AN OPTIMAL CONTROL APPROACH FOR HYBRID SYSTEMS

P. RIEDINGER, C. IUNG, F. KRATZ

Abstract. In this paper optimal control for hybrid systems will be discussed. While defining hybrid systems as causal and consistent dynamical systems, a general formulation for an optimal hybrid control problem is proposed. The main contribution of this paper shows how necessary conditions can be derived from the maximum principle and the Bellman principle. An illustrative example shows how optimal hybrid control via a set of Hamiltonian systems and using dynamic programming can be achieved. However, as in the classical case, difficulties related to numerical solutions exist and are increased by the discontinuous aspect of the problem. Looking for efficient algorithms remains a difficult and open problem which is not the purpose of this contribution.

1. Introduction

In the automatic control area two kinds of dynamical systems have been separately considered: continuous time and discrete event systems depending on the nature of the variables. "Discrete variables" means variables taking their values in a discrete set and "continuous variables" means variables taking their values in a continuous set. The behavior of such entities is described by dynamical systems such as automata and differential equations respectively. Each of these systems has indeed specific analysis tools defined with discrete and continuous metrics. Physical systems do not have this strict separation and there is often more or less interaction between discrete and continuous variables. Think for example of threshold phenomena which change the continuous dynamic, or of a batch process including continuous dynamical systems. This intricate interaction is well-known in electromechanical systems where averaging methods are commonly used to approximate the behavior of such systems [1, 2].

The notion of hybrid dynamical systems appears in [3] where the necessity to develop a theory melting continuous and discrete signals is stressed. Today, it is admitted that hybrid systems are dynamical systems in which discrete and continuous variables interact. Generally speaking it can be said that the discrete part plays the role of the supervisor of the continuous part. When a discrete event occurs, it may change the continuous dynamic and thus discontinuities in the vector field and /or in the continuous state may appear. This leads to a non linear behavior even if the basis models are linear.

In the last decade, there has been a great interest in studying such systems with regard to their potential applications as well as to the theoretical point of view [4]-[10]. Despite an extensive effort, it appears that the wide class of hybrid systems excludes a unified approach but several theoretical frameworks according to the objectives and the task are more realistic [11]. It could be further added that the questions relevant to the continuous aspect deal more with the non smooth analysis and a poor discrete part. In other words, the control of the continuous dynamic is determined by a switching between continuous models [12]. On the other hand, verification tools, supervisory control and computer science are used in hybrid automata but generally with a poor continuous dynamic such as integrators [13].

In this paper, an optimization problem for hybrid systems is investigated. Hybrid systems are introduced essentially as causal and consistent dynamical systems and the purpose is to look for

Key words and phrases. Hybrid Systems, Hybrid Automata, Optimal Control, Maximum Principle, Dynamic programming.

an optimal control law including discrete transitions and minimizing a given criterion. In [14], an approach mixing dynamic programming and Hamilton-Jacobi-Bellman equations is proposed to solve a hybrid optimal problem. A comment on this paper related to the proposed hysteresis example shows that the analytical solution can be calculated using the Maximum Principle and explains the numerical difficulties that can often be encountered in the optimal control of Hybrid Systems [15]. A stochastic point of view is expressed in [16, 17].

This paper is organized as follows: In section 2, after a brief description of optimal control problems, the Maximum Principle which states necessary conditions of optimality is recalled. Then a hybrid framework is proposed and a general hybrid optimal control problem is formulated. In this formulation all hybrid phenomena such as state jumps, switching dynamics and variable dimensions of the state space as well as autonomous and controlled events, are included.

Special attention is given to this problem and the main result shows that the Maximum Principle of Pontryagin yields to a complete set of necessary conditions to study hybrid optimal control problems. As smoothness requirement for continuous vector fields and boundary conditions are used, the result shows clearly how the discrete part interacts with the continuous one and avoids tedious explanations if no differentiability are met. Note that neither the existence of a solution nor algorithms to tract the solution are addressed in the paper.

In section 3, a simple hybrid control example is given to explain and to show how the necessary conditions can be applied using a mixed dynamic programming and Hamiltonian approach.

Readers familiar with the Maximum Principle of Pontryagin could go directly to section 2.2.

Notations: We denote by:

- $\underline{q} = \{1, ..., q\}$ an enumerate integer set,
- $\bar{Y}$ the closure of Y in X, for a given set Y in a space X
- ∂Y the boundary of Y.

2. Optimal Control of Hybrid Systems: A Pontryagin approach

In this section, the Maximum Principle of Pontryagin (MP) [20] is presented as a tool which gets a complete set of necessary conditions for solving optimal control problems in hybrid systems framework.

2.1. **Preliminary : the continuous case.** Optimal control refers to the best way to force a system to change from one state to another.

Problem 1: *Let us consider dynamical systems which can be represented by a differential equation on time interval* $[a, b]$ *and initial conditions such as:*

$$\dot{x}(t) = f(x(t), u(t), t), \tag{2.1}$$

$x(a) = x_0$ *with state* $x(t) \in \mathbb{R}^n$*, and control input* $u(t) \in U$ *a subset included in* $\mathbb{R}^m$*,* f *is given and the point* x_0 *is chosen within a given set* S_0*. Then the problem can be stated as follows:*

choose x_0 *and* u *so that the functional or cost function*

$$J = \phi(x(b), b) + \int_a^b L(x(t), u(t), t)dt \tag{2.2}$$

is minimized, subject to the condition that the final state $x(b)$ *lies in a given set* S_1*.*

Note that the final time b can be free. The cost function could reflect the consumption of energy, time and so on. Finally, and more generally, state constraints $c(x(t), t) \geq 0$ can be allowed. This will not be mentioned so as to avoid a tedious explanation. However readers may consult the following references about optimal control [21, 22, 23].

In many cases, optimal control problem can be viewed as the classical calculus of variations but there is no equivalence due to smoothness requirements of the calculus of variations. When Pontryagin introduced the Maximum Principle, he explained that "the calculus of variations offers no solutions for a whole range of problems of importance for modern technology". He mentioned in particular that the case of a closed control domain cannot meet Weierstrass's condition which occurs in the calculus of variations. So, as it is often the case, optimal controls taking values at the boundary ∂U (bang bang control) are not tractable with the calculus of variations.

Nowadays, progress in non-smooth analysis gives rise to an extended version of optimization problems in which the consideration of non-differentiability can be expressed, see [24].

Now, the following assumption is made,

Assumption (A_0)

i. The control domain U is a bounded subset of $\mathbb{R}^m$.
ii. The vector fields $f(x, u, t)$ and $L(x, u, t)$ are continuous functions on the direct product $\mathbb{R}^n \times \bar{U} \times [a, b]$ and continuously differentiable with respect to the state variable and the time variable.
iii. The boundaries ∂S_0, ∂S_1 are described by a set of continuously differentiable equality constraints $\partial S_0 := \{x, t : C_0(x, t) = 0\}$ and $\partial S_1 := \{x, t : C_1(x, t) = 0\}$ where $C_i : \mathbb{R}^n \times [a, b] \longrightarrow \mathbb{R}^{r_i}$, $(r_i < n)$ $i = 0, 1$,
iv. ϕ is continuous and continuously differentiable.

Necessary condition: The Maximum Principle (smooth version) [20]

Define the Hamiltonian function as

$$H(p_0, p(t), x(t), u(t), t) = p^T(t) f(x(t), u(t), t) - p_0 L(x(t), u(t), t) \tag{2.3}$$

and the Hamiltonian system as

$$\dot{x}(t) = \frac{\partial H}{\partial p}(p_0, p(t), x(t), u(t), t) \tag{2.4}$$

$$-\dot{p}(t) = \frac{\partial H}{\partial x}(p_0, p(t), x(t), u(t), t) \tag{2.5}$$

where p_0 is a positive constant ($p_0 \geq 0$). It can be noted that (2.4) is equivalent to (2.1).

Theorem 2.1. *If $u^*(.)$ and $x^*(.)$ are respectively an admissible optimal control and the corresponding trajectory for the problem 1 with assumption A_0, then there exists an absolutely continuous curve $p^*(.)$ and a constant $p_0^* \geq 0$, $(p_0^*, p^*(t)) \neq (0, \mathbf{0})$ on $[a, b]$, such that the uplet $(p_0^*, p^*, x^*, u^*)(.)$ satisfies the Hamiltonian system (2.4)-(2.5) almost everywhere as well as the following maximum conditions:*

for almost all $t \in [a, b]$,

$$H(p_0^*, p^*(t), x^*(t), u^*(t), t) \quad = \sup_{u \in U} H(p_0^*, p^*(t), x^*(t), u, t) \tag{2.6}$$

there exists a vector π_a^* such that:

$$\begin{aligned} p^*(a) &= \frac{\partial C_0^T}{\partial x}(x^*(a), a)\pi_a^* \\ H^*(a) &= -\frac{\partial C_0^T}{\partial t}(x^*(a), a)\pi_a^* \end{aligned} \tag{2.7}$$

there exists a vector π_b^* such that:

$$\begin{aligned} p^*(b) &= -p_0^* \frac{\partial \phi}{\partial x}(x^*(b), b) + \frac{\partial C_1^T}{\partial x}(x^*(b), b)\pi_b^* \\ H^*(b) &= p_0^* \frac{\partial \phi}{\partial t}(x^*(b), b) - \frac{\partial C_1^T}{\partial t}(x^*(b), b)\pi_b^* \end{aligned} \tag{2.8}$$

Remark 2.2. (1) The terms "almost everywhere" or "almost all t" can be omitted if the control is restricted to the class of piecewise continuous control functions. Then conditions (2.4)-(2.5) hold everywhere except perhaps for a finite number of points but at those points the one-sided limits exist and are equal [25].
(2) In most applications $p_0 \neq 0$, in which case one may take $p_0 = 1$ (normalization).

(3) (2.7)-(2.8) are called the transversality conditions. It express that the Hamiltonian and the costate must be transverse to the constraint.
(4) If f and L are independent of t, then H is constant along an optimal trajectory.

The Trajectories that satisfy (2.6), (2.7) and (2.8), are called extremal. These conditions provide a set of candidates for optimal control but say nothing about the existence question. This is why the MP assumes that an optimal control exists. In addition a non empty set of candidates without any solution can exist.

2.2. **Switched-Hybrid optimal control.**

2.2.1. *A Class of hybrid systems.* In order to bring up discrete phenomena into the continuous dynamics, a controlled-autonomous hybrid system in accordance with the terminology used by Branicky [14] may be defined as follows:

For a given finite set of discrete state $\underline{Q} = \{1, ..., Q\}$, there is an associated collection of continuous dynamics defined by differential equations

$$\dot{x}(t) = f_q(x(t), u(t), t) \tag{2.9}$$

where

- $q \in \underline{Q}$
- the continuous state $x(.)$ takes its values in $\mathbb{R}^{n_q}$ $(n_q \in \mathbb{N})$,
- the continuous control $u(.)$ takes its values in a control set U_q included in $\mathbb{R}^{m_q}$ $(m_q \in \mathbb{N})$.
- the vector fields f_q are supposed defined on $\mathbb{R}^{n_q} \times \mathbb{R}^{m_q} \times [a, b]$, $\forall q \in \underline{Q}$

Here, f_q, $q \in \underline{Q}$ meets classical hypothesis that guarantee the existence and uniqueness of the solution (i.e. f_q is globally lipschitz continuous). Note that the state space as well as the control space depend on the discrete state q and have variable dimensions with respect to it.

The discrete dynamic is defined using a transition function ν of the form:

$$q(t^+) = \nu(x(t^-), q(t^-), d(t), t) \tag{2.10}$$

with $q(.)$ the discrete state $(q(t) \in \underline{Q})$ and $d(.)$ the discrete control $(d : [a, b] \to \underline{D}$ where $\underline{D} = \{1, ..., D\}$ is a finite set). ν is a map from $X \times \underline{Q} \times \underline{D} \times [a, b]$ to $\underline{Q}$ where X is a subset of $\mathbb{R}^{n_1 + \dots + n_Q}$. More precisely, the set X takes the form:

$$X = \bigcup_{j=1}^{Q} \{0\}^{\sum_{k=1}^{j-1} n_k} \times \mathbb{R}^{n_j} \times \{0\}^{\sum_{k=j+1}^{Q} n_k} \subset \mathbb{R}^{n_1 + \dots + n_Q}.$$

For convenience, we should have to replace $x(t)$ in the above equation (2.10) by

$$\tilde{x}(t) = (0, \dots, 0, x(t), 0, \dots, 0) \in X$$

with $x(t) \in \mathbb{R}^{n_{q(t^-)}}$.

The discrete variable $q(.)$ is a piecewise constant function of the time. This mention is indicated by t^- and t^+ in (2.10) meaning just before and just after time t.

The value of the transition function ν depends on two kinds of discrete phenomena which can affect the evolution of $q(.)$: firstly, some changes in the discrete control $d(.)$ and secondly some boundary conditions on (x, t) of the form $C_{(q,q')}(x(t), t) = 0$ which modify the set of attainable discrete states. It is supposed that $\forall (q, q') \in \underline{Q}^2$, $C_{(q,q')} : \mathbb{R}^{n_q} \times [a, b] \longrightarrow \mathbb{R}^{r_q}$, $(r_q < n_q)$. These boundary conditions can represent thresholds, hysteresis, saturations, time delay between two switches, ... and refer to the manner the continuous dynamic interacts with the discrete part.

A set of jump functions

$$x(t^+) = \Phi_{(q,q')}(x(t^-), t) \tag{2.11}$$

which resets the continuous state when a discrete transition occurs from q to q' is also considered. For the remainder, we assume that the function $\Phi_{(q,q')} : \mathbb{R}^{n_q} \times [a,b] \to \mathbb{R}^{n_{q'}}$, $\forall (q,q') \in \underline{Q}^2$ is sufficiently regular i.e. continuously differentiable.

Now starting from a position (x_0, q_0), the continuous state $x(.)$ evolves in $\mathbb{R}^{n_{q_0}}$ according to the continuous control $u(.)$ and the state equation $\dot{x}(t) = f_{q_0}(x(t), u(t), t)$.

If at time t_1, $x(.)$ reaches boundary condition $C_{(q_0,q_1)}(x(t_1), t_1) = 0$ and/or a change in the discrete control $d(.)$ is produced which lead to a new discrete state $q_1 = \nu(x(t_1^-), q_0, d(t_1), t_1)$ then the continuous state jumps in $\mathbb{R}^{n_{q_1}}$ to $x(t_1^+) = \Phi_{(q_0,q_1)}(x(t_1^-), t_1)$ and evolves with a new vector field $f_{q_1}(x(t), u(t), t)$. And so on. Then, an hybrid trajectory on time interval $[a,b]$ can be viewed as the data of a piecewise constant function $q(.)$ and a piecewise continuous function $\widetilde{x}(.)$ in $X \subset \mathbb{R}^{n_1 + \cdots + n_Q}$ obtained according to equations (2.9)(2.10) and (2.11).

Equations (2.9)(2.10) and (2.11) denote a hybrid system and must be understood as a causal and consistent dynamical system. It is causal in the sense that the knowledge of the state $(x(t), q(t))$ at given time t is sufficient to built the trajectory for future time t. Consistence means unicity of the trajectory for a given admissible control and initial conditions. So, we do not deal with the well-posedness of the problem [26]. And we assume that the system is designed in such a way that it neither tolerates several discrete transitions at a given time nor zeno phenomena i.e. an infinite switching accumulation points.

This hybrid model covers a very large class of hybrid phenomena such as systems with switched dynamics, jumps on the states and variable continuous state space dimensional. It takes into account autonomous and/or controlled events.

For example, it is not difficult to see that this state representation can cover Branicky's hybrid systems or hybrid automata. In the case of Branicky's modelling, boundary conditions can be used to define the autonomous, controlled and destination sets. In the second case, let us consider the following hybrid automata (Figure 1) as an illustrative example.

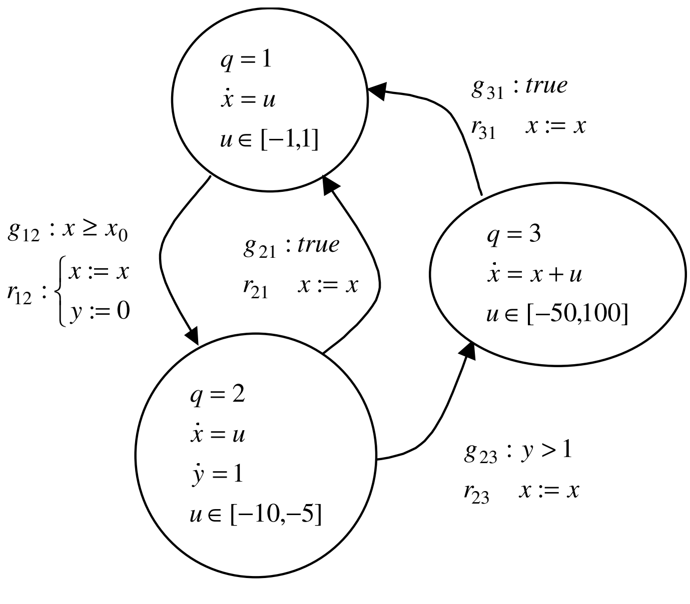

FIGURE 1. Hybrid Automata

There are two one-dimensional systems for $q = 1$ and $q = 3$ and one two-dimensional system for $q = 2$. From $q = 1$, we can pass to $q = 2$ if the set condition $g_{12} : x \geq x_0$ is satisfied and the continuous state is reset according the jump condition r_{12}. For $q = 2$, we can see that y is a counter which forbids a transition to the mode $q = 3$ as long as a time delay is not reached. Jumps to $q = 1$ from $q = 2$ or $q = 3$ are always allowed since the guards g_{21} and g_{31} are true.

Now, the discrete transition function ν can be described by the following (table 1).

TABLE 1. q^+ *versus* $q^-, d, g(x)$

	$d=1$	$d=2$	$d=3$
$q^-=1$	$q^+=1$	$if\ g_{12}=true\quad q^+=2$ $else\quad q^+=1$	$q^+=1$
$q^-=2$	$q^+=1$	$q^+=2$	$if\ g_{23}=true\quad q^+=3$ $else\quad q^+=2$
$q^-=3$	$q^+=1$	$q^+=3$	$q^+=3$

For example, this table can be read as follow : If at a time t, the value of discrete control $d(.)$ is set to 1 from any discrete state $q(t^-)$ =1,2 or 3 then a switch is produced to $q(t^+) = 1$ since this transition is always allowed from anywhere. A contrario, if $d(.)$ is set to 2 and the discrete state is $q(t^-) = 1$ the switch is only produced if the guard g_{12} is true or when it will become.

2.2.2. *Problem formulation.* Consider a hybrid system (2.9)(2.10) and (2.11) under the following assumptions:

- Assumption A_0 (see section 2.1) is supposed to be true for each subsystems (2.9)
- Assumption A_1: $\forall(q, q') \in \underline{Q}^2$ the jump functions $\Phi_{(q,q')}(.,.)$ and the boundary constraint $C_{(q,q')}(.,.)$ are continuous and continuously differentiable.

Let $[t_0 = a, t_1, ..., t_i, ..., t_m = b]$ and $[q_0, q_1, ..., q_i, ..., q_m]$ (recall that $q_i \in \underline{Q}$ and b can be infinite as well as m in this case) be the sequence of switching times and the associated mode sequence associated to the control $(u, d)(.)$ on time interval $[a, b]$.

Thus, a hybrid criterion can then be introduced as:

$$J(u,d) = \int_a^b L_{q(t)}(x(t), u(t), t)dt \tag{2.12}$$

$$= \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} L_{q_i}(x(t), u(t), t)dt \tag{2.13}$$

where $q_i \in \underline{Q}$. For all $q \in \underline{Q}$, we assume that L_q is defined and continuous on the direct product $\mathbb{R}^{n_q} \times \bar{U}_q \times [a, b]$ and continuously differentiable in the state variable and in time. L_q depends on the discrete state. So, different criteria associated with each mode are possible.

The optimal control $(u, d)(.)$ is the control that minimizes cost function J over time interval $[a, b]$.

In our case, the vector field and the integral criterion are not continuous that's why the above necessary conditions are not adapted. The topic is now to derive necessary conditions about the optimal control and trajectory for which optimal switching times are included.

2.2.3. *Necessary conditions.* The MP [20] is a principle which can be applied to various optimization problems with different specific and technical conditions (boundary, free final time, constraints, etc.) [27]. Hence, according to the regularity assumptions on the data, several versions of MP can be stated. In our formulation (2.9), (2.10), (2.11) and (2.13) allow a direct use of a smooth version of MP with an additional dynamic programming argument in order to consider discrete transitions. To this purpose, let us define the Hamiltonian function associated to each mode q as:

$$H_q(p, p_0, x, u, t) = p^T f_q(x, u, t) - p_0 L_q(x, u, t) \tag{2.14}$$

and the Hamiltonian system as

$$\dot{x} = \frac{\partial H_q}{\partial p} \qquad \dot{p} = -\frac{\partial H_q}{\partial x} \tag{2.15}$$

where p_0 is a positive constant ($p_0 \geq 0$).

Now ,we have the following theorem:

Theorem 2.3. *If $(u^*, d^*)(.)$ and $(x^*, q^*)(.)$ are respectively an admissible optimal control and the corresponding trajectory for the problem (2.9), (2.10), (2.11) and (2.13), then there exists a piecewise absolutely continuous curve $p^*(.)$ and a constant $p_0^* \geq 0$, $(p_0^*, p^*(t)) \neq (0, \mathbf{0})$ on $[a, b]$, so that:*

1. the sextuplet $(p^*, p_0^*, x^*, q^*, u^*, d^*)(.)$ satisfies the associated Hamiltonian system (2.15) almost everywhere (a.e.)

2. at any time t, the following maximum condition holds for $(p^*, p_0^*, x^*, q^*)(t)$:

$$H_{q^*}(p^*, p_0^*, x^*, u^*, t) = \sup_{u \in U_{q^*}} H_{q*}(p^*, p_0^*, x^*, u, t) \tag{2.16}$$

3. at switching time t_i, $i = 0, \ldots, m$, the following transversality conditions are satisfied: there exist a vector π_i^* such that

$$p^*(t_i^-) = \begin{bmatrix} [\frac{\partial \Phi_{(q_{i-1},q_i)}}{\partial x}(x(t_i^-), t_i)]^T & 0 \end{bmatrix} \nabla V_{q_i}(x(t_i^+), t_i) + [\frac{\partial C_{(q_{i-1},q_i)}}{\partial x}(x(t_i^-), t_i)]^T \pi_i^* \tag{2.17}$$

$$H^*(t_i^-) = -\begin{bmatrix} [\frac{\partial \Phi_{(q_{i-1},q_i)}}{\partial t}(x(t_i^-), t_i)]^T & 1 \end{bmatrix} \nabla V_{q_i}(x(t_i^+), t_i) - [\frac{\partial C_{(q_{i-1},q_i)}}{\partial t}(x(t_i^-), t_i)]^T \pi_i^* \tag{2.18}$$

with $\nabla V_{q_i}(x(t_i^+), t_i) = \begin{bmatrix} p^{*T}(t_i^+) & -H^*(t_i^+) \end{bmatrix}^T$.

Proof. Consider the value function $V_q(x, t) : \mathbb{R}^{n_q} \times [a, b] \longrightarrow \mathbb{R}^+$ corresponding to the optimal cost starting from position (q, x) at time t with the optimal control $(u^*, d^*)(.)$ on $[t, b]$:

$$V_q(x, t) = \int_t^b L_{q^*(\tau)}(x^*(\tau), u^*(\tau), \tau) d\tau \tag{2.19}$$

with $q^*(\tau^+) = \nu(x^*(\tau^-), q^*(\tau^-), d^*(\tau), \tau)$.

It is well known in the continuous case that the value function satisfies almost everywhere the Hamilton-Jacobi-Bellman equation that is:

$$p_0^* \frac{\partial V}{\partial t} = H(-p_0^* \frac{\partial V}{\partial x}, p_0^*, x^*, u^*, t) \text{ a.e} \tag{2.20}$$

Let $p^* = -p_0^* \frac{\partial V}{\partial x}$ so that along the optimal trajectory, we have

$$p_0^* \frac{\partial V}{\partial t} = H(p^*, p_0^*, x^*, u^*, t) \text{ a.e.}$$

which means that the maximum principle can be used to solve the partial differential equations (2.20) along with the extremal trajectories via the Hamiltonian system (2.15).

Let $[t_0 = a, t_1, ..., t_i, ..., t_m = b]$ and $[q_0, q_1, ..., q_i, ..., q_m]$ be the optimal switching time sequence and mode sequence associated to the optimal control $(u^*, d^*)(.)$ obtained from the initial position (x_0, q_0). Consider the first time t_1 along the optimal trajectory where there is a value change in the discrete state $q(.)^*$ and write the Bellman principle at this time:

$$V_{q_0}(x_0, t_0) = \min_{u|_{(t_0,t_1)} \in U_{q_0}} \{ \int_{t_0}^{t_1} L_{q_0}(x, u, t) dt + V_{q_1}(x(t_1^+), t_1) \} \tag{2.21}$$

with t_1, $x(t_1^+)$ and q_1 such that

$$V_{q_1}(x(t_1^+),t_1) = \min_{d(t_1)\in D} V_q(x(t_1^+),t_1) \tag{2.22a}$$

where

$$q = \nu(x(t_1^-),q(t_1^-),d(t_1),t_1) \tag{2.22b}$$

$$x(t_1^+) = \Phi_{(q_0,q)}(x(t_1^-),t_1) \tag{2.22c}$$

Even if equation (2.21) can be sufficient for the demonstration, equations (2.22a) explain how the optimal discrete control $d(t_1)$ is chosen: among the possibilities (D), $d(t_1)$ must give the minimum of V from the next initial state $(x(t_1^+,q_1)$. This step uses a dynamic programming argument and are of importance for the optimization procedure.

If this first change in q is due to a boundary condition, we must add the associated constraint:

$$C_{(q_0,q_1)}(x(t_1^-),t_1) = 0 \tag{2.23}$$

Now, looking at equation (2.21) as a simple continuous control problem on time interval $[t_0,t_1]$ with terminal cost function $V_{q_1}(x(t_1^+),t_1)$ yields to the following (according to (2.6), (2.7) and (2.8)):

- the continuous control must satisfy the maximum condition,

$$H_{q_0}(p_0^*,p^*(t),x^*(t),u^*(t),t) = \sup_{u\in U_{q_0}} H_{q_0}(p_0^*,p^*(t),x^*(t),u,t) \tag{2.24}$$

- the following transversality conditions hold:
  there exists π_1^* such that

$$p^*(t_1^-) = -p_0^*\frac{\partial(V_{q_1}\circ\Upsilon_{(q_0,q_1)})}{\partial x}(x(t_1^-),t_1) + \left[\frac{\partial C_{(q_0,q_1)}}{\partial x}(x(t_1^-),t_1)\right]^T \pi_1^* \tag{2.25}$$

$$H^*(t_1^-) = p_0^*\frac{\partial(V_{q_1}\circ\Upsilon_{(q_0,q_1)})}{\partial t}(x(t_1^-),t_1) - \left[\frac{\partial C_{(q_0,q_1)}}{\partial t}(x(t_1^-),t_1)\right]^T \pi_1^* \tag{2.26}$$

  with vector π_1^* equals to zero if there is no boundary conditions and $\Upsilon_{(q_0,q_1)}(x,t) = \begin{bmatrix} \Phi^T_{(q_0,q_1)}(x,t) & t \end{bmatrix}^T$. This results is directly obtained applying equation (2.8).

As

$$-p_0^*\frac{\partial(V_{q_1}\circ\Upsilon_{(q_0,q_1)})}{\partial x}(x(t_1^-),t_1) = \begin{bmatrix} \left[\frac{\partial\Phi_{(q_0,q_1)}}{\partial x}(x(t_1^-),t_1)\right]^T & 0 \end{bmatrix} \nabla V_{q_1}(x(t_1^+),t_1) \tag{2.27}$$

$$-p_0^*\frac{\partial(V_{q_1}\circ\Upsilon_{(q_0,q_1)})}{\partial t}(x(t_1^-),t_1) = \begin{bmatrix} \left[\frac{\partial\Phi_{(q_0,q_1)}}{\partial t}(x(t_1^-),t_1)\right]^T & 1 \end{bmatrix} \nabla V_{q_1}(x(t_1^+),t_1) \tag{2.28}$$

$$\text{with } \nabla V_{q_1}(x(t_1^+),t_1) = \begin{bmatrix} -p_0^*\left[\frac{\partial V_{q_1}}{\partial x}(x(t_1^+),t_1)\right]^T & -p_0^*\frac{\partial V_{q_1}}{\partial t}(x(t_1^+),t_1) \end{bmatrix}^T \tag{2.29}$$

$$= \begin{bmatrix} p^{*T}(t_1^+) & -H^*(t_1^+) \end{bmatrix}^T \text{ by (2.20)} \tag{2.30}$$

we get equations (2.17) and (2.18). Repeating this from $V_{q_1}(x(t_1^+),t_1)$ for next switching time yields to a complete set of necessary conditions. □

Remark 2.4. Equations (2.17) and (2.18) must be obviously adapted according to the final and initial constraints under the state (x,q) at time $t=a$ and $t=b$ (not specified in our case). See equations (2.7) and (2.8).

Remark 2.5. The notations (2.17), (2.18) imply : π_i^* must be equal to zero if t_i is a controlled switching time without boundary conditions.

Remark 2.6. As the state $x(.)$, the costate $p(.)$ should also be of different dimensions for different subsystems.

Remark 2.7. In practice many problems only require optimal solutions under fixed number of switchings and/or fixed order of active subsystems. As the above necessary conditions deal with a very large formulation of the optimal control problem, the total number of switchings and the order of active subsystems seem to be a priori unknown. However, it is possible to impose them as well as the switching time : it only depends on how the discrete transition function ν is specified. For example if one wants to impose the order of the active subsystems, the automata should be written in such a way that there is only the expected sequence. To specify the number of switchings, one has to write a tree with the good degree of depth. Moreover the boundary constraints can be used to impose the switching times.

Remark 2.8. In a classical optimal control problem, one may need to solve Boundary Value Problem (BVP). At this stage, it can be observed that the above necessary conditions may lead to a multi stage BVP corresponding to the discrete transitions. Transversality conditions give just a relation between initial and final values of the Hamiltonian and of the costate at each switching times without any information about when these switches occur. In fact due to discrete dynamic, the key to compute the solution is dynamic programming. But the task can be very hard to practice since bifurcation in the trajectory must be taken into account each time a discrete transition is allowed i.e. in the regions of the state space and time space for each subsystems. In addition the degree of freedom in the trajectory at each switching time due to the choice of π_i yields also to bifurcations.

Remark 2.9. In the case of switched system, when the dynamics can be described using a single system:

$$\dot{x}(t) = \sum_{q=1}^{Q} \alpha_q(t) f_q(x(t), u(t), t) \in \mathbb{R}^n$$

with $\alpha_q(t) \in \{0, 1\}$ and $\sum_{q=1}^{Q} \alpha_q(t) = 1, \forall t$. At any time, the active subsystem is selected via the values of the α_q's. Then the discrete transition function is degenerated to

$$q(t^+) = \nu(x(t^-), q(t^-), d(t), t) \equiv d(t)$$

with the discrete control set $\underline{D} = \underline{Q}$. In which case, the switching strategy must satisfy the following condition: at any time t, the following maximum condition holds for $(p^*, p_0^*, x^*, q^*)(t)$:

$$H_{q^*}(p^*, p_0^*, x^*, u^*, t) = \max_{q \in \underline{Q}} \sup_{u \in U_q} H_q(p^*, p_0^*, x^*, u, t) \tag{2.31}$$

It means that the active subsystem at any time is the one which has the largest Hamiltonian function. This situation has been studied in [18, 19].

Remark 2.10. Note that we can take into account jump costs:

$$\Psi_{(q,q')}(x, t) \tag{2.32}$$

every time a discrete transition occurs in a straightforward way. To do this, we need only to add the cost Ψ in (2.21) in order to determine new transversality conditions.

3. Illustrative example

Let's focus again on the hybrid automata given in Fig 1. Our purpose is to illustrate how the previous necessary conditions can be used to get a solution. So we will choose a simple problem for which the solution can be easily tracted. The control problem we propose to solve is a time optimal control one.

In the case of time optimization cost function J takes the form :

$$J = \int_a^b dt = b - a \tag{3.1}$$

with $L_q \equiv 1$ and $\phi_q \equiv 0$, for all q.

Starting from a position $(x, q)(0) = (0, 1)$, we want to reach position $x(b) = x_f$ for a non specified discrete state q in a minimum transfer time. To do this, we assume the following:

- in g_{12}, $x_0 = 2$ and the constraint C_{12} is equal to $C_{12}(x) = x - x_0 = 0$
- in g_{23}, $y_1 = 1$ and the constraint C_{23} is equal to $C_{23}(x, y) = y - y_1 = 0$
- $x_f = 10$

At this point, it can be observed that the motion in the different modes can be characterized according to the following items (see Fig 1.):

- slow in mode $q = 1$
- fast in mode $q = 3$
- going to the left in mode $q = 2$

So the optimal trajectory is certainly obtained with a switch to mode $q = 3$. But as we start in mode $q = 1$, a penalty transition phase in mode $q = 2$ occurs yielding to a motion to the left during a minimum time interval of one seconde. Then it can be supposed that the optimal trajectory is obtained with three stages:

(1) start with a control $u = 1$ in mode $q = 1$ until the boundary condition $x = x_0$ is reached,
(2) switch to mode $q = 2$ with $u = -5$ during a the minimum time transfer (one second) in this mode
(3) at last take the high way in mode $q = 3$ with a control to $u = 100$.

In order to prove this, we can define for each value of q, the associated Hamiltonian system, adjoint system and the jump functions with following table 2:

TABLE 2

	Hamiltonian function	Adjoint System	Jump Function
$q = 1$	$H_1 = p_x u - p_0$	$\dot{p}_x = 0$	$\Phi_{12}(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x$
$q = 2$	$H_2 = p_x u + p_y - p_0$	$\dot{p}_x = 0$ $\dot{p}_y = 0$	$\Phi_{23}(x, y) = \Phi_{21}(x, y) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
$q = 3$	$H_3 = p_x(x + u) - p_0$	$\dot{p}_x = -p_x$	$\Phi_{31}(x) = x$

Observe that in both modes $q = 1$ and $q = 2$, the costate is constant since $\dot{p} = -\frac{\partial H_q}{\partial x} = 0$, $q = 1, 2$ and for $q = 3$, $\dot{p} = -p$.

At a jump time, we have the following transversality conditions which form a recursive rule between initial and final values:

The key to solve this problem is dynamic programming.

Starting from the end, observe that there are only modes $q = 1$ and 3 for which end point x_f can be reached since in mode $q = 2$ the set of continuous control $\{-10, -5\}$ implies a decreasing motion.

So, if the trajectory is finishing in mode $q = 3$, as $p(.)$ is free at this time according to the maximum condition (2.16) we must choose a strictly positive value to $p(b)$ in order to guarantee

TABLE 3. Transversality conditions

$q=1\to 2$	$q=2\to 3$	$q=2\to 1$	$q=3\to 1$
$p_x^- = p_x^+ + \pi_{12}$ if $x^-=2$ $p_x^- = p_x^+$ if $x^- > 2$	$p_x^- = p_x^+$ $p_y^- = \pi_{23}$ if $y^-=1$ $p_y^- = 0$ if $y^- > 1$	$p_x^- = p_x^+$ $p_y^- = 0$	$p_x^- = p_x^+$
$H_1 = H_2$	$H_2 = H_3$	$H_2 = H_1$	$H_3 = H_1$

that x increases, $H_3 = 0$ and $(p, p_0) \neq (0,0)$. In this case and while $q=3$, the control takes value $u(t) \equiv 100$ since $p(t)$ keeps a constant sign.

A switch from $q=2$ to $q=3$ arises when in this mode 2 the value of counter y is greater than 1. At this time, the following conditions must be respected:

$$
\begin{aligned}
&p_x^- = p_x^+\\
&p_y^- = \pi_{23} \quad \text{if } y^- = 1\\
&p_y^- = 0 \quad \text{if } \ y^- > 1\\
&H_2 = H_3
\end{aligned}
$$

If $y^- > 1$, then $p_x^- = p_x^+$ and $H_2 = H_3$ yield to:

$$
\begin{aligned}
&p_x^- > 0\\
&u^- = x + 100
\end{aligned}
$$

But (2.16) implies:

$$
\begin{aligned}
u &= -5\\
x &= -105
\end{aligned}
$$

From this, we can assume a switch cannot occur from mode $q=1$ to $q=2$ according to the associated transversality conditions:

$$
\begin{aligned}
&p_x^- = p_x^+ + \pi_{12} \text{ if } x^- = 2\\
&p_x^- = p_x^+ \qquad\quad \text{if } x^- > 2\\
&H_1 = H_2
\end{aligned}
$$

Indeed, if the switch is obtained when $x^- > 2$ then $p_x^- = p_x^+$ and $H_1 = H_2$ are incompatible with (2.16). In the second case, if $x^- = 2$, then an increasing motion implies $p_x^- > 0$ and (2.16), $u^- = 1$. Or $H_1 = H_2$ does not hold in this condition since it implies $p_x^- = -5p_x^+$.

Then, only a switch from $q=2$ to $q=3$ must be produced when $y^- = 1$.

With similar arguments, the following tree can be built (Figure 2).

This tree summarizes all the extremal trajectories ending in mode $q=3$. In each location, we mention discrete state q, the continuous control and the transversality condition at the next switching time. Recursive information is used to determine the discontinuities of the costate, possibly the continuous state position and if the transition is compatible with the necessary conditions.

Now, it is not difficult to see that starting from position $(x,q)(0) = (0,1)$ the unique trajectory ending in $q=3$ is obtained by switching the first time $x(.)$ hits C_{12} and C_{23} with respectively the control $u(t) \equiv 1$, -5, 100.

In the same way, there is one extremal trajectory which ends in the position x_f in mode $q=1$. It is obtained with no switch and $u(t) \equiv 1$.

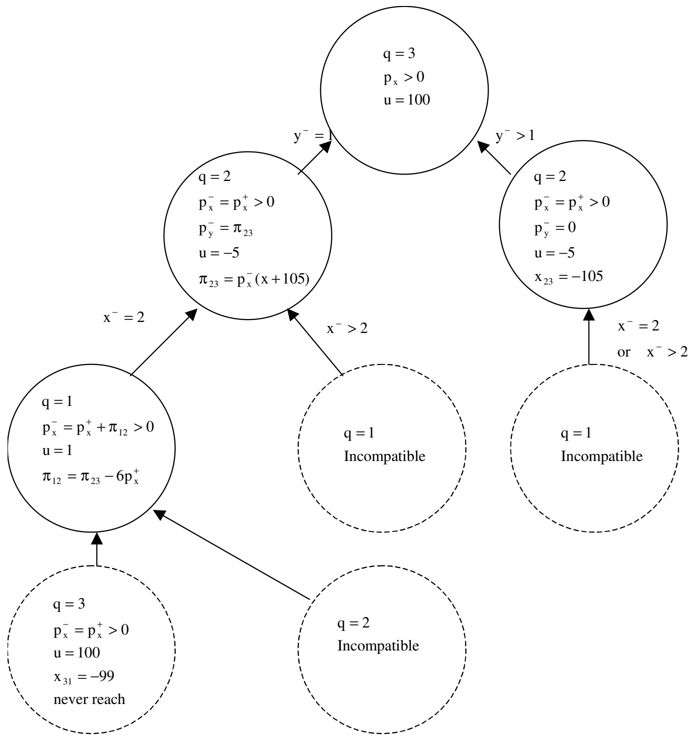

Figure 2. Pre-synthesis: Discrete trajectories ending in mode $q = 3$

Now the optimal one depends on the value of x_f. In our case, the best trajectory is obviously the first one.

4. Conclusion

In this article, it has been shown that the MP approach can be used in order to get some necessary conditions in context of hybrid optimization problems. This point has been illustrated with an example. Avoiding the difficulties encountered in solving the HJB's partial derivative equations, the Maximum Principle allows to build extremal trajectories using Hamiltonian systems. But the discrete dynamic yields to bifurcations in the continuous trajectory each time a transition is allowed. These bifurcations are obtained with the transversality conditions. So we must proceed as in dynamic programming by evaluating the cost function along the different branches of the tree to determine which ones are optimal. Even if the discrete state is finite, the number of branches can be infinite. This is a new and big difficulty encountered in hybrid optimization. Today, we have a much of interest to build efficient algorithms in this situation.

References

[1] N. N. Bogoliubov & Y. A. Mitropolsky (1961), *Asymptotic Methods in the Theory of Nonlinear Oscillations,* N.Y. Gordon and Breach.

[2] J. A. Sanders & F. Verhulst (1985), Averaging Methods in Nonlinear Dynamical Systems, N.Y. Springer .

[3] A. Levis, S. Marcus, W. Perkins, P. Kokotovic, M. Athans, R. Brockett & A. Willsky (1987) Challenge to control, a collective view, *IEEE Trans. on Aut. Cont.,* 32(12), 274-285.

[4] R. L. Grossman, A. Nerode, A. P. Ravn & H. Rischel (1993), *Hybrid Systems,* vol 736 of LNCS, Springer.

[5] P. Antsaklis, W. Kohn, A. Nerode & S. Sastry (1995), *Hybrid systems II,* vol 999 of LNCS, Springer.
[6] R. Alur, T. Henzinger & E. Sontag (1996), *Hybrid systems III-Verification and Control,* vol 1066 of LNCS, Springer.
[7] P. Antsaklis, W. Kohn, A. Nerode & S. Sastry (1997), *Hybrid systems IV,* vol 1273 of LNCS, Springer.
[8] P. Antsaklis, W. Kohn, M. Lemmon, A. Nerode & S. Sastry (1998), *Hybrid systems V*, LNCS, Springer.
[9] Special Issue on Hybrid Control Systems (1998), *IEEE Trans. on Aut. Cont.,* 43(4).
[10] A Special Issue on Hybrid Systems (1999), *Automatica*, 35(3).
[11] P. J. Antsaklis & A. Nerode (1998), Hybrid Control Systems: An Introductory Discussion to the Special Issue, *IEEE Trans. on Aut. Cont.,* 43(4), 457-460.
[12] H. Ye, A. N. Michel & L. Hou (1995), Stability Theory for Hybrid Dynamical Systems, 34th CDC, 2679-2684.
[13] B. Lennartson, M. Tittus, B. Edgardt & S. Pettersson (1996), Hybrid Systems in Process Control, *IEEE Cont. Syst.,* 45-56.
[14] M. S. Branicky, V. S. Borkar & S. K. Mitter (1998), A Unified Framework for Hybrid Control: Model and Optimal Control Theory,*IEEE Trans. on Aut. Cont.,* vol(43)(1).
[15] P. Riedinger & C. Iung (1999), Optimal Control for Hybrid Systems: An Hysteresis Example, proc.IEEE System, Man and Cybernetics, I.188-I.193.
[16] M. K. Ghost, A. Arapostathis & S. I. Marcus (1992), Optimal Control of hybrid system with pathwise average cost, 31st CDC, 1061-1066.
[17] M. Nguyen, & V. Gaitsgory (1997), Asymptotic Optimization for a Class of Nonlinear Stochastic Hybrid Systems on Infinite Time Horizon, ACC'97.
[18] P. Riedinger, F. Kratz, C. Iung & C. Zanne (1999), Linear Quadratic Optimization for Hybrid Systems, proc. IEEE Conf. on Decision and Control, 3059-3064.
[19] P. Riedinger, C. Zanne & F. Kratz (1999), Time Optimal Control of Hybrid Systems, proc. American Control Conference, 2466-2470.
[20] L.S. Pontryagin,V. G. Boltyanskii, R.V. Gamkrelidze & E. F. Mishchenko (1964), *The Mathematical Theory of Optimal Processes,* Pergamon.
[21] L. Young (1980), *Optimal Control Theory*, Cambridge University Press.
[22] F. Lewis (1986), *Optimal Control*, Wiley N.Y.
[23] A. E. Bryson & Y.C. Ho (1969), *Applied Optimal Control*, Gin and Co.
[24] F.H.Clarke (1990), *Optimization and Nonsmooth Analysis*, SIAM.
[25] J. Macki & A. Strauss (1982), *Introduction to Optimal Control Theory*, Springer-Verlag.
[26] Y.L. Lootsma, A.J. van der Schaft & M. K. Camlibel (1999), Uniqueness of solutions of linear relay systems, in Automatica 35(3) 467-478.
[27] B. Jakubczyk & W. Respondek (1998), *Geometry of feedback and optimal control,* Marcel DEKKER.

Centre de Recherche en Automatique de Nancy (CRAN) CNRS UPRES 7039A,, ENSEM 2, avenue de la forêt de Haye, 54516 Vandoeuvre-les-Nancy Cedex France

E-mail address: E-mail: Pierre.Riedinger@ensem.inpl-nancy.fr, Fax: (33) 3 83 59 56 44