2 + \frac{E - x_2}{L}\lambda_3 \end{bmatrix}$$

First, we must consider the singular solution. Applying Proposition 22, the following equations set helps to obtain the singular arcs:

$$\begin{aligned}
S_1(x) &= det([f(x), g^1(x), ad_f g^1(x), ad_{g^2} g^1(x)]) \\
S_2(x) &= det([f(x), g^2(x), ad_f g^2(x), ad_{g^1} g^2(x)]) \\
S_3(x) &= det([f(x), g^1(x), ad_f g^1(x), ad_{g^3} g^1(x)]) \\
S_4(x) &= det([f(x), g^3(x), ad_f g^3(x), ad_{g^1} g^3(x)]) \\
S_5(x) &= det([f(x), g^2(x), ad_f g^2(x), ad_{g^3} g^2(x)]) \\
S_6(x) &= det([f(x), g^3(x), ad_f g^3(x), ad_{g^2} g^3(x)])
\end{aligned}$$

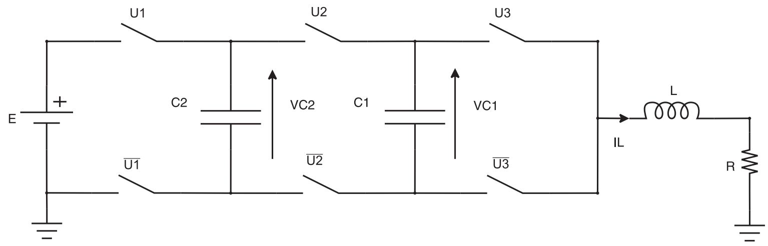

Figure 9: Multilevel converter. The state variables are the capacitor voltages and the current load. The control is the switch position of U_1, U_2 and U_3.

$$S_7(x) = det([g^1(x), ad_f g^1(x), ad_{g^3} g^1(x), ad_f^2 g^1(x) + \sum_{j \in M} [g^j, ad_f g^1](x) u_j])$$
$$S_8(x) = det([g^2(x), ad_f g^2(x), ad_{g^3} g^2(x), ad_f^2 g^2(x) + \sum_{j \in M} [g^j, ad_f g^2](x) u_j])$$
$$S_9(x) = det([g^1(x), ad_f g^1(x), ad_{g^2} g^1(x), ad_f^2 g^1(x) + \sum_{j \in M} [g^j, ad_f g^1](x) u_j])$$
$$S_{10}(x) = det([g^3(x), ad_f g^3(x), ad_{g^2} g^3(x) ad_f^2 g^3(x) + \sum_{j \in M} [g^j, ad_f g^3](x) u_j])$$
$$S_{11}(x) = det([g^2(x), ad_f g^2(x), ad_{g^1} g^2(x), ad_f^2 g^2(x) + \sum_{j \in M} [g^j, ad_f g^2](x) u_j])$$
$$S_{12}(x) = det([g^1(x), ad_f g^1(x), ad_{g^3} g^1(x), ad_f^2 g^1(x) + \sum_{j \in M} [g^j, ad_f g^1](x) u_j])$$

$$S_{13}(x) = det([g^1(x), ad_f g^1(x), ad_f^2 g^1(x) + [g^1, ad_f g^1](x) u_1])$$
$$S_{14}(x) = det([g^2(x), ad_f g^2(x), ad_f^2 g^2(x) + [g^2, ad_f g^2](x) u_2])$$
$$S_{15}(x) = det([g^3(x), ad_f g^3(x), ad_f^2 g^3(x) + [g^3, ad_f g^3](x) u_3])$$

Table 1 shows how to compute singular arcs. This table is composed by:

- In the first column, the values of the set M.
- In the second column, the drift term including f and terms $g^i(x)u^i$, if $i \notin M$. u^i is fixed to the values 0 or 1 to compute the solution.
- In the last column, the equations set that gives the solution (x, u).

All solutions in this example are obtained for $p = 1$. For $p \geq 2$, necessary conditions yield an empty set and so no singular arcs exist.

The admissible λ such that it is orthogonal to the components of $\{h_k(x, u),\ k = 0, 1, ...\}$ is *a posteriori* determined. The minimum condition (5) of the MP is checked as well as the second order conditions.

Table 1: Singular surfaces. This table shows all the possibles singular controls and the equations to obtain them.

M	f_{drift}	Singular arc
$\{1,2,3\}$	$f(x)$	$\bigcap_l \{(x,u), S_l(x,u)=0\}\ l=1,\ldots,12$
$\{1,2\}$	$f(x)+g^3(x)u_3$	$\{S_1=0\}\cap\{S_2=0\}\cap\{S_7=0\}\cap\{S_8=0\}$
$\{1,3\}$	$f(x)+g^2(x)u_2$	$\{S_3=0\}\cap\{S_4=0\}\cap\{S_9=0\}\cap\{S_{10}=0\}$
$\{2,3\}$	$f(x)+g^1(x)u_1$	$\{S_5=0\}\cap\{S_6=0\}\cap\{S_{11}=0\}\cap\{S_{12}=0\}$
$\{1\}$	$f(x)+g^2(x)u_2+g^3(x)u_3$	$\{S_{13}=0\}$
$\{2\}$	$f(x)+g^1(x)u_1+g^3(x)u_3$	$\{S_{14}=0\}$
$\{3\}$	$f(x)+g^1(x)u_1+g^2(x)u_2$	$\{S_{15}=0\}$

5.2.3 State feedback

All the trajectories are generated from the singular arcs. After obtaining regular and singular arcs numerically following proposition 24, an artificial feed-forward neural network is used to interpolate the solutions. The network is trained with 15 neurons in the hidden layer and sigmoid functions. The inputs are the three errors between each reference and each state variable $[x - x_{ref}] = [v_{c1} - v_{c1,ref}, v_{c2} - v_{c2,ref}, i_l - i_{l,ref}]^T$ and the outputs are the three controls u_1, u_2 and u_3. Nominal charge value is used.

The method presented in this article has been validated in simulation with the following nominal parameter values: $C_1 = C_2 = 45\mu F$, $L = 0.5H$, $R = 30\Omega$.

Once the optimal state feedback control is determined, it defines a state space partition. Then, the state space is divided into regions and the borders of each region may be partially composed by singular arcs.

In this example, control $u(x)$ is updated at a sampling frequency of $1/T_e = 40$ kHz.

The nominal source voltage is $E = 30$V. The system modes are listed in Table 2.

Table 2: System modes.

Mode	u_1	u_2	u_3
1	0	0	0
2	0	0	1
3	0	1	1
4	0	1	0
5	1	1	0
6	1	1	1
7	1	0	1
8	1	0	0

A few relevant performance indices were selected for simulation tests:

1. Start-up transient. Figures 10 and 11 show the system's response with

nominal parameters. The control law achieves the reference in 0.018 s.

2. Line transient. The supply voltage is subject to variations during operation. Figures 12 through 13 and Figures 14-15 show the transient when a supply variation is applied: From $E = 45$ V to $E = 30$ V at $t = 0.025$ s and back again at $t = 0.03$ s.

   The figures show that the control law can compensate the different supply changes. Despite the variation ($E = 30 \rightarrow 45$ V), the modes generating the limit cycle do not change. However, the duration of each mode is modified by the control law.

3. Load transient. The load is subject to variations during operation. The control law is only computed for the nominal value $R = 30\Omega$. Figures 16 and 17 show the transient when a change of $R = 35\Omega$ to $R = 20\Omega$ occurs at $t = 0.025$ s. Figures 18 and 19 show the response for $R = 20\Omega$ to $R = 35\Omega$ at $t = 0.03$ s. The results show that the regulation, even when the load varies, is achieved. It can be seen that the cycle is different (modes and its duration change).

6 Conclusions

In this article, a methodology to compute an optimal state feedback control law for low-order switched affine systems has been proposed. A large class of power converters is included into this class of systems.

As it has been shown, our approach consists in extending the discrete controls set to its convex hull. The bang- bang solutions in this formulation are also the original problem's solutions. Otherwise, there exist singular arcs corresponding to Fillipov solutions which are not admissible for the switched system.

Nevertheless, as density theorems have proven, these Fillipov solutions can be approximated by chattering control on the optimal surface when either a maximum switched frequency is fixed or a hysteresis band is imposed.

The key contribution shows that singular surfaces can be algebraically obtained by finding the roots of a determinants set. Second order conditions are jointly used to reduce the potential candidates' number.

For infinite time quadratic or time optimal criteria, it is shown that the optimal trajectory synthesis can be numerically computed by integrating backward in time the Hamiltonian system as a single initial value problem. This avoids the classical two-points boundary value problem.

Once all the solutions are generated, a neural network learns them as a simple input-output function which is easy to implement in real time. The procedure yields a state feedback control law with a very few level approximation due to the discrete nature of the control values.

The application on two converter examples shows the method's applicability and its efficiency. This synthesis can be extended to take into account the changes in parameters.

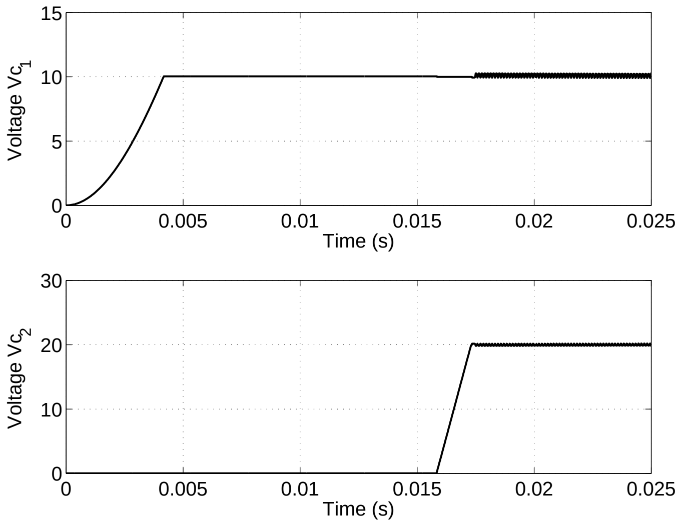

Figure 10: Start up transient response (voltages).

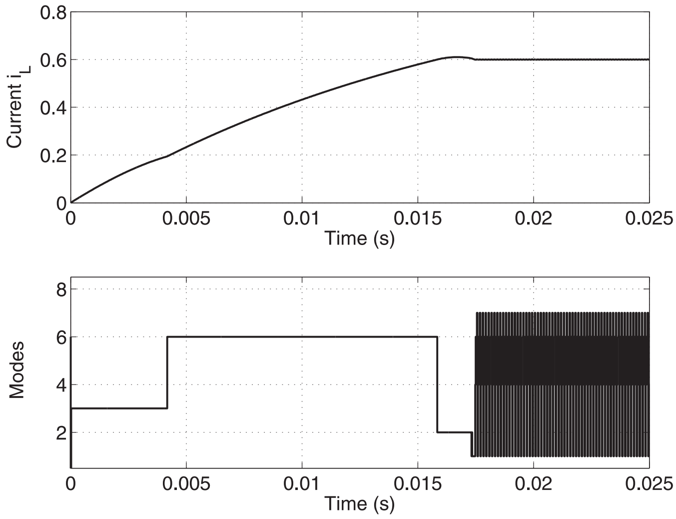

Figure 11: Start up transient response (current).

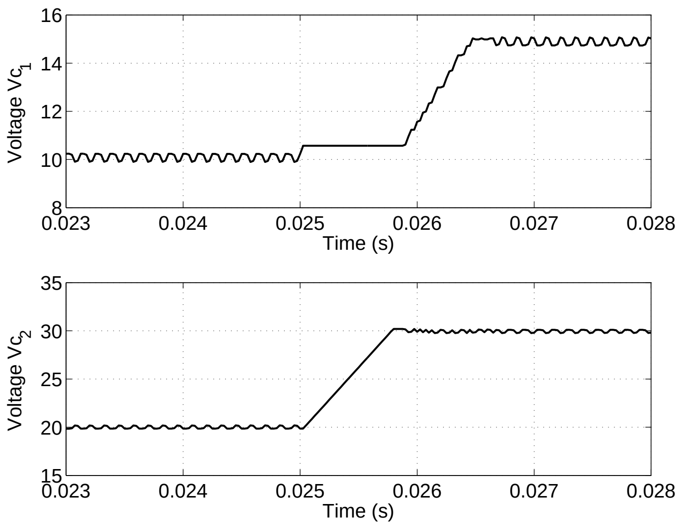

Figure 12: Transient response to a step in the source voltage from $E = 30V$ to $E = 45V$ at $t = 0.025$ s (voltage).

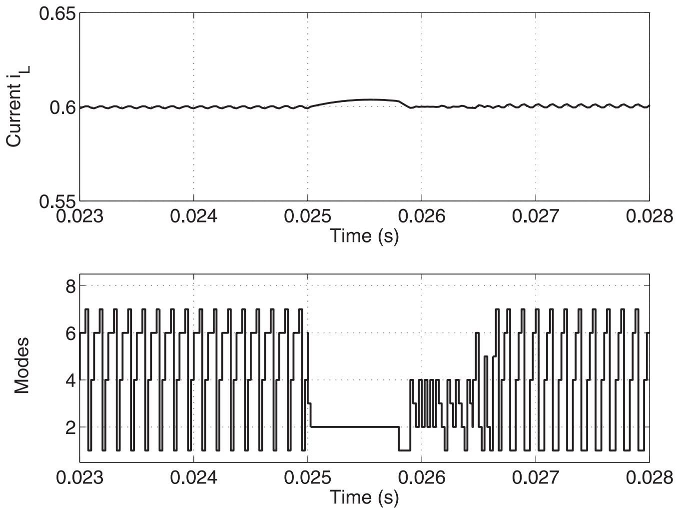

Figure 13: Transient response to a step in the source voltage from $E = 30V$ to $E = 45V$ at $t = 0.025$ s (current).

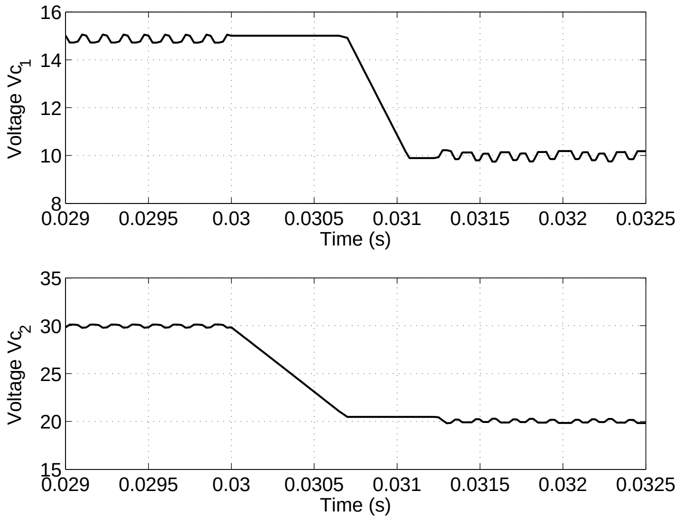

Figure 14: Transient response to a step in the source voltage from $E = 45V$ to $E = 30V$ at $t = 0.03$ s (voltage).

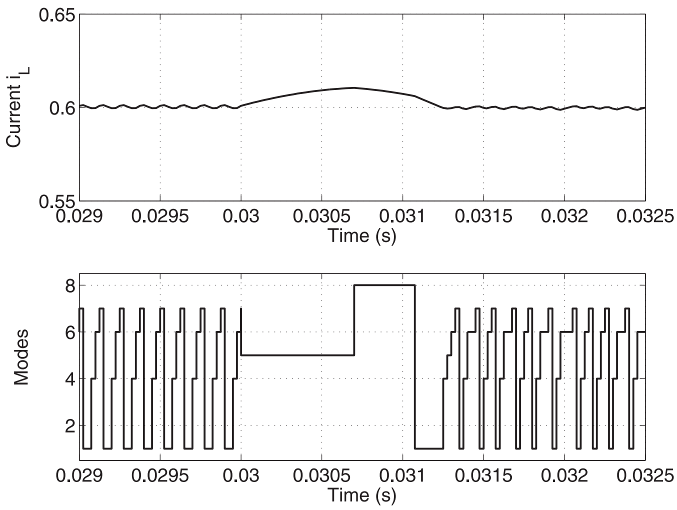

Figure 15: Transient response to a step in the source voltage from $E = 45V$ to $E = 30V$ at $t = 0.03$ s (current).

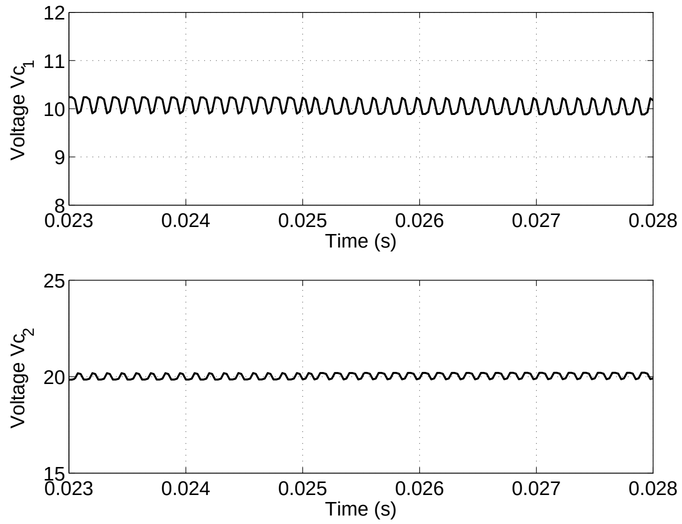

Figure 16: Transient response to a step in the load resistance from $R = 30\Omega$ to $R = 20\Omega$ at $t = 0.025$ s (voltage).

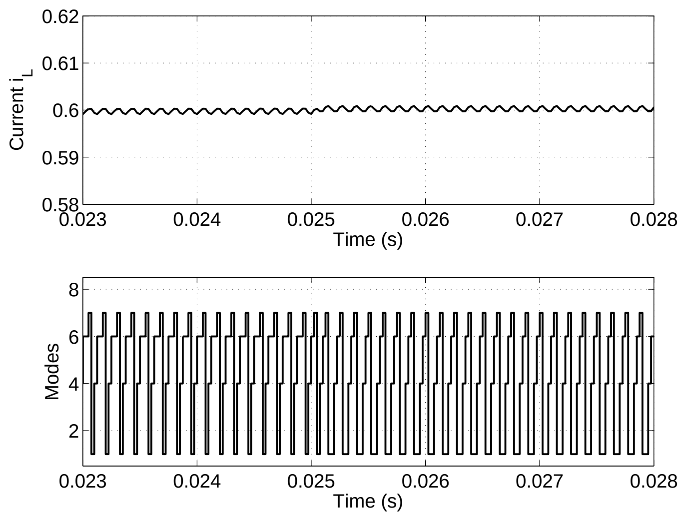

Figure 17: Transient response to a step in the load resistance from $R = 30\Omega$ to $R = 20\Omega$ at $t = 0.025$ s (current).

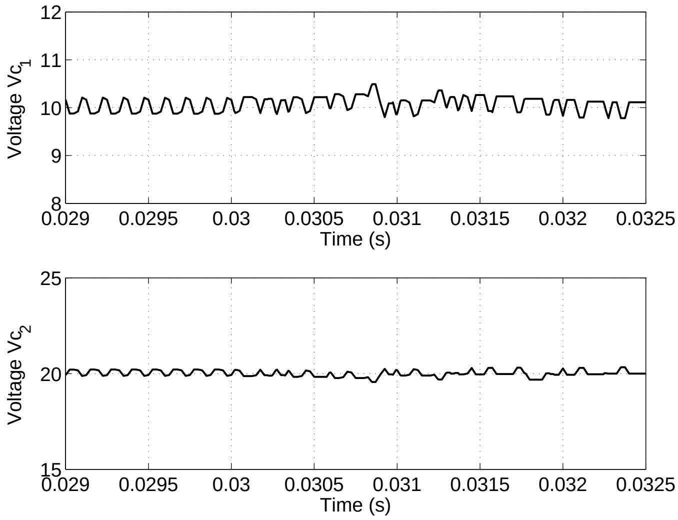

Figure 18: Transient response to a step in the load resistance from $R = 20\Omega$ to $R = 35\Omega$ at $t = 0.03$ s (voltage).

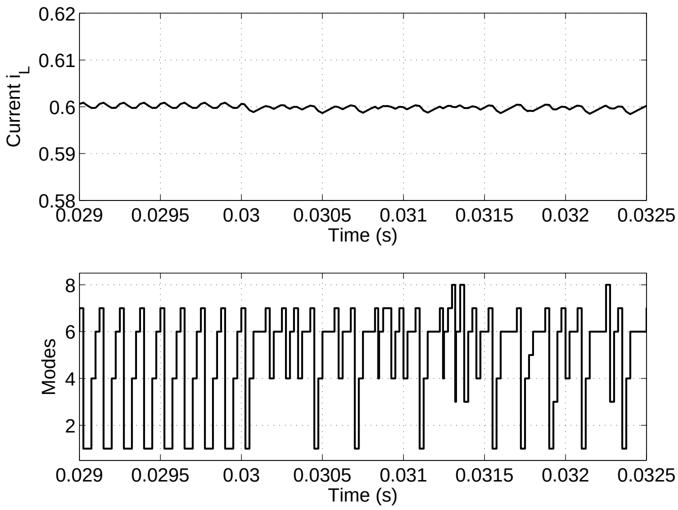

Figure 19: Transient response to a step in the load resistance from $R = 20\Omega$ to $R = 35\Omega$ at $t = 0.03$ s (current).

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