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Abstract

The problem of robust and guaranteed stabilization is addressed for switched *affine* systems using sampled state feedback controllers. Based on the existence of a control Lyapunov function for a relaxed system, we propose three sampled-data controls. Global attracting sets, computed by solving a sequence of optimization problems, guarantee practical and global asymptotic stabilization for the whole system trajectories. In addition, robust margins with respect to parameters uncertainties and non uniform sampling are provided using input-to-state stability. Finally, a buck-boost converter is considered to illustrate the effectiveness of the proposed approaches.
Keywords: switched affine systems, stabilization of hybrid systems, input-to-state stability, robust control

1. Introduction

During the past decades, hybrid systems have attracted a large interest from the scientific community. Indeed, a wide range of systems can be modeled in a hybrid context: physical systems involving impacts, multi-model approaches, electrical circuits containing switching elements (diodes, transistors,...), etc. Here, we study a particular class of hybrid systems: switched affine systems. It consists in a finite collection of affine subsystems which are selected by a switching rule. Usually, the control design of these systems relies on an averaged model [1]. Averaging methods [2, 3] are largely used in power electronics in order to provide feedback strategies using Pulse Width Modulation (PWM) control. Advanced methods such as passivity based control [4], sliding modes [5, 6], optimal [7, 8, 9] and predictive control [10, 11, 12, 13, 14] are attractive in order to improve dynamic performances. These methods, in which a direct selection of the active subsystem is made, provide strategies to take into account the discontinuities introduced by switchings.

Two main problems are related to the stability of switched systems: the first concerns the stability conditions for an arbitrary switching law and the second concerns the switching strategy which keeps the system stable. In this paper, we treat the latter problem. Several surveys are available [15, 16, 17, 18] on this topic. Most of the available techniques for either analyzing the stability or synthesizing control laws are based on Lyapunov functions: quadratic [19], multiple [20, 21, 22], piecewise quadratic [23, 24], etc. In [17], the author presents a Lie algebra approach for the study of switching systems. The role of dwell time and the impact of time-delay have also been emphasized in [25, 26]. Even various techniques are employed, most of them deal with switched systems whose subsystems share a common equilibrium.

Unlike these studies, we address the case where no common equilibrium can be defined. A large class of systems having a practical interest, such as DC-DC power converters is covered by this framework. Based on the existence of a common quadratic Lyapunov function, a continuous time stabilizing switching strategy is provided in a recent paper [27]. In [28], in a discrete time framework, a positively invariant set [29] formed by the union of bounded ellipsoids is determined and used in a predictive control algorithm to steer the state inside. However, the method uses a LMI formulation to compute these ellipsoids which introduces some conservatism in the result. Indeed, LMIs imply that the switched system possesses a switching sequence S of a prescribed length for which a property of uniform stability w.r.t. the initial condition is satisfied. So, the computed invariants are not particularly tight around the target.

In this paper, based on the existence of a Control Lyapunov Function (CLF) for a relaxed system - obtained by relaxing the control domain to its convex hull -, robust stability for sampled switched strategies is investigated. In this framework, the referred targets, named operating points, are defined as the equilibria of the relaxed system. Assuming that a continuous time CLF is known for the relaxed system, different sampled switched strategies are deduced. A method which computes estimation of tight positive invariant sets around the targets is given. The global and practical asymptotic stabilization is thus guaranteed.

Precisely, we prove that positive invariant sets can be obtained by solving optimization problems. Since no assumption is made on the CLF, this problem is in general non trivial and non-linear. Fortunately, when the target defines a stable equilibrium of the relaxed system, a quadratic Lyapunov function can be easily exhibited and the optimization problem reveals to be a quadratically constrained quadratic program (QCQP) for which efficient solvers exist.

The robustness aspects of the proposed sampled switched strategies in case of non uniform sampling and parameter uncertainties are also studied and discussed. The Input-to-State Stability (ISS) formulation [30, 31] is used in order to provide stability margins.

The paper is organized as follows. Section 2 gives notations and definitions used throughout the paper. The system description is given and the operating points are defined in Section 3. In Section 4, we propose three different sampled-data controls for the switched system, deduced from a known CLF for the relaxed system. Using this CLF, a set of optimization problems is also formulated. In Section 5, we prove that the solutions of these problems allow to define global attracting sets for the sampled switched affine system. Section 6 provides some relations of inclusions between these attracting sets. An extension of those results in the case of parameter uncertainties and non-uniform sampling is given in Section 7. The computational aspects are addressed in Section 8. A buck-boost converter is used in Section 9 to illustrate our results. We show that the stability is guaranteed even in presence of parameter uncertainties. To conclude, Section 10 summarizes the results of this paper and their interest in the research field of switched affine systems.

2. Notations

Let $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}_*$ denote the set of real, natural and strictly positive natural numbers, respectively. Moreover, for any $a \in \mathbb{N}$, let $\mathbb{N}_{\leq a}$ denotes the set $\{k \in \mathbb{N} \mid k \leq a\}$. $\|\cdot\|$ is the Euclidian norm of a vector and $\|\cdot\|_\infty$ the infinite norm of a function. In this paper, systems are of the form $\dot{x}(t) = f(x(t), u(t))$ where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz continuous. So, for a given input u, there is a unique solution of the initial value problem and is denoted $x(t, x_0, u)$ for each initial state x_0.

Definition 1 ($\mathcal{N}_0$, $\mathcal{K}$ and $\mathcal{K}_\infty-$functions)**.** *A function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{N}_0-$function, if it is continuous, nondecreasing and satisfies $\alpha(0) = 0$. Moreover, α is a $\mathcal{K}-$function if $\alpha \in \mathcal{N}_0$ and is strictly increasing. α is a $\mathcal{K}_\infty-$function if it is an unbounded $\mathcal{K}-$function.*

Definition 2 ($\mathcal{KL}-$function)**.** *A class $\mathcal{KL}-$function is a function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \geq 0$ and $\forall r \geq 0,\ \beta(r, t) \to 0$ as $t \to +\infty$.*

Definition 3 (ISS)**.** *A system of the form $\dot{x} = f(x, u)$ is said to be Input-to-State Stable (ISS) if there exist some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that $\|x(t, x_0, u)\| \leq \beta(\|x_0\|, t) + \gamma(\|u\|_\infty),\ \forall u\ \forall x_0$.*

Definition 4 (Practical stability)**.** *A system of the form $\dot{x} = f(x, p)$ where p is a fixed parameter, is said to be practically stable if there exist some $\beta \in \mathcal{KL}$ and a positive constant $c(p)$ such that $\|x(t, x_0, u)\| \leq \beta(\|x_0\|, t) + c(p),\ \forall x_0$.*

Definition 5 (0-GAS)**.** *A system of the form $\dot{x} = f(x, u)$ is said to be 0-Globally Asymptotically Stable (0-GAS), if there exists some $\beta \in \mathcal{KL}$ such that $\|x(t, x_0, 0)\| \leq \beta(\|x_0\|, t),\ \forall x_0$.*

Definition 6 (AG)**.** *A system of the form $\dot{x} = f(x, u)$ has the Asymptotic Gain property (AG), if there exists some $\alpha \in \mathcal{N}_0$ such that $\limsup_{t\to+\infty} \|x(t, x_0, u)\| \leq \alpha(\|u\|_\infty),\ \forall u\ \forall x_0$.*

Definition 7 (CLF)**.** *For a system of the form $\dot{x} = f(x, u)$, a control Lyapunov function is a function V that is continuous, differentiable, positive-definite, proper, and such that for all x, there exists u for which the directional derivative along the trajectory satisfies $\dot{V}(x; u(x)) := \frac{\partial V}{\partial x} \cdot f(x, u(x)) \leq -\gamma(\|x\|)$ where γ is a class $\mathcal{K}-$function.*

3. System Description

A switched *affine* system has the form:

$$\dot{x}(t) = A_0 x(t) + B_0 + \sum_{i=1}^{m} u^i(t)(A_i x(t) + B_i) \tag{1}$$

where $u^i(t),\ i = 1, \ldots, m$ are component values of the control $u(t) \in U = \{0, 1\}^m$ and $x(t) \in \mathbb{R}^n$ represents the state value at time t. A_i and B_i are real matrices of appropriate dimensions. As previously claimed, the most studied case in the literature is the particular case $B_i = 0$, $\forall i \in \mathbb{N}_{\leq m}$. In this situation, all the subsystems (following the finite values of u)

share a common and unique equilibrium: the origin. The aim of this paper is to explore the case where B_i are all distinct from 0 and for which no common equilibrium can be defined, e.g. DC-DC converters.

System (1) belongs to the class of nonsmooth systems for which the notion of solution can be properly defined and generalized in the sense given by Fillipov [32, 33]. The generalized solutions are defined by considering the following relaxed system:

$$\dot{x}(t) = A_0 x(t) + B_0 + \sum_{i=1}^{m} u^i(t)(A_i x(t) + B_i) \quad \text{with } u^i(t) \in [0, 1] \tag{2}$$

where the control domain is now the convex hull $\text{co}(U)$ of the original one. A link between the solutions of the system (1) and those of the system (2) can be established by a density theorem in infinite time [34]:

Theorem 1. *If z is a global solution of (2) starting from z_0 and $\varepsilon : [0, +\infty) \to (0, +\infty)$ is continuous, then there exists a solution x of (1), starting from $x_0 \in B(z_0, \varepsilon(0))$ such that $\|z(t) - x(t)\| < \varepsilon(t)$ for all $t \in [0, +\infty)$.*

Therefore, switching laws $u \in L^\infty([0, +\infty), U)$ (where L^∞ denotes the Banach space of all essentially bounded measurable functions) exist such that the trajectory of the system (2) can be approached as close as desired by the one of the system (1). For this reason, the *operating points set* of the switched system (1) denoted by X_{ref}, is defined as the set of equilibrium points of the system (2):

$$X_{ref} = \left\{ x_{ref} \in \mathbb{R}^n : A_0 x_{ref} + B_0 + \sum_{i=1}^{m} u^i_{ref}(A_i x_{ref} + B_i) = 0,\ u^i_{ref} \in [0, 1] \right\}. \tag{3}$$

This set defines the control targets for the state of the system (1).

It is worth noting that none of the controls $u_{ref} \in \text{co}(U) \backslash U$ from (3) and corresponding to an equilibrium x_{ref}, is admissible for the switched system (1). The outcome is that the switched system state x cannot be maintained on x_{ref} by a control taking its values in U (unless the time duration between switchings tends towards 0). Consequently, if the target for the switched system is an operating point x_{ref}, the asymptotic behavior of the trajectories of (1) is characterized by:

- a cycle near x_{ref} if a *dwell time* condition is applied on the switchings (i.e. a lower limit exists for the time duration between switchings);
- an infinite switchings sequence with a vanishing time duration between switchings as $t \to \infty$.

These features concern indirectly most of the control designs which use an averaged model [2, 3, 35].

4. Sampled control strategies

Many control strategies like optimal [7, 11, 9], predictive [13, 36], sliding mode [37] or stabilizing controls [38, 6, 5] can be investigated to steer the state of the system (1) near a defined operating point $x_{ref} \in X_{ref}$. As shown in [9], some singularities known as *singular arcs* that render particularly hard the optimal control synthesis, appear in the resolution of optimal control problems for the system (1) or (2). This certainly explains why discrete time predictive formulation with control restricted to U or Lyapunov based approaches seem to be more tractable. Our aim is not to discuss the advantages and/or drawbacks of each method. The paper focuses on two practical aspects: the use of sampled switched laws and the guarantee of stability margins.

For a given $x_{ref} \in X_{ref}$, let us define the change of coordinates $z = x - x_{ref}$. The system (1) or (2) can be rewritten in the form:

$$\dot{z} = A(u)z + B(u) \tag{4}$$

with $A(u) = A_0 + \sum_{i=1}^{m} u^i A_i$ and $B(u) = A_0 x_{ref} + B_0 + \sum_{i=1}^{m} u^i (A_i x_{ref} + B_i)$.

Suppose that the following assumption is satisfied:

Assumption 1. *A control Lyapunov function V is known for the system (4) with a control domain relaxed to $co(U)$.*

Assumption 1 implies that a continuous time state feedback strategy:

$$u^* = \kappa(z) \in co(U) \tag{5}$$

exists such that the system (4) is globally asymptotically stable (GAS). Once the origin is reached, i.e. the target x_{ref} in $x-$coordinates, the relation $u^* = u_{ref}$ necessarily holds.

From Assumption 1, we can deduce three sampled switched control strategies:

1. **Pulse-Width Modulation strategy** is a simple way to apply a control defined by (5) to the switched system. For a given sample period T_s, the i^{th} component of the control u is approximated by:

$$v^i(t, T_s) = \sum_{k=0}^{\infty} \mathbb{I}_{[t_k,\ t_k+u^i_k T_s[}(t),\ \ \forall t \in \mathbb{R}^+ \tag{6}$$

   where $\mathbb{I}_A(.)$ stands for the indicator function which takes the value 1 when $t \in A$ and 0 otherwise, $t_k = kT_s$ and $u_k = u(t_k)$.

2. **Steepest descent strategy** consists in choosing at time t_k, $k \in \mathbb{N}$, the most decreasing direction of the CLF V among the finite values given by $u \in U$:

$$v(t, T_s) = \sum_{k=0}^{\infty} u_k\ \mathbb{I}_{[t_k,\ t_{k+1}[}(t),\ \ \forall t \in \mathbb{R}^+ \tag{7}$$

   where
$$u_k = \arg\min_{u \in U} \dot{V}(z_k; u) \tag{8}$$

   with $\dot{V}(z_k; u)$ the derivative of V in the direction given by $A(u)z_k + B(u)$.

3. **Predictive strategy** minimizes, over a horizon N_H and among a finite set of sequences, $V(z_{k+N_H})$ from the current position z_k:

$$v(t, T_s) = \sum_{k=0}^{\infty} u_k\ \mathbb{I}_{[t_k,\ t_{k+1}[}(t),\ \ \forall t \in \mathbb{R}^+ \tag{9}$$

   where
$$u_k = \arg_1 \min_{u_k, u_{k+1}, \cdots, u_{k+N_H-1} \in U^{N_H}} V(z_{k+N_H}) \tag{10}$$

   with $\arg_1$ the first argument of the optimal sequence $u_k, u_{k+1}, \cdots, u_{k+N_H-1}$.

Note that all the proposed switching strategies define explicitly or implicitly a state feedback control law:

$$v(t, T_s) = \kappa_s(t, z(t), T_s). \tag{11}$$

Which stability guarantees can be given for these three strategies? To answer this question, consider the closed loop obtained from one of the three feedbacks v. The resulting exact discretization of (4) at time t_k, $k \in \mathbb{N}$, can be written as:

$$z_{k+1} = A_s(\kappa_s)z_k + B_s(\kappa_s). \tag{12}$$

When the chosen strategy is the predictive or the steepest descent, $A_s(\kappa_s) = e^{A(u_k)T_s}$ and $B_s(\kappa_s) = \left(\int_0^{T_s} e^{A(u_k)(T_s-\tau)}d\tau\right)B(u_k)$ with u_k given in (8) or (10). For the PWM strategy, the right side expression is obtained recursively since this strategy defines a piecewise constant control on each interval (t_k, t_{k+1}) depending on the value u_k, given in (6).

Definition 8 (Level sequence and sublevel set sequence)**.** *For a sequence $\{z_0, \cdots, z_N\}$ of length $N+1$ generated by the system (12) from an initial condition z_0, let us define the level sequence by:*

$$\mathcal{L}_k(z_0) = V(z_k),\ k \in \mathbb{N}_{\leq N}, \tag{13}$$

and the sublevel set sequence by:

$$\mathcal{S}_{\mathcal{L}_k}(z_0) = \{z : V(z) \leq \mathcal{L}_k(z_0)\},\ k \in \mathbb{N}_{\leq N}. \tag{14}$$

Following the fact mentioned at the end of the previous section that a switched system cannot be maintained on x_{ref}, it is clear that non-monotone decreasing sequences $\mathcal{L}_k(z_0)$, $k \in \mathbb{N}_{\leq N}$, may exist for some z_0. Intuitively and as one can expect, cyclic path is followed near the operating point x_{ref}. Thus, the notion of practical stability seems convenient to characterize an attracting set w.r.t. the period T_s.

To get some insights: for a sequence $\{z_0, \cdots, z_N\}$, generated by the system (12) from an initial condition z_0, one can search w.r.t. z_0, the highest level $\mathcal{L}_N(z_0)$ at the end of the sequence that can be reached from a lower level $\mathcal{L}_0(z_0)$. Denote this optimization problem by $\mathcal{P}_N$:

$$\mathcal{P}_N : \quad \max_{z_0 \in \mathbb{R}^n} \ \mathcal{L}_N(z_0) \tag{15}$$

$$\text{s.t.} \quad z_{k+1} = A_s(\kappa_s)z_k + B_s(\kappa_s),\ k \in \mathbb{N}_{\leq N-1} \tag{16}$$

$$\mathcal{L}_N(z_0) \geq \mathcal{L}_0(z_0) \tag{17}$$

Remark 1. *For all* $N \geq 1$, *the constraints (16) and (17) can be trivially satisfied with an initial condition* $z_0 = 0$. *Therefore* $z_0 = 0$ *is always a feasible argument for* $\mathcal{P}_N$.

If z_0^* is an optimal argument of $\mathcal{P}_N$, then $\mathcal{L}_N^* = \mathcal{L}_N(z_0^*)$ denotes the optimum and $\mathcal{S}_{\mathcal{L}_N}^* = \mathcal{S}_{\mathcal{L}_N}(z_0^*)$ the corresponding sublevel set.

Definition 9. *The problem* $\mathcal{P}_N$ *is said to be bounded if the optimum* $\mathcal{L}_N^*$ *is finite.*

From the definition of $\mathcal{P}_N$, any sequence $\{z_0, \cdots, z_N\}$ with z_0 outside $\mathcal{S}_{\mathcal{L}_N}^*$ clearly satisfies $\mathcal{L}_N(z_0) < \mathcal{L}_0(z_0)$. The next section uses this feature, connects the asymptotic properties of the system (12) to the set $\mathcal{S}_{\mathcal{L}_N}^*$, $N \in \mathbb{N}_*$, and proves practical stability results for the system (12).

5. A sufficient condition for global and practical stabilization

Let us begin by recalling some definitions:

Definition 10. *A set* Ω *is said to be positively invariant for the system (12), if for all* $z_0 \in \Omega$, *the state sequence* $z_k \in \Omega,\ k \in \mathbb{N}_*$.

Definition 11. *A trajectory is said to approach a set* Ω, *if the distance* $d(z_k, \Omega) = \min_{\omega \in \Omega} \|z_k - \omega\| \to 0$ *as* $k \to \infty$.

Definition 12. *A closed positively invariant set* Ω *is said to be a global attracting set of (12), if for all initial conditions* $z_0 \in \mathbb{R}^n$, *the trajectories approach* Ω.

Now, some properties concerning the sublevel sets $\mathcal{S}_{\mathcal{L}_N}^*$, $N \in \mathbb{N}_*$ can be established:

Theorem 2. *Under Assumption 1, if the problem* $\mathcal{P}_N$ *is bounded, then* $\mathcal{S}_{\mathcal{L}_N}^*$ *is a global attracting set for all trajectories of the system (12).*

Proof. Consider a trajectory $(z_k)_{k \in \mathbb{N}}$ of the system (12) obtained from an arbitrary initial condition z_0. First, let us prove that $\mathcal{S}_{\mathcal{L}_N}^*$ is a positive invariant set for all infinite subsequences $(z_{pN+r})_{p \in \mathbb{N}}$, $r \in \mathbb{N}_{\leq N-1}$. Suppose that a state $z_r \in \mathcal{S}_{\mathcal{L}_N}^*$ exists such that $V(z_r) \leq \mathcal{L}_N^* < V(z_{r+N})$. Then, z_r leads to a bounded solution for $\mathcal{P}_N$ better than the optimum, which is absurd. Consequently, if $V(z_r) \le \mathcal{L}_N^*$ then $V(z_{pN+r}) \le \mathcal{L}_N^*$, for all $p \in \mathbb{N}$, which means that $\mathcal{S}_{\mathcal{L}_N}^*$ is a positive invariant set for infinite subsequences of the form $(z_{pN+r})_{p\in\mathbb{N}}$, $r \in \mathbb{N}_{\le N-1}$.

Now, let show that $\mathcal{S}_{\mathcal{L}_N}^*$ is a global attracting set for the system (12). Assume that an index $s \in \mathbb{N}_{\le N-1}$ exists such that $z_s \notin \mathcal{S}_{\mathcal{L}_N}^*$, then two cases must be distinguished:

- either an index $p_0 \in \mathbb{N}_*$ exists such that $V(z_{p_0N+s}) \le \mathcal{L}_N^*$ then the positive invariance property of $\mathcal{S}_{\mathcal{L}_N}^*$ implies $\forall p \ge p_0 \in \mathbb{N}_*$, $z_{pN+s} \in \mathcal{S}_{\mathcal{L}_N}^*$;
- or this index p_0 does not exist, then, following the definition of $\mathcal{L}_N^*$, a strict decreasing of the sequence $V(z_{pN+s})$, $\forall p \in \mathbb{N}_*$ is mandatory. The relation $V(z_{pN+s}) > V(z_{(p+1)N+s}) > \mathcal{L}_N^*$ necessarily holds, for $p \in \mathbb{N}_*$. As V is also continuous and the sequence is bounded, a limit ℓ_s exists such that $\lim_{p\to\infty} V(z_{pN+s}) = \ell_s$. Assume $\ell_s > \mathcal{L}_N^*$. By compactness, a subsequence $(z_{\varphi_s(pN+s)})_{p\in\mathbb{N}}$ (with $\varphi_s : \mathbb{N} \to \mathbb{N}$ strictly increasing), that converges to a limit point y_s can be extracted. Moreover, y_s necessarily satisfies $V(y_s) = \ell_s$. Considering y_s^+ the N^{th} iterate of y_s by (12), the relation $V(y_s^+) = V(y_s) = \ell_s$ also holds. However, $V(y_s^+) = V(y_s)$ implies that y_s is a possible argument for $\mathcal{P}_N$ which is absurd. Therefore the sequence $(z_{pN+s})_{p\in\mathbb{N}}$ fulfills $\lim_{p\to\infty} V(z_{pN+s}) = \mathcal{L}_N^*$.

Since all subsequences of the form $(z_{pN+s})_{p\in\mathbb{N}}$, $s \in \mathbb{N}_{\le N-1}$, follow one of the two aforementioned cases, then the whole sequence $(z_k)_{k\in\mathbb{N}}$ approaches $\mathcal{S}_{\mathcal{L}_N}^*$ i.e. $\limsup_{k\to\infty} V(z_k) \le \mathcal{L}_N^*$. Therefore, from Definition 12 and the fact that $\mathcal{S}_{\mathcal{L}_N}^*$ is a positively invariant set, $\mathcal{S}_{\mathcal{L}_N}^*$ is a global attracting set for the system (12). □

Corollary 1. *A sufficient condition for the global and practical stabilization of the sampled switched system (12) is that an integer N exists such that $\mathcal{P}_N$ is bounded.*

6. Relations of inclusion and smallest attracting set

A natural question concerns how the sets $\mathcal{S}_{\mathcal{L}_N}^*$, $N \in \mathbb{N}_*$, are imbricated.

Theorem 3. *Assume problem $\mathcal{P}_N$ is bounded. Then $\forall p \in \mathbb{N}_*$, the problem $\mathcal{P}_{pN}$ is bounded and the following inclusions hold:*

$$\mathcal{S}_{\mathcal{L}_{pN}}^* \subseteq \mathcal{S}_{\mathcal{L}_N}^*.$$

Proof. Following Remark 1, the set of candidates is not empty since for all N, $z_0 = 0$ is always an initial candidate. Consider a sequence z_k, $k \in \mathbb{N}_{\leq pN}$, for any positive integer p, and suppose that $z_{pN} \notin \mathcal{S}^*_{\mathcal{L}_N}$. As in the proof of Theorem 2, the subsequence $V(z_{jN})$, $j \in \mathbb{N}_{\leq p}$, is necessarily strictly decreasing and then, this sequence is not feasible for $\mathcal{P}_{pN}$. It implies that the optimal sequence z^*_k, $k \in \mathbb{N}_{\leq pN}$, for $\mathcal{P}_{pN}$ fulfills $z^*_{pN} \in \mathcal{S}^*_{\mathcal{L}_N}$ (and a fortiori z^*_0). Then $\mathcal{P}_{pN}$ is bounded since $\mathcal{P}_N$ is bounded. □

One might expect a strict inclusion between the sets $\mathcal{S}^*_{\mathcal{L}_N}$. This is not the case in general and it is easy to exhibit an example showing that a relation as $\mathcal{L}^*_N \geq \mathcal{L}^*_{N+1}, \forall N \in \mathbb{N}_*$, cannot hold. However, the upper bound $\mathcal{L}^*_1 \geq \mathcal{L}^*_N$, $\forall N \in \mathbb{N}_*$ remains valid when $\mathcal{P}_1$ is bounded.

Corollary 2. *Assume that a non empty set of integers I exists (necessarily infinite following Theorem 3) rendering $\mathcal{P}_i$, $i \in I$, bounded. Then $\mathcal{S}_\infty = \bigcap_{i \in I} \mathcal{S}^*_{\mathcal{L}_i} = \liminf_{i \to \infty} \mathcal{S}^*_{\mathcal{L}_i}$ is the smallest attracting set of the system (12) given by the set of problems $\mathcal{P}$.*

7. Robust stabilization

In order to investigate the robustness of the proposed sampled-data controllers, an input-to-stable stability property with the sample time as input is given hereafter. Then, in the next subsection, a generalization of this result is provided in the case of parameter uncertainties and non-uniform sampling.

7.1. Input-to-state stability w.r.t. the sample time

Theorem 4. *Under Assumption 1 and assuming for every sampled period T_s, $0 < T_s \leq T_{s_{max}}$, that an integer $N(T_s)$ exists such that the problem $\mathcal{P}_N$ is bounded, the system (12) when $T_s \to 0$ is $0-GAS$.*

In order to establish the proof of this theorem, consider the following definition:

Definition 13 (Supporting hyperplane)**.** *A hyperplane $\mathcal{H}$ of dimension $(n-1)$ is said to support a closed and convex set $M (\subset \mathbb{R}^n)$ on point $y \in \left(\partial M \cap \mathcal{H}\right)$ if M is completely located in one of the two closed half-spaces determined by $\mathcal{H}$ (where ∂M is the boundary of M). If* *a vector λ is inward-pointing normal to this supporting hyperplane $\mathcal{H}$ of M on point y then $\lambda^T y = \inf_{z\in M} \lambda^T z$.*

Proof. First, we show that the system (12) is GAS whatever the chosen switched strategy is, when $T_s \to 0$. For the PWM strategy, since $\lim_{T_s\to 0} v(t, T_s) = u(t) = \kappa(z(t))$ holds almost everywhere, the control law corresponds exactly to the feedback given by the CLF. So, from Assumption 1, the system is $0-$GAS when $T_s \to 0$.

For the N_H-step predictive strategy, the best decreasing value for $V(z_{k+N_H})$ from $V(z_k)$, when T_s vanishes, corresponds to the direction given by $\arg\min_{u\in U} \dot{V}(z_k; u)$ which precisely corresponds to the steepest descent. A first order Taylor expansion can be used to prove this point. So it only remains to prove the fact that the steepest descent strategy is GAS when $T_s \to 0$.

Notice that the instantaneous switching law from a current position z along the trajectory is given by $\arg\min_{u\in U} \dot{V}(z; u)$. Now, if

$$\min_{u\in U} \dot{V}(z; u) \le \dot{V}(z; \kappa(z)) < -\gamma(\|z\|) \tag{18}$$

where γ is a class $\mathcal{K}-$function, the GAS property holds. Note that the existence of the function γ is deduced from the definition of a CLF.

In order to prove the left side inequality of (18), observe that along the trajectory, the derivative of V is given by $\dot{V}(z(t); u) = \frac{\partial V}{\partial z}^T f(z, u)$. For a fixed z, $f(z, u) = A(u)z + B(u)$ is affine w.r.t. u. So, the set defined by $\{f(z, u), u \in \mathrm{co}(U)\}$ matches with the set $\Lambda = \mathrm{co}\{f(z, u), u \in U\}$ and is a closed polyhedron. Let $\lambda = \frac{\partial V}{\partial z}(z)$ and $\mathcal{G}$ its supporting hyperplane on Λ. Denote $u^* = \arg\min_u \lambda^T f(z, u)$. Then, on the point $\rho = f(z, u^*)$, we have $\lambda^T \rho = \min_{w\in\Lambda} \lambda^T w$. Two cases must be distinguished: either ρ is single, then ρ is a vertex of the polyhedron Λ and $u^* \in U$, or ρ is non single, then ρ belongs to an edge or a face of the polyhedron Λ. At least one vertex δ exists such as $\delta \in \partial\Lambda \cap \mathcal{G}$ (Figure 1). So, a control $u^* \in U$ always exists such that (18) holds.

□

Corollary 3. *Assuming for every sampled period T_s, $0 < T_s \le T_{s_{max}}$, an integer $N(T_s)$ exists such that problem $\mathcal{P}_N$ is bounded. Then, the system (12) is input-to-state stable w.r.t. the*

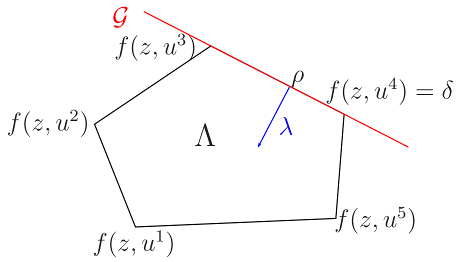

Figure 1: Supporting hyperplane $\mathcal{G}$ on Λ.

class of constant input T_s.

Proof. A classical result [30] states that ISS is equivalent to 0-GAS property (cf. Theorem 4) and asymptotic gain property (the solutions are ultimately bounded) i.e. $\limsup_{k\to\infty} \|z_k(z_0, T_s)\| \leq \gamma(T_s)$ where γ is a $\mathcal{N}_0-$function for $0 < T_s \leq T_{s_{max}}$. From the definition of $\mathcal{S}_\infty$, define the associated level $\mathcal{L}_\infty = \liminf_{i\to\infty} \mathcal{L}_i^*$. If $\mathcal{L}_\infty(T_s)$ is a class $\mathcal{N}_0-$function for $0 < T_s \leq T_{s_{max}}$, the result is given by taking $\gamma(T_s) = \mathcal{L}_\infty(T_s)$. If not, it is always possible to define a class $\mathcal{N}_0-$function γ by choosing $\gamma(T_s) \geq \sup_{0<T\leq T_s} \mathcal{L}_\infty(T)$ since $\sup_{0<T\leq T_s} \mathcal{L}_\infty(T)$ is bounded and nondecreasing for all $T_s \leq T_{s_{max}}$. □

Remark 2. *Note that this ISS result is given for the class of constant input T_s.*

7.2. Non-uniform sampling and parameter uncertainties

An improvement can be obtained if the class of switching laws is relaxed in the following manner: define a minimum (resp. maximum) dwell time δ_{min} (resp. δ_{max}) as the minimum (resp. maximum) duration between two switchings. Let us define the switching time sequence $t_k,\ k \in \mathbb{N}$, with duration constraints:

$$\delta_{min} \leq \tau_k = |t_{k+1} - t_k| \leq \delta_{max}, \tag{19}$$

corresponding to the time instants where the system (4) switches from one mode to another.

Assume also bounded parameter uncertainties θ on the matrices A_i and B_i in (4). Without loss of generality, the uncertainties are given in the form:

$$-\theta_{max} \leq \theta \leq \theta_{max}. \tag{20}$$

We relax the problem $\mathcal{P}_N$ by:

$$\mathcal{P}_N(\delta_{min}, \delta_{max}, \theta_{max}) : \quad \max_{z_0, \theta, \tau_k} \; \mathcal{L}_N(z_0) \tag{21}$$

$$\text{s.t. } z_{k+1} = A_s(\star) z_k + B_s(\star), \; k \in \mathbb{N}_{\leq N-1} \tag{22}$$

$$\delta_{min} \leq \tau_k = |t_{k+1} - t_k| \leq \delta_{max} \tag{23}$$

$$-\theta_{max} \leq \theta \leq \theta_{max} \tag{24}$$

$$\mathcal{L}_N(z_0) \geq \mathcal{L}_0(z_0) \tag{25}$$

where $(\star) = (\tau_k, \theta, \kappa_s)$.

Remark 3. *κ_s remains unchanged and based on the unperturbed model (4).*

Remark 4. *In this relaxed problem, the optimization depends on the initial condition z_0, the switching time sequence t_k and the parameter uncertainties θ. Notice that the set of constraints (22) is now time dependent following the integration duration τ_k.*

Remark 5. *All the results concerning the attracting sets $\mathcal{S}^*_{\mathcal{L}_k}$ remain valid since the given proofs do not depend on how the closed loop sequence $(z_k)_{k \in \mathbb{N}}$ is obtained from an initial guess z_0.*

Property 1. *The optimal value of $\mathcal{P}_N(\delta_{min}, \delta_{max}, \theta_{max})$ is non-decreasing w.r.t. δ_{max} or θ_{max} and non-increasing w.r.t. δ_{min}.*

Proof. It is clear that an optimal argument $(z_0^*, \theta^*, \tau_k^*, \; k \in \mathbb{N}_{\leq N-1})$ for $\mathcal{P}_N(\delta_{min}, \delta_{max}, \theta_{max})$ is also an admissible argument for $\mathcal{P}_N(\delta_{min}, \delta_{max} + \delta, \theta_{max})$ for all $\delta \geq 0$. The rest of the announced properties is also trivially established. □

Corollary 4. *Assume $(\Delta_{max}, \Theta_{max}) > 0$ exists such that for all $\delta_{min} > 0$ ($\Delta_{max} \geq \delta_{max} \geq \delta_{min}$), an integer $N(\Delta_{max}, \delta_{min}, \Theta_{max})$ exists such that the problem $\mathcal{P}_N(\delta_{min}, \Delta_{max}, \Theta_{max})$ is bounded. Then the system (12) with relaxed switching laws (19) and parameter uncertainties (20), is ISS with input (τ, θ) corresponding to the switching duration sequence $\tau = (\tau_0, \tau_1, \cdots)$ and the parameter uncertainties θ.*

Proof. The proof uses, as in Corollary 3, the equivalence between ISS and (0-GAS+AG) properties [30]. Taking $\delta_{min} \leq \delta_{max} \to 0$ and $\theta_{max} \to 0$, 0-GAS property expressed in Theorem 4 remains valid with this class of relaxed switching laws and bounded uncertainties. The AG property i.e. $\limsup_{k\to\infty} \|z_k(z_0, \theta, \tau_i, i \in \mathbb{N}_{\leq k-1})\| \leq \gamma(\|(\tau,\theta)\|_\infty)$ with $\|(\tau,\theta)\|_\infty = \max\ (\sup_k \tau_k, \theta)$, follows from the fact that the function $\phi(\delta_{max}, \theta_{max}) = \sup_{0<\delta_{min}\leq\delta_{max}} \mathcal{L}_\infty(\delta_{min}, \delta_{max}, \theta_{max})$ is bounded and non-decreasing w.r.t. δ_{max}, for all $\delta_{max} \leq \Delta_{max}$ and respectively θ_{max}, for all $\theta_{max} \leq \Theta_{max}$. Then, it is always possible to define a class $\mathcal{N}_0-$function γ by choosing for example $\gamma(s) \geq \phi(s, s)$ with $s = \|(\tau,\theta)\|_\infty$. □

8. Computational aspects

This section discusses some computational aspects that can be encountered when one solves the optimization problems $\mathcal{P}_N$. Since no assumption is made about the known CLF and since the state feedback is generally a discontinuous function of the state, the optimization problems $\mathcal{P}_N$ are non-linear and non-smooth.

Nevertheless, if the predictive or steepest strategies are considered, the feedback law leads to a partition of the state space w.r.t. the control values $u(z) \in U$. Then, the smoothness requirement can be achieved if $\mathcal{P}_N$ is solved for every fixed switching sequences. In this context, additional constraints related to the chosen switching strategy must be added. Precisely for a fixed sequence:

- Steepest strategy: at each time t_k, the control u_k* of the chosen sequence has to verify $2^m - 1$ constraints:

$$\dot{V}(z_k; u_k*) \leq \dot{V}(z_k; u), u \in U, u \neq u_k*,\ k \in \mathbb{N}_{\leq N-1}. \tag{26}$$

  Therefore, for a fixed sequence of length N, $N(2^m - 1)$ constraints are added to $\mathcal{P}_N$. The problem is clearly smooth in this case, if the CLF is.

- N_H-predictive strategy: at each time t_k, the control u_k* of the chosen sequence has to verify the constraints:

$$\min_{u_k*, u_{k+1}, \cdots, u_{k+N_H-1} \in U^{N_H}} V(z_{k+N_H}) \leq \min_{u_k, u_{k+1}, \cdots, u_{k+N_H-1} \in U^{N_H}} V(z_{k+N_H}) \tag{27}$$

with $u_k \neq u_k*,\ k \in \mathbb{N}_{\leq N-1}$. The left minimization is done over $2^{m(N_H-1)}$ elements and the right one over $2^{mN_H} - 2^{m(N_H-1)}$ elements for each N element of the fixed sequence. As the left and right terms are continuous but not differentiable everywhere, a direct search algorithm is needed in order to solve the problem (except the case $N_H = 1$: where the smoothness requirement is achieved).

This caution can be avoided if, at each time t_k, the sequence $u_k*, u_{k+1}, \ldots, u_{k+N_H-1}$ in the left term is fixed in advance. This procedure implies to define a set of additional optimization problems corresponding to all possible sequences at all time t_k. Then, the total number of optimization problems becomes 2^{mNN_H}.

- PWM strategy: since the state feedback laws $u(z)$ are generally discontinuous functions of the state, optimization problems are non-smooth. Nevertheless, there are two cases where $\mathcal{P}_N$ can be solved without numerical issues: if $u(z)$ is continuous or if $u(z) \in U$ almost everywhere and allows to define a partition of the state space. In this case, the same previous methodology can be applied.

Now, we have shown that the smoothness requirement can be met. It can be underlined that, for many practical cases, quadratic Lyapunov function candidates can be exhibited. For example, in (4) as $B(u_{ref}) = 0$, if $A(u_{ref})$ is Hurwitz then there exists a quadratic Lyapunov function associated to the system $\dot{z} = A(u_{ref})z$ which can be used with one of the given strategies. Passivity based control is another way to get such quadratic CLF. It means that the objective function and the constrains are quadratic functions. So, a quadratically constrained quadratic program (QCQP) can be used. QCQP is a wide-studied problem in the optimization literature having a large number of applications [39]. Relaxations of QCQP based on semidefinite programming (SDP) and the reformulation-linearization technique (RLT) can be an efficient way to solve it. Global optimization solvers, such as GloptiPoly [40], that solve non convex global optimization problem of minimizing a multivariable polynomial function subject to polynomial inequality, equality or integer constraints, are particularly efficient for QCQP. GloptiPoly allows to solve a series of convex relaxations of increasing size, whose optima are guaranteed to converge monotonically to the global optimum. The result is an

Table 1: Compared computation time

Solver	Computation Time (s)		
	$\mathcal{L}_1^*$	$\mathcal{L}_4^*$	$\mathcal{L}_6^*$
NL Matlab	0.98	399.2	4078
Gloptipoly	1.36	3.91	14.9

extremely fast solver. A comparison between the Non-Linear Solver `fmincon` of Matlab and the solver `GloptiPoly` is performed on the example given in the next section. The results are summarized for the steepest strategy case in table (1).

Now, if parameter uncertainties and non uniform sampling are taken into account, the level set computed matches to the worst case for the dynamics. In this case, the program is not a QCQP but accurate polynomial approximations of (22) can be obtained using Taylor expansion of $e^{(A+\Delta A)(T_s+\delta T_s)}$ where ΔA and δT_s define the uncertainties. More accurate, a polytopic approximation of the dynamic like in [26] could be another way to deal with this issue. If a polytopic approximation is used then the problem becomes again a QCQP. So, the proposed solver remains adapted for both cases.

9. Application

9.1. DC-DC converter description

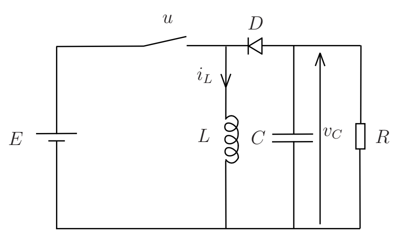

Figure 2: Buck-boost converter

Consider a buck-boost converter (Figure 2) whose state equation in continuous conduction mode (the current passing through the inductance never falls to zero) is given by:

$$\dot{x} = A_0 x + B_0 + u(A_1 x + B_1)$$

where $x = [i_L, v_C]^T$ and

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} \qquad A_1 = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix}$$
$$B_0 = [\frac{E}{L}, 0]^T \qquad B_1 = [-\frac{E}{L}, 0]^T$$

with $R = 50\ \Omega$, $C = 220\ \mu F$, $L = 20\ mH$ and $E = 6\ V$.

Let the target be $x_{ref} = [0.24, -6]^T \in X_{ref}$ corresponding to $u_{ref} = 0.5$. As $A(u_{ref}) = A_{ref} = A_0 + u_{ref}A_1$ is Hurwitz and as $B(u_{ref}) = 0$, the solutions $P = P^T > 0$ of $A_{ref}^T P + PA_{ref} + Q = 0$ with $Q = Q^T > 0$ allow to define quadratic CLFs $V(z) = z^T Pz$ for the system (4). Taking $Q = 180 \times Id$, one gets: $P = \begin{bmatrix} 91.05 & 0.04 \\ 0.04 & 1 \end{bmatrix}$. In the next two subsections, the results of the proposed approaches are illustrated through the steepest and predictive strategy. The sample time is $T_s = 2.5.10^{-5}$s.

9.2. Attracting set estimations for the sampled strategies

Figure 3 shows a system trajectory using the steepest descent feedback law and attracting sets determined by $\mathcal{P}_N$ for $N = 1$ (red dashed line) and $N = 2$ (magenta solid line). Clearly, $\mathcal{S}^*_{\mathcal{L}_2}$ is an accurate approximation of the all system trajectories. Using Glotipoly software, the computation times are respectively 1.36 s and 0.97 s.

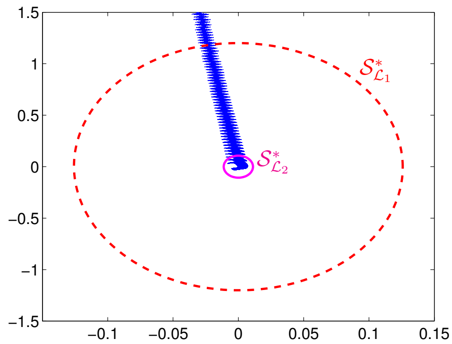

Figure 3: Trajectory in the state-space and attracting sets for $N = 1$ and $N = 2$

For a receding horizon N_H fixed to $N_H = 2$, Figure 4 shows the case of the predictive strategy. The black solid ellipse is the estimation for a sequence of length $N = 1$. The estimation is still large comparing to the limit cycle. For a sequence of length $N = 8$, a better approximation is obtained. The computation times, still using Glotipoly are respectively 1.6 s and 29.10^3 s.

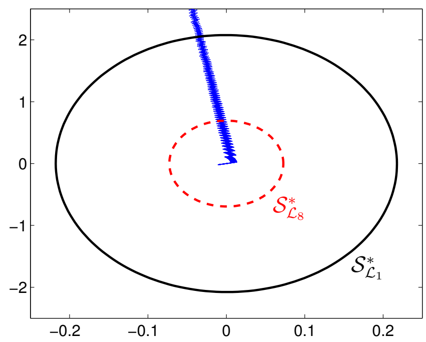

Figure 4: Trajectory in the state-space and attracting sets for $N = 1$ and $N = 8$

Figure 5 represents the evolution of $\mathcal{L}^*_N$ in function of N for the steepest (solid blue line) and the predictive strategies (dashed red line). Observe that the relations $\mathcal{S}^*_{\mathcal{L}_{pN}} \subseteq \mathcal{S}^*_{\mathcal{L}_N}$, $\forall p \in \mathbb{N}_*$ hold as stated in Theorem 3. While the relation $\mathcal{L}^*_N \geq \mathcal{L}^*_{N+1}$ does not hold in general. Figure 5 also shows that the above given approximations of the attracting sets are accurate although in the case of predictive control, this estimation appears not particularly tight around the cycle. This can be clearly justified by the fact that there exists at least one sequence starting inside the sublevel set that reaches the level. This sequence is obviously the solution of $\mathcal{P}_N$. In view of the evolution of the curve in Figure 5, an increase of N seems not to lead to a better estimation of $\mathcal{S}_\infty$.

In Figure 6, the evolution of $\mathcal{L}^*_2$ w.r.t. T_s is drawn for both strategies. This figure clearly illustrates the ISS property of the system. Finally, Figure 7 shows the exponential growth of the computation time for the two strategies.

9.3. Robust attracting set estimations for sampled strategies

Suppose now that all parameters L, R, E, C are known with 5% of uncertainties and that the sample time T_s is time dependent with variation of 5% around its nominal value. The problem $\mathcal{P}_N(\delta_{min}, \delta_{max}, \theta_{max})$ gives the attracting set in the worst case. Figure 8 shows in solid line the level sets corresponding to $\mathcal{L}^*_N$ for $N = 1$ (red line) and $N = 8$ (black line) and

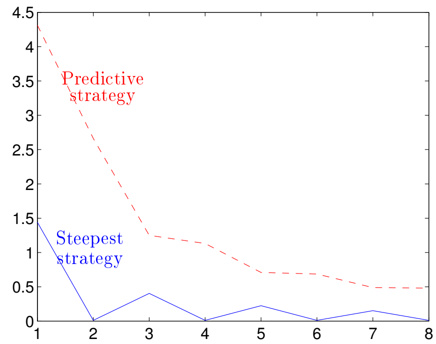

Figure 5: $\mathcal{L}_N^*$ versus N

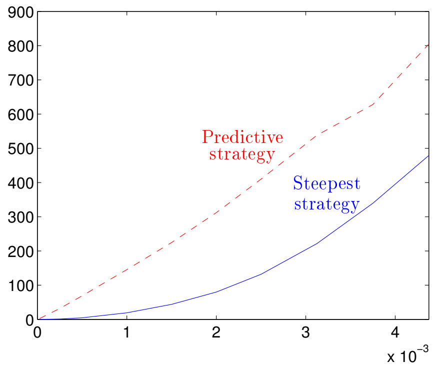

Figure 6: $\mathcal{L}_2^*$ versus $T_s \in [2.5, 4500]\mu$s

in dashed line the respective level sets for the system without uncertainties. This figure also shows two trajectories, simulated with a uniformly distributed random law for the sample time variations, and two parameters sets inside the 5% of uncertainties.

It is worthy noticing that, as expected, the attracting sets for the system with parameters variations are bigger than the ones for the nominal system. However, the boundedness of the optimization problem guarantees the stability of the perturbed system.

10. Conclusion

In this paper, robust stability for the class of switched affine systems has been investigated. Based on the existence of a CLF for the relaxed control problem, sampled switched strategies have been proposed to stabilize the switched affine system around an operating point.

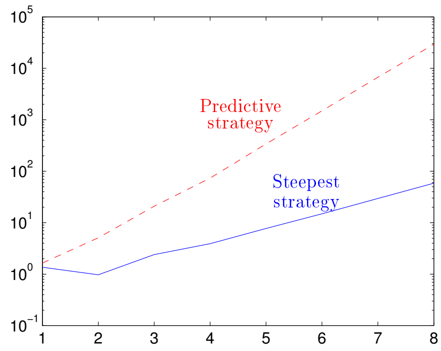

Figure 7: Computation time (in seconds) versus N

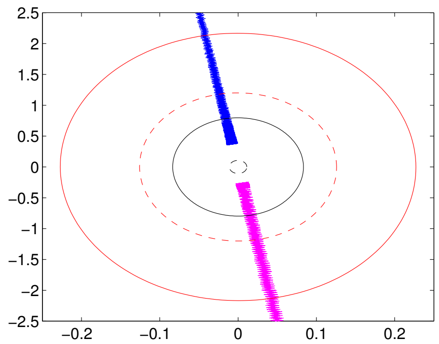

Figure 8: Trajectories in the state-space and attracting sets for $N = 1$ and $N = 8$ in case of uncertainties

The proposed framework allows to compute tight global attracting sets for the whole system trajectories, by solving a set of constrained optimization problems. Numerical aspects have been discussed and it has been shown that practically, the optimization problems reveal to be QCQP or non convex polynomial optimization problems for which efficient global optimization solvers exist. In addition, ISS results with respect to the sample time and the parameter uncertainties are formulated. In doing so, some stability margins are guaranted.

The numerical illustration given on a buck-boost converter shows that quadratic CLF can be easily designed for DC-DC converters. Applying the steepest or predictive strategies, numerical results also showed that it is not necessary to consider a high order in $\mathcal{P}_N$ to get a good accuracy in the over-approximation of $\mathcal{S}_\infty$.

As future work, a comparison between optimal control and the given switching laws would be of interest in order to measure the ratio performances over design easiness.

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