Time Optimal Control of Hybrid Systems

P. Riedinger, C. Zanne, F. Kratz
Centre de Recherche en Automatique de Nancy (CRAN)
CNRS UPRES 7039 A
ENSEM 2, avenue de la forêt de Haye
54516 Vandoeuvre-les-Nancy Cedex France
Pierre.Riedinger@ensem.u-nancy.fr

Abstract

In this paper we discuss about optimal control for hybrid systems. First, we present a general formulation of optimal control problem for hybrid systems. Such a formulation allows to use straightaway the Maximum Principle of Pontryagin. In a second time, we focus on time optimal control for hybrid systems defined by a finite family of controlled linear systems and show how switching time can be achieved. An illustrative example ends this paper.

1 Introduction

In automatic control area two kinds of dynamical systems have been separately considered: continuous time and discrete event depending on the nature of the variables. Discrete variables mean variables that take values in a discrete set and likewise for continuous variables, that is to say they take values in a continuous set. The behavior of such entities is described by dynamical systems such as automata and differential equations respectively.

Their integration into one system ensures interaction between them. Generally it can be said that the discrete part plays the role of the supervisor of the continuous part. When discrete event occurs, it may change the continuous dynamic and therefore discontinuities in the vector field and /or in the continuous state appear. This leads to non linear behavior even if the basis models are linear.

In the last decade, there was a great interest in studying such systems with regard to their potential applications as well as to the theoretical point of view [1-6]. Despite an extensive effort, it appears palpably that the wide class of hybrid systems excludes an unified approach but rather several theoretical frameworks according to the objectives and the task [7]. It could be further added that questions relevant to continuous aspect deal more with the non smooth analysis with a poor discrete part. In other words the control of the continuous dynamic is determined by switching between continuous models [8]. On the other hand, verification tools, supervisory control and computer science are used in full like in hybrid automata but with a poor continuous dynamic such as integrators [9].

In this article optimization problem for continuous part of hybrid systems is investigated. Linear controlled hybrid systems in accordance with terminology employed by Branicky [10] is used and for a given criteria the target is to look for an optimal control law. This paper is organized as follows. In section 2 optimal control problem for hybrid systems is formulated. Such a formulation leads to consider the MP as an efficient tool for handling the problem under study. Section 3 focuses on time optimal control. In this section it is shown how optimal switching schemes can be obtained. Finally the proposed results are illustrated on a second order hybrid system in section 4.

2 Optimal Problem

Let us start by a general description of controlled hybrid systems and the associated optimal problem. We have in mind systems that switch among vector fields over time, that is:

$$\dot{x}(t) = f_k(x(t), u_k(t)), k = 1, ..., s \tag{1}$$

where $x(t) \in R^n$, $u_k(t) \in U_k$ a bounded set included in R^{m_k}. Here, we suppose that f_k and $\frac{\partial f_k}{\partial x}$ are continuous functions on the direct product $R^n \times \bar{U}_k$. To be more general weak assumptions may be considered as threshold, jumps on the states and Lipschitz conditions, see [11] and [12] for more details.

System (1) can also be written as:

$$\dot{x}(t) = \sum_{k=1}^{s} i_k(t) f_k(x(t), u_k(t)) \tag{2}$$

where $i_k(t)$ refers to the k^{th}-component of $i(t)$ and $i(t)$ is a Boolean vector ($i(t) \in \{0,1\}^s$) so that there is one and only one component of $i(t)$ equal to 1 i.e. $i(t) \in D$ where

$$D = \left\{ \alpha \in \{0,1\}^s \,/ \sum_{k=1}^{s} \alpha_k = 1 \right\}.$$

Control inputs are both the discrete input $i(.)$, which can be seen as piecewise-constant function, and the classical inputs $u_k(.)$. So, discrete input $i(.)$ allows to choose the active mode at any time.

Now, let us introduce an hybrid criteria

$$J = \sum_{k=1}^{s} \int_{t_0}^{t_f} i_k(t) L_k(x(t), u_k(t)) dt \qquad (3)$$

where L_k and $\frac{\partial L_k}{\partial x}$ are defined and continuous on the direct product $R^n \times \bar{U}_k$. Then optimal control is the control that minimizes the functional J.

This formulation allows to use straightaway the Maximum Principle (MP) of Pontryagin [13] to get necessary conditions for optimal trajectories. The main advantage of MP against variational method results in its extremely large admissible control classes. Indeed, only measurable control inputs that take values into a bounded set are needed which is clearly the case here.

3 Time optimal control

In this section, we emphasize on time optimal control with MP for switched and hybrid systems where each vector field is linear (i.e. $f_k(x, u_k) = A_k x + B_k u_k$). In this case the functions L_k take the form $L_k(x, u_k) \equiv 1$ and J becomes $J = t_0 - t_f$.

Given a fixed initial position x_0 and final position x_1 in the state space, the problem may now be stated as follows: find among admissible controls, the controls which take the state from the initial position x_0 to the final position x_1 within a minimal transfer time. We assume the reader is familiar with the MP, so the Hamiltonian system is given by

$$\dot{x}(t) = \sum_{k=1}^{s} i_k(t) \left(A_k x(t) + B_k u_k(t) \right) \qquad (4)$$

$$\dot{\lambda}(t) = - \sum_{k=1}^{s} i_k(t) A_k^T \lambda(t) \qquad (5)$$

with the Hamiltonian function for (4) and (5)(time mention is omitted)

$$H(\lambda, x, u, i) = \sum_{k=1}^{s} i_k \lambda^T (A_k x + B_k u_k) \qquad (6)$$

Applying the MP principle a necessary condition for the control $(i, u)(t)$ and trajectory $x(t)$ to be time-optimal is that there exists an absolutely continuous function $\lambda(t)$, never zero, solving (5) almost everywhere (a.e.) on $[t_0, t_f]$ such that:

$$\begin{aligned} i.\ H(\lambda, x, u, i) &= \sup_{j,v} \sum_{k=1}^{s} j_k \lambda^T (A_k x + B_k v_k)\ a.e., &(7)\\ ii.\ H(\lambda, x, u, i) &\geq 0 \text{ on } [t_0, t_f]. &(8) \end{aligned}$$

Remark: The term "a.e." can be omitted if the control is restricted to the class of piecewise continuous control functions. Then (5) holds everywhere except perhaps for finitely many points but at those points the one-sided limits exist and are equal [14].

Trajectories such that (4)(5) and (6) are satisfied, are called extremal. Conditions (6)-(8) provide a set of candidates for an optimal control but say nothing about existence question. This is why MP assumes that an optimal control exists. In addition, a non empty set of candidates without solution can exist.

The following is intended for this specific study and will state which assumptions are needed.

We choose bounded closed convex polyhedron domain U_k for each control $u_k(.)$ as usually for linear time-optimal processes.

Recall that the Lie bracket of the two vector fields f and g is again a vector field, denoted by $[f, g]$, which can be expressed as:

$$[f(x), g(x)] = \frac{\partial g}{\partial x}(x) f(x) - \frac{\partial f}{\partial x}(x) g(x). \qquad (9)$$

The k^{th}-iterated Lie bracket of f on g is defined as (see [15] for more details):

$$ad_f^k g(x) = [f(x), ad_f^{k-1} g(x)] \qquad (10)$$

with initial term

$$ad_f^0 g(x) = g(x). \qquad (11)$$

Let $e_k^1, e_k^2, \ldots, e_k^q$, be all the vertices of the polyhedron U_k. Assume that the distribution of vector fields

$$\Delta_{k,\ell}^{j,i}(x) = span \left\{ ad^m_{f_k(x, e_k^j)} \left(f_k(x, e_k^j) - f_\ell(x, e_\ell^i) \right), m = 0, 1, \ldots \right\} \qquad (12)$$

has constant dimension n for all x and all indices k, ℓ, i, j with restriction $i \neq j$ if $k = \ell$.

One can notice that if $k = \ell$ and $i \neq j$ then $w = e_i^k - e_j^k$ is a vector directed along a rib of U_k. So if $\dim \Delta_{i,j}^{k,k}(x) = n$ then the vectors $B_k w, A_k B_k w, \ldots, A_k^{n-1} B_k w$ are linearly independent in R^n i.e. in other words, the vector $B_k w$ cannot belong to a true subspace of R^n which is invariant with respect to the operator A_k.

Condition (12) avoids singular trajectories which arise when the partial derivative of the Hamiltonian function H with respect to the control variable u vanishes identically on a set of positive measure [16]. Thus, this condition leads to piecewise-constant function $u_k(.)$ which takes its values on the vertices of the polyhedron U_k. So, if a time-optimal control exists, it is a bang-bang control.

Consider now an extremal trajectory $x(.)$ at a given time t (not a switching time) with extremal control ($u_k(t) = e_k^j$, $i_k(t) = 1$). So we may see in accordance with (7) that, for all ℓ and i ($i \neq j$ if $k = \ell$),

$$\lambda^T(t)(A_k x(t) + B_k e_k^j) > \lambda^T(t)(A_\ell x(t) + B_\ell e_\ell^i). \quad (13)$$

Let

$$P_{k,\ell}^{j,i}(t) = \lambda^T(t)((A_k - A_\ell)x(t) + B_k e_k^j - B_\ell e_\ell^i) \quad (14)$$

$P_{k,\ell}^{j,i}(t)$ are absolutely continuous function. If we denote by T_k^j the interval of time where the control ($u_k(t) = e_k^j$, $i_k(t) = 1$) is solution of (7), $P_{k,\ell}^{j,i}(t)$ is analytic on T_k^j since $x(.)$ and $\lambda(.)$ are solutions of equation (4) and (5) with constant coefficients. A switching time occurs at the first time where there exist indices such that $P_{k,\ell}^{j,i}(t)$ vanishes. Switching times can be obtained by the application of the following theorem.

Theorem 1 *$P_{k,\ell}^{j,i}(t)$ satisfies the homogenous linear ODE with constant coefficient defined by the characteristic polynomial SQ where S is the symmetric of the minimal polynomial of A_k and Q the minimal polynomial of $A_k^T \otimes I_n - I_n \otimes A_k$, ($\otimes$ denote Kronecker product, [17]). Further, general solution has the form*

$$P_{k,\ell}^{j,i}(t) = \sum_m p_m(t)e^{-\alpha_m t} + \sum_m c_m(t)e^{\omega_m t} \quad (15)$$

where α_m, ω_m are respectively distinct eigenvalues of the matrices A_k and $A_k^T \otimes I_n - I_n \otimes A_k$, and p_m, c_m are polynomials such that the degree is less than multiplicity of the associated eigenvalue.I_n refers to the identity matrix in the space R^n.

Proof: Consider two differentiable vector fields f and g and form the analytic function $P(t)$ as

$$P(t) = \lambda^T(t)g(x(t)) \quad (16)$$

where dynamics of x and λ are given by

$$\dot{x} = f(x) \quad (17)$$

$$\dot{\lambda} = -\left[\frac{\partial f}{\partial x}\right]^T \lambda. \quad (18)$$

So, on differentiating $P(t)$ successively with respect to t, we get :

$$\frac{d^m P}{dt^m}(t) = \lambda^T(t) ad_f^m g(x(t)) \quad (19)$$

First, suppose $f(x) = Mx$ and $g(x) = Nx$ linear so $ad_f^1(.) : R^{n^2} \to R^{n^2}$ appears as a first order differential operator which assigns at a given matrix N, the matrix $NM - MN$. Let $col(.)$ the stacking operator applied to matrix $A = [A_1, ..., A_n]$ by $col(A) = [A_1^T, ..., A_n^T]^T$ where $A_1, ..., A_n$ are the columns of A. Then an interesting property is: $col(ACB) = (B^T \otimes A)col(C)$. Thus, if we form $col(N)$ and $col(NM - MN)$, it is easy to check that $ad_f^1(.)$ is defined by the matrix :

$$\tilde{M} = M^T \otimes I_n - I_n \otimes M, \quad (20)$$

with eigenvalues

$$\omega_j = \alpha_k - \alpha_i, \quad j = 1, ..., n^2, \quad i, k = 1, ..., n \quad (21)$$

and associated eigenvectors

$$V_j = V_k \otimes V_i \quad (22)$$

where α_i, α_k, denote eigenvalues of the matrix M and V_k, V_i denote respectively eigenvectors of the matrix M^T and M. Let Q be the minimal polynomial of $M^T \otimes I_n - I_n \otimes M$, also it is obvious that, for all N,

$$Q(ad_M)(N) = \sum_m \mu_i ad_M^m(N) = 0, \quad (23)$$

where μ_i's are coefficients of Q. In this case, we can conclude, that the scalar function $P(t)$ satisfies the ODE

$$Q(P)(t) = \sum_m \mu_i \frac{d^m P}{dt^m}(t) = 0. \quad (24)$$

Second, more generally, we must consider affine vector fields $f(x) = Mx + c$ and $g(x) = Nx + d$. Similarly we obtain

$$Q(ad_{f(x)})(g)(x) = Q(ad_{Mx})(N)(x) + V \quad (25)$$

where the additional term V is a constant vector. As $Q(ad_{Mx})(N)(x)$ is identically zero, applying differential operator ad_f to this relationship successively, we get :

$$ad_f^k Q(ad_f)(g)(x) = (-1)^k M^k V \quad (26)$$

By Caley-Hamilton theorem, we conclude :

$$SQ(ad_f)(g)(x) = 0 \quad (27)$$

where S is the symmetric polynomial of the minimal polynomial of M (i.e. $S(-x) = -R(x)$ if R is the minimal polynomial of M). This completes the proof of Theorem 1. ∎

Remarks:

1. Condition (12) is no more than (15) has initial conditions not identically null.
2. In (25), the constant vector V is obtained by the relationship

$$\frac{d^m P}{dt^m}(t) = \lambda^T ad^m_{Mx}(N)x + \lambda^T(-1)^m \sum_{k=1}^{m}(-1)^k ad^{k-1}_{Mx}(M^{m-k}N)c + \lambda^T(-1)^m M^m d \qquad (28)$$

4 Application

Consider now an hybrid system built with two second-order controlled-systems, that is:

$$\dot{x}(t) = \begin{cases} A_1 x(t) + B_1 u_1(t), & if\ i = [1\ 0] \\ A_2 x(t) + B_2 u_2(t), & if\ i = [0\ 1] \end{cases} \qquad (29)$$

Matrices A_1 and A_2 are defined by, $A_1 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & 4 \\ -3 & 2 \end{pmatrix}$ and $B_1 = B_2 = I_2$. Let the control domain be defined by $U = U_1 = U_2 = [-1, 1]^2$.

Let us consider the problem of fastest descent to the origin $O = (0, 0)$ from a given initial state. In this present case (5) leads to the following system:

$$\dot{\lambda}(t) = \begin{cases} -A_1^T \lambda(t), & if\ i = [1\ 0] \\ -A_2^T \lambda(t), & if\ i = [0\ 1] \end{cases} \qquad (30)$$

For a given x, if we draw the set of point form by $A_k x + B_k U$, $k = 1, 2$, it is clear from Figure 1 that, if the vector λ is located at the angle α, the maximum of H is attained at the vertex $A_1 x + e_1$. A switching time occurs when λ leaves this angle location. In linear case, it is easy to determine when this event occurs but here, α evolves with time due to the dependence upon x. Theorem 1 gives the solution to find this time. In fact, assume x and λ are given, the analytic form (15) of functions $P^{1,j}_{1,\ell}(t)$, $j = 1, 2, 4$ and $\ell = 1, 2 (j \neq 1$ if $\ell = 1)$ is identified. It works because all the roots ((15) and (21)) and all initial conditions (19) are known. Thus, we can determine the first time where one of them vanishes. This one matches at a switching time.

Now, with an appropriate final choice of λ at the origin point O, we can build all extremal trajectories finishing at this point and determine precisely time interval during which a control is effective (Fig. 2).

Remember that optimal trajectories are a subset of extremal trajectories. Thus, if for a given initial state there is only one extremal trajectory that solves the problem, we can conclude at optimality under existence assumption. However, if we find several extremal trajectories, we must decide which one is optimal. Thus,

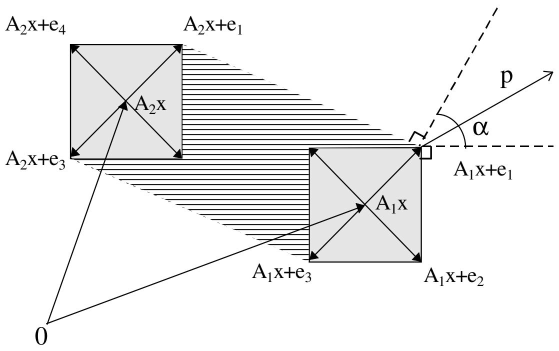

Figure 1: Optimal trajectories

an important and largely open problem is still to detect when an extremal trajectory ceases to be optimal.

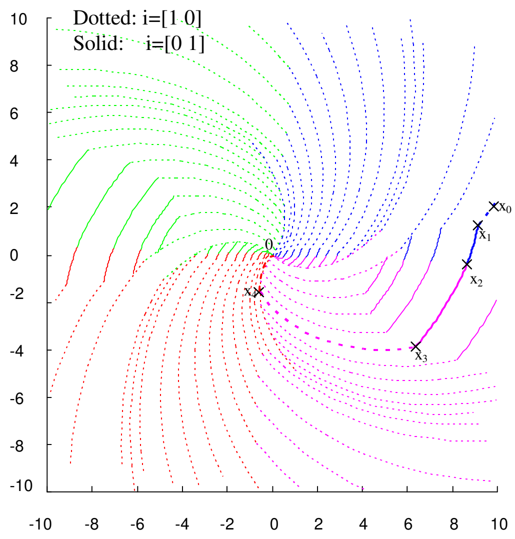

Figure 2: Optimal trajectories

Remark: On Figure 2, one can see optimal trajectories for the fastest descent problem. For example, if we start at the position $x_0 \approx (10, 2)$, time optimal control is $i = [1\ 0]$, $u = e_3$ until we reach position x_1 where the control switchs to $i = [0\ 1]$, $u = e_3$ and so on. Finally, we obtain the following bang bang control sequence:

$i = [1\ 0], [0\ 1], [0\ 1], [1\ 0], [1\ 0]$;

$u = e_3, e_3, e_4, e_4, e_1$.

5 Conclusion

In this article, we show that an MP based approach is an efficient way to solve optimization problem in context of hybrid system. This point has been illustrated with time optimal problem in the particular case where each mode is described by linear controlled system. It has been shown that switching time can be retrieved using a theoretical result which states "iterated Lie bracketts of affine vector fields satisfy a linear combination". Our results have been illustrated on a numerical application. At last, future work will address the difficult question of existence of solution and singular trajectories for linear controlled hybrid systems.

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