Linear Quadratic Optimization for Hybrid Systems

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Abstract

A general optimal hybrid control problem which enables a direct Maximum Principle approach is presented. Necessary conditions at switching time for controlled and autonomous switching are derived in the case where a linear quadratic criteria is used. Further extra condition for stability of switched systems is addressed. Finally an illustrative example of optimal hybrid control problem is treated.

1 Introduction

In automatic control area two kind of dynamical systems have been separately considered: continuous and discrete depending on the nature of the variables. Discrete variables mean variables that take values in a discrete set and likewise for continuous variables, that is to say they take values in a continuous set. The behavior of such entities is described by dynamical systems such as automata and differential equations respectively. The main reason of this separation states essentially in theoretical point of view. Each of these systems has indeed specific analysis tools defined with a discrete and continuous metrics. Physical systems do not have this strict separation and there is often a more or less interaction between discrete and continuous variables. Think for example at threshold phenomena which changes the continuous dynamic or at recipes in a batch process whose dynamics are continuous. This intricate interaction is well-known in electromechanical systems and some approaches consist to approximate the system using methods such as averaging methods [1]-[2]. But in general there is no theoretical framework.

The notion of hybrid dynamical systems appears in [3] where the necessity to develop a theory melting continuous and discrete signals is stressed. Today, it is admitted that hybrid systems are dynamical systems in which discrete and continuous variables interact. Their integration into one system ensures interaction between them. Generally it can be said that the discrete part plays the role of the supervisor of the continuous part. When discrete event occurs, it may change the continuous dynamic and therefore discontinuities in the vector field and /or in the continuous state appear. In the last decade, there was a great interest in studying such systems with regard to their potential applications as well as to the theoretical point of view [4]-[8]. In this article optimization problem for continuous part of hybrid systems is investigated. Linear hybrid systems are considered and for a given criteria the target is to look for an optimal control law. In [9], a dynamic programming approach is proposed to solve an hybrid optimal problem. In this paper, we suggest the use of the Maximum Principle (MP) of Pontryagin [10] for solving such a problem.

This paper is organized as follows. In section 2, the Pontryagin's approach for solving optimal control problem is presented and a general hybrid optimal control problem is formulated. The main result shows that the Maximum Principle (MP) is an efficient tool to study hybrid optimal control problems. In section 3, the work on Linear Quadratic (LQ) optimal control is focused on for linear hybrid systems. Transversality conditions at switching times are stressed. The main result shows that solving a linear quadratic hybrid problem consists in solving a sequence of Differential Riccati Equations(DRE). A stability result for switched systems less conservative than one the proposed in [11] is stated in section 4. Finally all this results are applied in section 5 to a second order hybrid controlled system for which a closed loop control including optimal switching time is built.

2 Hybrid Optimization: Pontryagin approach

Let us start by a general description of controlled hybrid systems and associated optimal problems. We have in mind systems that switch among vector fields over time, in the form

$$\dot{x}(t) = f_k(x(t), u_k(t)), k \in \underline{K} = \{1, ..., K\} \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the continuous state and $u_k(t) \in U_k$ a bounded set included in $\mathbb{R}^{m_k}$, is the continuous control. Here, f_k and $\frac{\partial f_k}{\partial x}$ are assumed to be continuous functions on the direct product $\mathbb{R}^n \times \bar{U}_k$ where $\overline{U}_k$ denote the closure of U_k. More generally, weaker hypotheses can be considered (Lipschitz conditions and jumps on the states), see [12], [13] and [14] for more details. The index $k \in \underline{K}$ or mode is referred as the discrete state whose dynamic is obtained using a transition function,

$$k(t) = \phi(x(t), k(t^-), d(t), t) \qquad (2)$$

where $d(t) \in \underline{D} = \{1, ..., D\}$ is a discrete input. Note that the relations (1) and (2) can be described by an hybrid automata. In order to conciliate the discrete dynamic with the continuous dynamic, the time is chosen as a continuous variable and the discrete variables k and d are considered as piecewise constant functions. Thus the transition function ϕ appears piecewise constant from the right. This mention is indicated by t^- in (2). Two categories of hybrid phenomena can be represented in this formulation: the autonomous switching and the controlled switching [9]. All these changes are reflected on the discrete state $k(t)$ or mode of the system at the time t using the transition function.

Let $[0, \tau_1, ..., \tau_i, ..., T]$ be the sequence of switching time and associated mode sequences $[k_0, k_1, ..., k_i, ..., k_m]$ ($k_i \in \underline{K}$). It is clear that these sequences depend on the control $(u, d)(.)$. A hybrid criteria can then be introduced as:

$$J = \sum_{i \geqslant 0} \int_{\tau_i}^{\tau_i+1} L_{k_i}(x(t), u_{k_i}(t))dt \qquad (3)$$

where $k_i \in \underline{K}$. For all $k \in \underline{K}$, L_k, is defined and continuous on the direct product $\mathbb{R}^n \times \bar{U}_k$ and continuously differentiable in the state variable. So, different criteria with respect to each mode are possible. Then, the optimal control $(u, d)(.)$ is the control that minimizes the cost function J over time interval $[0, T]$.

MP is a principle which can be applied to various optimization problems with different specific and technical conditions (boundary, free final time, constraints, etc.). Hence, according to the regularity assumptions on the data, several versions of MP can be stated. In our case formulation (1)(2) and (3) allows a direct use of a smooth version of MP with an additional dynamic programming argument in order to consider autonomous switching. To this purpose, define the Hamiltonian function associated to each mode k as (mention of time is omitted):

$$H_k(\lambda, \lambda_0, x, u) = \lambda^T f_k(x, u_k) - \lambda_0 L_k(x, u_k) \qquad (4)$$

and the Hamiltonian system as

$$\dot{x} = \frac{\partial H_k}{\partial \lambda} \qquad - \dot{\lambda} = \frac{\partial H_k}{\partial x} \qquad (5)$$

where λ_0 is a positive constant ($\lambda_0 \geq 0$). Now ,we have the following theorem:

Theorem 1 *If $(u, d)(.)$ and $(x, k)(.)$ are respectively an admissible optimal control and the corresponding trajectory for the problem (1)(2) and (3), then there exists a piecewise absolutely continuous curve $\lambda(.)$ and a constant $\lambda_0 \geq 0$, $(\lambda_0, \lambda(t)) \neq (0, 0)$ on $[0, T]$, such that:*

1. the sixtuplet $(\lambda, \lambda_0, x, k, u, d)(.)$ satisfies the associated Hamiltonian system (5) almost everywhere

2. for given $(\lambda, \lambda_0, x, k)(t)$ at given time t, the following maximum condition holds if t is not a switching time

$$H_k(\lambda, \lambda_0, x, u) = \max_{d \in \underline{D}} \left(\sup_{v \in U_m} H_m(\lambda, \lambda_0, x, v) \right) \qquad (6)$$

$$\text{where } m = \phi(x, k, d, t) \qquad (7)$$

3. at switching time τ_i (i.e. when (6) is not obtained for the k-mode) the following transversality conditions are satisfied:

a. if τ_i is an autonomous switching time on the manifold defined by a p components vector ($p < n$), $C_k(x, t) = 0$, from the mode k to the mode j then:

$$\lambda_j^+ = \lambda_k^- - \left.\frac{\partial C_k^T}{\partial x}\pi\right|_{\tau_i} \qquad (8)$$

$$H_j^+ = H_k^- + \left.\frac{\partial C_k^T}{\partial t}\pi\right|_{\tau_i} \qquad (9)$$

with π a p dimensional vector.

b. if τ_i is a controlled switching time from the mode k to the mode j then:

$$\lambda_j^+ = \lambda_k^- \qquad (10)$$

$$H_j^+ = H_k^- \qquad (11)$$

Proof: It is an extension to the MP approach of a result given in ([16],chap.3). ∎

Remarks: 1) It can be noted that $H_k(\lambda, \lambda_0, x, u) = \sup_{v \in U_{m_k}} H_k(\lambda, \lambda_0, x, v)$ is exactly the optimal condition holding for a single differential system. Hence the active available mode is the one which has the largest Hamiltonian.

2) Notice that (8)(9) imply discontinuities at time τ in adjoint variable λ and in Hamiltonian function H.

3) Trajectories such that the previous necessary conditions are satisfied are called *extremals*.

4) In most applications $\lambda_0 \neq 0$, in which case one may take $\lambda_0 = 1$.

Examples of optimal problem for hybrid systems have been studied in [15], [17] and [18]. In these papers it is shown how optimal switching scheme can be achieved. In the paper [9], the authors describe an algorithm to design optimal controls of hybrid dynamical systems. The proposed method is mainly based on dynamic programming and Hamilton-Jacobi-Bellman (HJB) equations. An hysteresis example, which is fully described in [19]-[20], illustrates fairly well the capability of the proposed algorithm to converge towards the optimal criteria in certain cases. Unfortunately, a bifurcation both in the trajectory and the criteria appears with respect to a parameter value in the criteria [18], then the shape of the trajectories found with this approach are very far from the optimal ones. Our method can then be used to solve the example given by Branicky et al. and get the optimal value of the criteria and the optimal trajectories. In general it is difficult to solve HJB equations due to non-smooth value function. The MP approach avoids this difficulty with the introduction of the Hamiltonian systems.

Another interesting case is certainly one where the criteria is a linear quadratic.

3 Linear Quadratic Criteria

In this section linear quadratic (LQ) criteria for hybrid linear systems is considered. Optimal control problem using a collection of quadratic criteria can be stated as follows. Minimize the quadratic criteria :

$$J = \frac{1}{2}\int_0^T \left(x^T(t) Q_{k(t)} x(t) + u^T_{k(t)}(t) R_{k(t)} u_{k(t)}(t) \right) dt \quad (12)$$

over time interval $[0, T]$ where R_k (respectively Q_k) are symmetric positive definite (respectively semi-definite) matrices and x is subject to the hybrid system :

$$\dot{x}(t) = A_k x(t) + B_k u_k(t) \qquad x(0) = x_0 \quad (13a)$$

$$k(t) = \phi(x(t), k(t^-), d(t), t) \qquad k(0) = k_0 \quad (13b)$$

where A_k, B_k are respectively $n \times n$ and $n \times m$ dimensional matrices and $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $k(t) \in \underline{K} = \{1, 2, ..., K\}$, $d(t) \in \underline{D} = \{1, 2, ..., D\}$ for some integer values K and D. Define the Hamiltonian function associated to each mode as:

$$H_k(\lambda, x, u) = \lambda^T(A_k x + B_k u_k) - \frac{1}{2}(x^T Q_k x + u_k^T R_k u_k) \quad (14)$$

and the Hamiltonian system:

$$\dot{x} = \frac{\partial H_k}{\partial \lambda} = A_k x + B_k u_k \quad (15a)$$

$$\dot{\lambda} = -\frac{\partial H_k}{\partial x} = -A_k^T \lambda + Q_k x. \quad (15b)$$

Remark: We take $\lambda_0 = 1$ in (14)(cf. remark bellow Theorem1).

The optimal condition for each Hamiltonian H_k with respect the continuous control u_k leads to:

$$u_k = R_k^{-1} B_k^T \lambda \quad (\frac{\partial H_k}{\partial u_k} = 0) \quad (16)$$

Assume now that there is a time interval $[\tau_i, \tau_{i+1}[$ where the active optimal mode is the k_i-mode, then

$$H_{k_i}(\lambda, x, u_{k_i}) > H_q(\lambda, x, u_q), \forall q \in \underline{s}(t), q \neq k_i, \quad (17)$$

where $\underline{s}(t) = \phi(x(t), k_i(t^-), \underline{D}, t)$ is the constant set of available modes on $[\tau_i, \tau_{i+1}[$ from k_i. If we denoted by $J_i(x(t), t)$ the value function J (The optimal cost starting from $(x(t), k_i)$ at time t) on $[\tau_i, \tau_{i+1}[$ then the Hamilton-Jacobi-Bellman equation gives rise to:

$$\frac{\partial J_i}{\partial t} = H_{k_i}(-\frac{\partial J_i}{\partial x}, x, u_{k_i}) \text{ on } [\tau_i, \tau_{i+1}[\quad (18)$$

where $\lambda = -\frac{\partial J_i}{\partial x}$. Hence if we choose J_i of the form $J_i(x(t), t) = \frac{1}{2}(x(t)^T P_i(t) x(t) + c_i)$ with $P_i(t)$ a symmetric matrix and c_i a constant, (18) leads to the Differential Riccati Equation (DRE):

$$\dot{P}_i = -P_i A_{k_i} - A_{k_i}^T P_i + P_i B_{k_i} R_{k_i}^{-1} B_{k_i}^T P_i - Q_{k_i} \quad (19)$$

with $\lambda = -P_i x$. Now the limit conditions depend to the type of the switch which is produced at time τ_{i+1}. If a controlled switching is obtained to the k_{i+1} mode then the following transversality must be satisfied:

$$H_{k_{i+1}}(-P_{i+1}x, x, u_{k_{i+1}}) = H_{k_i}(-P_i x, x, u_{k_i}) \quad (20a)$$

$$P_{i+1}x = P_i x. \quad (20b)$$

else an autonomous switching occurs on a manifold $C_{k_i}(x(\tau_{i+1}), \tau_{i+1}) = 0$ and we have:

$$P_{k_{i+1}}x = P_{k_i}x + \left.\frac{\partial C_{k_i}^T}{\partial x}\right|_{t=\tau} \pi \quad (21a)$$

$$H_{k_{i+1}}(-P_{i+1}x, x, u_{k_{i+1}}) = H_{k_i}(-P_i x, x, u_{k_i}) + \left.\frac{\partial C_{k_i}^T}{\partial t}\right|_{t=\tau} \pi \quad (21b)$$

At last as the value function must be continuous along optimal trajectories, the following additional condition must be held in both cases

$$x^T P_{i+1} x + c_{i+1} = x^T P_i x + c_i \quad (22)$$

Remark: The choice of an additional constant term c_i for each value function J_i is completely justified with the full proof of theorem 1.

Finally at the final time as x is free, $\lambda(T)$ must be taken to zero. Solving (12) and (13) consists now to determine switching time sequence $[0, \tau_1, \tau_2, ..., \tau_i, T]$ and mode sequence $[k_0, k_1, k_2, ..., k_i]$ such that :

a) (16)(17) and (19) are satisfied for all indices k_i on $[\tau_i, \tau_{i+1}]$

b) (20) or (21) (following controlled or autonomous switching is considered) and (22) is satisfied at switching time τ_i.

Remarks: 1) In (20) the matrix P_ℓ must be chosen such that $x \in Ker(P_\ell - P_k)$. So P_ℓ is not uniquely defined at this time. If two candidates are considered P_ℓ^2 and P_ℓ^1 then at future times it always holds $x(t) \in Ker(P_\ell^1(t) - P_\ell^2(t))$(since solution of (15) is unique). Therefore the trajectories are equal and we can choose $P_\ell = P_k$ in (20). Deduce in the same manner for (21).

2) The same result occurs if time varying matrices are considered in place of (13) and/or if the problem is supposed in infinite time $T = +\infty$).

Stability results can be derived from this optimal problem.

Theorem 2 *Suppose that Q_k are positive definite for all $k \in \underline{K}$. Starting from position (x_0, k_0), if a time sequence $[0, \tau_1, \tau_2, ..., \tau_i, ...]$ and associated mode sequences $[k_0, k_1, k_2, ..., k_i, ...]$ ($k_i \in \underline{K}$) are exhibited such that*

1. the origin can be reach in finite time by some control (controllable),

2. the time between two switches is bellow bounded ($\exists\ \epsilon > 0,\ \tau_{i+1} - \tau_i \geq \epsilon$),

3. the sequence of $P_i(t)$ ($i = 0, 1, 2, ...$) is uniformly bounded

$$(\exists\ \alpha > 0, \beta > 0, \quad \alpha \|x\| \leq x^T P_i(t) x \leq \beta \|x\| \quad \forall t, \forall i),$$

4. (16),(17),(19),(20) (21) and (22)-hold,

then the trajectory is asymptotically stable.

Proof: The proof is quite long and is omitted. It is obtained showing that: the cost function is bounded (hypothesis1), $\frac{dJ(x,t)}{dt} \to 0$ and x is bounded (hypothesis2 and 3), $\frac{dJ(x,t)}{dt} \to 0$ implies $x(t) \to 0$ ($Q_k > 0$). Note that $J_i(x,t) = \frac{1}{2}(x^T P_i x + c_i)$ is not a lyapunov function since $J_i(0,t) = c_i \neq 0$. ■

4 Stability of Switched Systems

Linear switched systems is the degenerate case of the previous control problem since only autonomous switching between linear systems are considered. It can be viewed as a control problem with a unique control (consequently an optimal control).

Let $\dot{x}(t) = A_k(t)x(t), k \in \underline{N} = \{1, ..., N\}$ with the switching rule defined by relations of type $C_k(x,t) = 0$. From an initial position (x_0, k_0), $x(t)$ evolves according to the switching rule which products switching time sequence $\{0, \tau_1, \tau_2, ..., \tau_i, ...\}$ and mode sequences $\{k_0, k_1, k_2, ..., k_i, ...\}$ with $k_i \in \underline{N}$. Stability of such systems can be stated from the following theorem.

Theorem 3 *Starting from position (x_0, k_0), if there exist two sequences of positive definite matrices $P_i(t)$, Q_{k_i} and a bounded associated sequence of number c_i such that*

1. the time between two switches is bellow bounded , ($\exists\ \epsilon > 0,\ \tau_{i+1} - \tau_i \geq \epsilon$),

2. $\exists\ \alpha > 0, \beta > 0, \quad \forall t, k, \quad \alpha \leq \|P_i(t)\| \leq \beta$,

3. P_i verifies:

$$\dot{P}_i = -P_i A_{k_i} - A_{k_i}^T P_i - Q_{k_i} \text{ on } [\tau_i, \tau_{i+1}]$$

at switching time τ_{i+1},

$$P_{i+1}x = P_i x + \left.\frac{\partial C_{k_i}^T}{\partial x}\right|_{t=\tau_{i+1}} \pi_{i+1}$$

$$x^T P_{i+1} A_{k_{i+1}} x + x^T A_{k_{i+1}}^T P_{i+1} x + x^T Q_{k_{i+1}} x = x^T P_i A_{k_i} x + x^T A_{k_i}^T P_i x + x^T Q_{k_i} x - 2\left.\frac{\partial C_{k_i}^T}{\partial t}\right|_{t=\tau_{i+1}} \pi_{i+1}$$

$$x^T P_{i+1} x + c_{i+1} = x^T P_i x + c_i$$

where π_{i+1} is a constant vector and c_i a constant, then the trajectory is asymptotically stable.

There has been recent work concerning Lyapunov stability of switched systems [11], [21], [22] summarized in [23]. All these approaches deal with the search of candidate Lyapunov functions, one for each mode, with a non increasing condition at switching times [21], [22]. As mentioned in [23] has underlying, the conditions obtained in this related work can yield to very conservative results.

Theorem 3 shows that less conservative results can be achieved if a time varying candidate function is considered. Two interesting cases can be stressed when the sequence of events (modes) is periodic or finite. Indeed if the sequence is a cycle of events, stability can be stated by looking for periodic matrices $P_k(t)$ which satisfy theorem 3. Fixed point algorithms are efficient in this situation. In other case if there is a finite sequence, it means that the last mode say k must satisfy an Algebraic Riccati Equation (ARE). Consequently the matrix $P_k(t)$ is known and we can build the rest of the sequence from an arbitrary switching time.

5 Switching time and Example

Suppose we get a problem of type (12)(13) without autonomous switching and with the particular transition function: $k(t) = \phi(x(t), k(t^-), d(t), t) \equiv d(t)$. So controlled switching could taken place anywhere at any time. MP define the control which must be used in terms of λ. Consider now an extremal trajectory $x(.)$ at a given time t (not a switching time) with closed loop control $u_k = -R_k^{-1} B_k^T P_k x$ and an active mode k. Let $\psi_{k,\ell}(t) = H_k(\lambda, x, u_k) - H_\ell(\lambda, x, u_\ell)$ where $\lambda = -P_k x$. A switching time occurs at the first time where there exists indice ℓ such that $\psi_{k,\ell}(t)$ vanishes. We can obtain these switching times by the help of the following theorem.

Theorem 4 *The function $\psi(t) = H_k(\lambda, x, u_k) - H_\ell(\lambda, x, u_\ell)$ satisfies the homogenous linear ODE with constant coefficients defined by the characteristic polynomial S_k where S_k is the minimal polynomial of $M_k \oplus M_k$, ($\oplus$ denote Kronecker sum, [24]) with* $M_k = \begin{pmatrix} A_k & B_k R_k^{-1} B_k^T \\ Q_k & -A_k^T \end{pmatrix}$.

Proof: Using (16), (15) can be rewritten as

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{pmatrix} A_k & B_k R_k^{-1} B_k^T \\ Q_k & -A_k^T \end{pmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

and $\psi(t)$ becomes $\psi(t) = \left\langle \begin{bmatrix} x \\ \lambda \end{bmatrix}, D \begin{bmatrix} x \\ \lambda \end{bmatrix} \right\rangle$ where

$$D = \begin{pmatrix} \frac{1}{2}(Q_\ell - Q_k) & 0 \\ A_k - A_\ell & \frac{1}{2}(B_k R_k^{-1} B_k^T - B_\ell R_\ell^{-1} B_\ell^T) \end{pmatrix}$$

First derivative of $\psi(t)$ leads to:

$$\psi(t) = \left\langle \begin{bmatrix} x \\ \lambda \end{bmatrix}, M_k^T D + D M_k \begin{bmatrix} x \\ \lambda \end{bmatrix} \right\rangle$$

This derivative appears as a first order differential operator which assigns to a given matrix D, the matrix $M_k^T D + D M_k$. Let $col(.)$ be the stacking operator applied to a given matrix $A = [A_1 ... A_n]$ by $col(A) = [A_1^T ... A_n^T]^T$ where $A_1, ..., A_n$ are the columns of A. Then an interesting property is: $col(ACB) = (B^T \otimes A) col(C)$. Thus, if we form $col(D)$ and $col(M_k^T D + D M_k)$, (33) can be equivalently rewritten as:

$$\psi(t) = \left\langle \begin{bmatrix} x \\ \lambda \end{bmatrix} \otimes \begin{bmatrix} x \\ \lambda \end{bmatrix}, M_k^T \oplus M_k^T col(D) \right\rangle$$

Differentiating $\psi(t)$ successively with respect to t, one gets:

$$\frac{d^n \psi}{dt^n}(t) = \left\langle \begin{bmatrix} x \\ \lambda \end{bmatrix} \otimes \begin{bmatrix} x \\ \lambda \end{bmatrix}, (M^T \oplus M^T)^n col(D) \right\rangle$$

One can conclude by Caley-Hamilton theorem. ■

This theorem enables a fast construction of extremal trajectory from a given x_0 and P_0 since switching time can be exactly determined. Hence using an appropriate algorithm (steepest descent, Newton) optimal trajectory can be retrieved.

In the sequel, we considered an optimal (LQ) problem over infinite time obtained from two vector fields :

$$\dot{x}(t) = (A_k x(t) + B_k u_k(t)), k = 1, 2 \qquad x(0) = x_0.$$

where $A_1 = \begin{pmatrix} -1 & 4 \\ -3 & 2 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$, $B_1 = B_2 = I_2$ and the associated (LQ) criteria :

$$J = \frac{1}{2} \int_0^{+\infty} (x^T Q_{k(t)} x + {u_{k(t)}}^T R_{k(t)} u_{k(t)}) dt$$

where

$$Q_1 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{pmatrix} \quad Q_2 = \begin{pmatrix} 2 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$
$$R_1 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.25 \end{pmatrix} \quad R_2 = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$$

Starting from a P_0 and x_0 and using theorem 4, one gets switching times sequence $[0, \tau_1, \tau_2, ..., \tau_i, ...]$ and mode sequence $[k_0, k_1, k_2, ..., k_i, ...]$. If there is a periodic cycle it can be detected with the asymptotic behavior. Hence using appropriate algorithm (steepest descent, Newton) optimal trajectory can be retrieved (Figure1). For

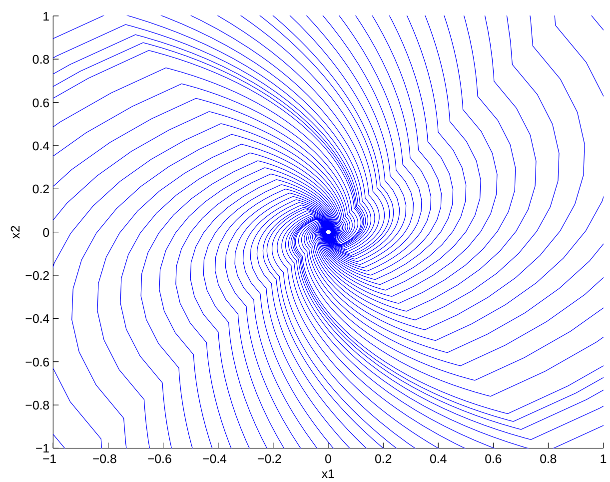

Figure 1: Optimal trajectories in the phase plane

this example, a cycle sequence is obtained such that: 1. A switch occurs from mode 1 to mode 2 for time duration of 1.37 seconds when the trajectory hits the line $D_{12} = \{x : x_2 = -1.29 x_1\}$. Furthermore at this time

$$P_2 = \begin{pmatrix} 0.7106 & 0.2412 \\ 0.2412 & 0.5610 \end{pmatrix}$$

2. A switch occurs from mode 2 to mode 1 for time duration of 0.55 seconds when the trajectory hits the line $D_{21} = \{x : x_2 = 1.19 x_1\}$. Furthermore at this time

$$P_1 = \begin{pmatrix} 0.8865 & 0.05916 \\ 0.05916 & 1.4342 \end{pmatrix}$$

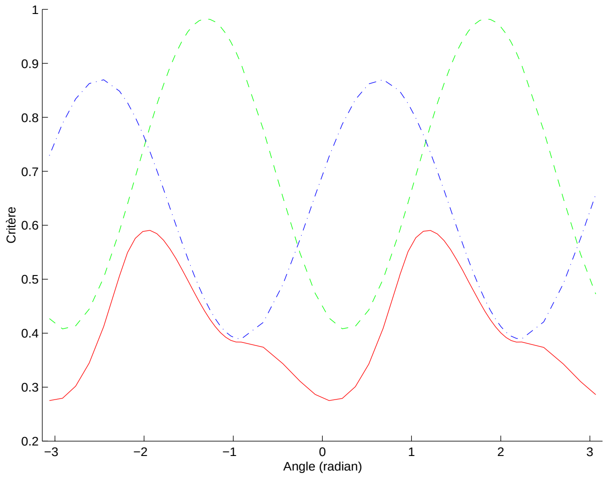

Figure 2: Optimal hybrid criteria (solid), Optimal criteria for the mode 1(dotted) and 2 (dash dotted) obtained for several initial conditions on the unit circle.

3. $P_k(t)$, $k = 1, 2$ is periodic of time period $T = 1.92$.

Conclusion about optimality can be made since this is the only one extremal trajectories which lead to a periodic switching scheme and symmetric solutions with respect to x. The figure 2 compare the optimal hybrid criteria with the one obtained for each mode separately (solving ARE associated). This figure shows that the result we obtained, is effectively better one.

6 Conclusion

A general optimal hybrid control problem which enables autonomous and controlled switching has been presented. A MP approach for solving such hybrid systems appears clearly as an efficient tool. Applying this principle, one gets necessary conditions satisfied by the solution. Additional transversality conditions at switching time are obtained. In context of a linear quadratic criteria, it is shown that closed loop hybrid control is determined by solving a sequence of Differential Riccati Equations. This last point leads to a stability result for switched systems. At last an illustrative example shows how to build numerically the closed loop control. Theorem 4 which express how switching time can be achieved in the case of controlled switching, is of importance for optimal control law synthesis.

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