Tutorial on Optimal Control Theory: Application to the LQ problem

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27 mars 2014

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Issue of the presentation

Provide some elements to cope with optimization problems in the framework of optimal control theory :

- Euler-Lagrange equations
- Pontryaguin Maximum Principle (necessary conditions of optimality)
- Dynamic Programming (sufficient conditions of optimality)

How these tools may be applied to the Linear Quadratic problem ?

Outline of the talk

Formulation of the problem

Motivation

Variational approach : Euler-Lagrange equation

Pontryaguin Minimum Principle

Dynamic Programming

Linear Quadratic Control Problem

Conclusion and Open Questions

Optimal control problem

Dynamical System :

$$\dot{x}(t) = f(x(t), u(t), t), \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^r. \tag{1}$$

$$x(t_0) = x_0. \tag{2}$$

Criterion to minimize :

$$J(x, u, x_0) = \underbrace{K_f(x(t_f), t_f)}_{Terminal\ cost} + \int_{t_0}^{t_f} \underbrace{L(x(t), u(t), t)}_{Instantaneous\ cost} \mathrm{d}t \tag{3}$$

- $t_f < +\infty$, the time horizon is finite ; t_f free or fixed.
- $t_f = +\infty$, the time horizon is infinite, (then $K_f(x(t_f), t_f) = 0$).

Optimization problem :

$$\min_{u(t) \in \mathbb{R}^r} J(x, u, x_0)$$

under the constraint (1).

Motivation and typical examples of criteria

- Stability : at $t = t_f$, $x(t_f)$ should be the closest as possible to the origin : $K_f(x(t_f), t_f) = \frac{1}{2}x'(t_f)K_f x(t_f)$; $L(x, u, t) = 0$.
- Trajectory tracking : $x(t)$ should tracks $x_{\text{desired}}(t)$. Minimize the error $\|x_{\text{desired}}(t) - x(t)\|$.
- Potential energy ; input's cost energy : $K_f(x(t_f), t_f)$ and $L(x, u, t)$ are quadratic :

$$\begin{aligned} K_f(x(t_f), t_f) &= \frac{1}{2}x'(t_f)K_f x(t_f); \\ L(x(t), u(t), t) &= \frac{1}{2}(x'(t)Q(t)x(t) + u'(t)R(t)u(t)). \end{aligned}$$

- Economy : Monetary and fiscal policy interaction (discount rates).

$$Q(t) = e^{2\alpha t}Q; \qquad R(t) = e^{2\alpha t}R.$$

- Time optimality : $K_f(x(t_f), t_f) = 0$, $L(x(t), u(t), t) = 1$, that is $J(x, u, x_0) = t_f$.

Basic assumptions

- $f(x,u,t)$ and $L(x,u,t)$ are continuous functions.
- f is Lipschitz.
- for both f and L, all partial derivatives w.r.t. x and u exist and are continuous.
- L belongs to $\mathcal{C}^1$.

- In addition, here, $t \to u(t)$ is assumed to be continuous.
- t_f is fixed.

This implies the equation (1) admits one and only one solution.

Main idea : modifying the criterion

To avoid the problem of constrained optimization problem, the instantaneous criterion is modified to take into account the constraint $f(x(t), u(t), t) - \dot{x}(t) = 0$.

$$\int_{t_0}^{t_f} \lambda'(t)\Big(f(x(t), u(t), t) - \dot{x}(t)\Big)\mathrm{d}t = 0, \tag{4}$$

where $\lambda(t)$ is an arbitrary chosen column vector, called the Lagrange multiplier or costate-vector.

By introducing the Hamiltonian function

$$\mathcal{H}(x, u, t, \lambda) = L(x, u, t) + \lambda'(t) f(x, u, t), \tag{5}$$

we have a modified criterion to minimize

$$\tilde{J}(x, u, x_0) = K_f(x(t_f), t_f) + \int_{t_0}^{t_f} \Big[\mathcal{H}(x, u, t, \lambda(t)) - \lambda'(t)\dot{x}(t)\Big]\mathrm{d}t. \tag{6}$$

Note also that, by definition

$$\dot{x}(t) = \frac{\partial \mathcal{H}}{\partial \lambda(t)}(x(t), u(t), t) = f(x(t), u(t), t). \tag{7}$$

Refomulation of the criterion

By using the integration by parts

$$-\int_{t_0)}^{t_f} \lambda'(t)\dot{x}(t)\mathrm{d}t = -\lambda'(t_f)x(t_f) + \lambda'(t_0)x_0 + \int_{t_0}^{t_f} \dot{\lambda}'(t)x(t)\mathrm{d}t, \qquad (8)$$

one gets

$$\tilde{J} = \int_{t_0}^{t_f} \left[\mathcal{H}(x,u,t,\lambda(t)) + \dot{\lambda}'(t)x(t)\right]\mathrm{d}t + \left[K_f(x(t_f),t_f) - \lambda'(t_f)x(t_f)\right] + \lambda'(t_0)x_0. \qquad (9)$$

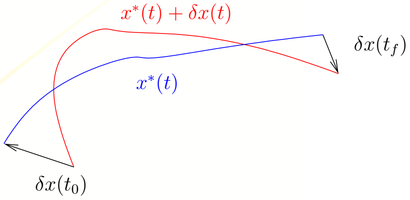
First order necessary conditions (I)

By assuming that $u^*(t)$ is a continuous optimal control input, solution of the optimization problem. The associated trajectory is denoted $x^*(t)$.

Considering

$$u(t) = u^*(t) + \delta u(t); \quad (10)$$

$$x(t) = x^*(t) + \delta x(t) \quad (11)$$



leads to the inequality

$$\tilde{J}(x^*, u^*, x_0) \leq \tilde{J}(x^* + \delta x, u^* + \delta u, x_0), \qquad \forall \delta x, \delta u. \qquad (12)$$

The first order necessary condition consists in the fact that $\tilde{J}(x^*, u^*, x_0)$ is an extremum.

First order necessary conditions (II)

$$\tilde{J}(x^* + \delta x, u^* + \delta u, x_0) - \tilde{J}(x^*, u^*, x_0)$$
$$= \int_{t_0}^{t_f} \Big[\frac{\partial \mathcal{H}}{\partial x}'(x^*, u^*, t, \lambda(t))\delta x + \frac{\partial \mathcal{H}}{\partial u}'(x^*, u^*, t, \lambda(t))\delta u + \dot{\lambda}'(t)\delta x\Big]\mathrm{d}t$$
$$+ \Big[\frac{\partial K_f}{\partial x}'(x^*(t_f), t_f) - \lambda'(t_f)\Big]\delta x(t_f) + \lambda'(t_0)\delta x_0 \geq 0. \quad (13)$$

$$\tilde{J}(x^* + \delta x, u^* + \delta u, x_0) - \tilde{J}(x^*, u^*, x_0)$$
$$= \int_{t_0}^{t_f} \Big[\Big(\frac{\partial \mathcal{H}}{\partial x}'(x^*, u^*, t, \lambda(t)) + \dot{\lambda}'(t)\Big)\delta x + \frac{\partial \mathcal{H}}{\partial u}'(x^*, u^*, t, \lambda(t))\delta u\Big]\mathrm{d}t$$
$$+ \Big[\frac{\partial K_f}{\partial x}'(x^*(t_f), t_f) - \lambda'(t_f)\Big]\delta x(t_f) + \lambda'(t_0)\delta x_0 \geq 0. \quad (14)$$

First order necessary conditions (III)

- The initial state x_0 is fixed, that is $\delta x_0 = 0$. No condition about $\lambda(t_0)$.
- The final state is free, that is $\delta x(t_f) \neq 0$, which implies the transversality condition

$$\lambda(t_f) = \frac{\partial K_f}{\partial x}(x^*(t_f), t_f). \tag{15}$$

- $\forall \delta x$, leads to

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x^*, u^*, t, \lambda(t)). \tag{16}$$

- $\forall \delta x$, leads to

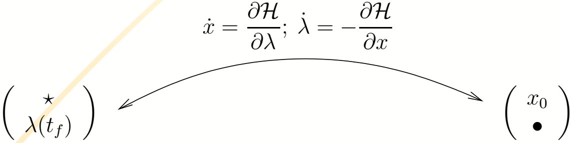
$$\frac{\partial \mathcal{H}}{\partial u}(x^*, u^*, t, \lambda(t)) = 0. \tag{17}$$

Interpretations of the first order necessary conditions

By definition, $\lambda(t) = \dfrac{\partial \mathcal{H}}{\partial f} = \dfrac{\partial \mathcal{H}}{\partial \dot{x}}$. We have the Euler-Lagrange equation

$$\dot{\lambda}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{H}}{\partial \dot{x}}\right)(x^*, u^*, t, \lambda(t)) = -\frac{\partial \mathcal{H}}{\partial x}(x^*, u^*, t, \lambda(t)). \qquad (18)$$

Two-point boundary-value problem, due to the initial state condition and to the final co-state condition (transversality condition).



Generally, $\dfrac{\partial \mathcal{H}}{\partial u} = 0$ leads to the expression of the optimal control u^*.

Numerical example (I)

A car is driving in straight line and should maximize the distance and simultaneously to minimize the energy of the input.

$$\ddot{\xi} = u; \quad \xi(0) = \dot{\xi}(0) = 0.$$

The criterion to minimize is

$$J = -\xi(t_f) + \int_0^{t_f} u^2(t)\mathrm{d}t.$$

Let denote $x = \begin{pmatrix} \xi \\ \dot{\xi} \end{pmatrix}$, we have $\dot{x}(t) = f(x, u, t) = \begin{pmatrix} \dot{\xi} \\ u \end{pmatrix}$,

$\lambda(t) = \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix}$ and $\mathcal{H} = u^2(t) + \lambda_1(t)\dot{\xi}(t) + \lambda_2(t)u(t)$.

Numerical example (II)

Applying the Euler-Lagrange relations leads to

$$
\begin{aligned}
\dot{\lambda}_1(t) &= -\frac{\partial \mathcal{H}}{\partial \xi} = 0, \\
\dot{\lambda}_2(t) &= -\frac{\partial \mathcal{H}}{\partial \dot{\xi}} = -\lambda_1(t), \\
u^*(t) &= -\frac{1}{2}\lambda_2(t), \\
\lambda_1(t_f) &= \frac{\partial K_f}{\partial \xi} = -1 \Rightarrow \lambda_1(t) = -1, \\
\lambda_2(t_f) &= \frac{\partial K_f}{\partial \dot{\xi}} = 0 \Rightarrow \lambda_2(t) = t - t_f.
\end{aligned}
$$

Which implies $u^*(t) = \dfrac{t_f - t}{2}$ and $x(t_f) = \dfrac{t_f^3}{6}$.

Second numerical example (I)

Let consider the system

$$\dot{x}(t) = -u(t); \quad x(0) = 2; \quad J = -x(t_f) + \frac{1}{2}\int_0^{t_f} (x^2(t) + u^2(t))\mathrm{d}t.$$

Then $\mathcal{H} = \frac{1}{2}(x^2 + u^2) - \lambda u$. It yields $\dot{x} = \lambda$ and $\dot{\lambda} = -x$ and $\lambda(t_f) = -1$.

$$\begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{pmatrix} x_0 \\ \lambda(0) \end{pmatrix}$$

For $t_f = \frac{\pi}{2}$,

$$-1 = \lambda(t_f) \neq \lambda(\frac{\pi}{2}) = -\sin(\frac{\pi}{2})x_0 + \cos(\frac{\pi}{2})\lambda(0) = -2.$$

There is a solution to the two-boundary problem only iff $\cos(t_f) \neq 0$.

Third numerical example (I)

Let consider the system with two inputs

$$\dot{x}(t) = u_1(t) + u_2(t); \quad x(0) = 0.$$

The criterion to minimize is

$$J = -x(t_f) + \int_0^{t_f} (u_1^2(t) - u_2^2(t))\mathrm{d}t.$$

Then $\mathcal{H} = u_1^2(t) - u_2^2(t) + \lambda(t)[u_1(t) + u_2(t)]$.

Third numerical example (II)

Applying the Euler-Lagrange relations leads to

$$\begin{aligned}
\dot{\lambda}(t) &= -\frac{\partial \mathcal{H}}{\partial x} = 0 \Rightarrow \lambda(t) = \mathrm{Cste}, \\
\frac{\partial \mathcal{H}}{\partial u_1} &= 0 \Rightarrow u_1(t) = -\frac{1}{2}\lambda, \\
\frac{\partial \mathcal{H}}{\partial u_2} &= 0 \Rightarrow u_2(t) = +\frac{1}{2}\lambda, \\
\lambda(t_f) &= \frac{\partial(-x_f)}{\partial x_f} = -1.
\end{aligned}$$

That implies $\lambda(t) = -1$, $u_1(t) = -u_2(t) = -\frac{1}{2}$ and $x(t) = 0$ and finally $J^* = 0$. Nevertheless for $u_1(t) = u_2(t) = 1$, we have $x(t) = 2t$ and $J = -2t_f < 0 = J^*$. J^* is not the minimum, but a saddle-point.

Further necessary conditions are required.

Second order necessary conditions

The solution consists in a mimimum and not only in an extremum.

$$\frac{\partial^2 \mathcal{H}}{\partial u^2} > 0.$$

Link with the convexity of the criterion and implicit function theorem.

Main idea : introducing the singular controls

Let us consider a control system (the link with our optimization problem, that is the relation between x and z will be clarified in the sequel) :

$$\dot{z}(t) = \tilde{f}(z, u, t), \quad z(t_0) = z_0 \in \mathbb{R}^\ell. \tag{19}$$

We denote z_u the trajectory solution of (19) on $[t_0, t_f]$ issue from z_0 and associated with the control u.
The associated end-point mapping E_{z_0,t_f} at time t_f is defined by

$$\begin{aligned} E_{z_0,t_f} : \quad \mathcal{U} \quad &\longrightarrow \quad \mathbb{R}^\ell \\ u \quad &\longmapsto \quad z_u(t_f), \end{aligned} \tag{20}$$

with z_u the trajectory solution of (19) associated to the control u, and $\mathcal{U}$ is the set of admissible controls.
Definition : We call singular a control $u(t)$ on $[t_0, t_f]$ with trajectory defined on $[t_0, t_f]$ such that the end-point mapping is singular at u, that is the Fréchet derivative of the end-point mapping is not surjective when evaluated on u.

Fréchet derivative of the end-point mapping (I)

Regularity : The end-point mapping is $\mathcal{C}^\infty$. The successive derivatives can be computed as follows.

$$\begin{array}{ccc} u & \xrightarrow{\tilde f} & z \\ u+\delta u & \xrightarrow{\tilde f} & z+\delta z \end{array}$$

$$\dot z + \dot{\delta z} = \tilde f(z+\delta z; u+\delta u) = \tilde f(z,u) + \frac{\partial \tilde f}{\partial z}(z,u)\delta z + \frac{\partial \tilde f}{\partial u}(z,u)\delta u$$

$$+ \frac{\partial^2 \tilde f}{\partial z \partial u}(z,u)(\delta z, \delta u) + \frac{1}{2}\frac{\partial^2 \tilde f}{\partial z^2}(z,u)(\delta z, \delta z) + \frac{1}{2}\frac{\partial^2 \tilde f}{\partial u^2}(z,u)(\delta u, \delta u) + \cdots$$

δz can be written as $\delta_1 z + \delta_2 z + \cdots$, where $\delta_1 z$ is linear in δu, $\delta_2 z$ is quadratic in δu and are given by identification as (where $z(t_0) = z_0$ is fixed) :

$$\dot{\delta_1 z} = \frac{\partial \tilde f}{\partial z}(z,u)\delta_1 z + \frac{\partial \tilde f}{\partial u}(z,u)\delta u; \quad \delta_1 z(t_0) = 0; \tag{21}$$

$$\dot{\delta_2 z} = \frac{\partial \tilde f}{\partial z}(z,u)\delta_2 z + \frac{\partial^2 \tilde f}{\partial z \partial u}(z,u)(\delta_1 z, \delta u) + \frac{1}{2}\frac{\partial^2 \tilde f}{\partial z^2}(z,u)(\delta_1 z, \delta_1 z)$$

$$+ \frac{1}{2}\frac{\partial^2 \tilde f}{\partial u^2}(z,u)(\delta u, \delta u); \quad \delta_2 z(t_0) = 0;$$

Fréchet derivative of the end-point mapping (II)

Then the Fréchet first derivative of the end-point mapping at u is given by

$$\mathrm{d}E_{z_0,t_f}(u).\delta u = \delta_1 z(t_f).$$

By denoting

$$A(t) = \frac{\partial \tilde{f}}{\partial z}(z_u(t), u(t)); \qquad B(t) = \frac{\partial \tilde{f}}{\partial u}(z_u(t), u(t));$$

The equation (21) becomes

$$\dot{\delta_1} z = A(t)\delta_1 z + B(t)\delta u; \quad \delta_1 z(t_0) = 0; \tag{22}$$

and if the transition matrix $M(t)$ is solution of $\dot{M}(t) = A(t)M(t)$ and $M(t_0) = Id$, then also

$$\mathrm{d}E_{z_0,t_f}(u).\delta u = \delta_1 z(t_f) = M(t_f)\int_{t_0}^{t_f} M^{-1}(s)B(s)\delta u(s)\mathrm{d}s.$$

The singular control is of corank 1 if there exists a unique vector φ (up to a scalar) such that

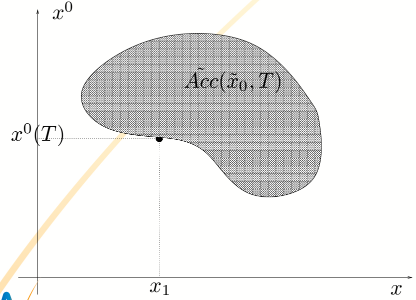
$$\varphi' \mathrm{d}E_{z_0,t_f}(u).\delta u = 0, \quad \forall \delta u$$

Back to the main idea

Consider $z = \begin{pmatrix} x(t) \\ x^0(t) \end{pmatrix}$, $z(t_0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ where

$$\begin{pmatrix} \dot{x}(t) \\ \dot{x}^0(t) \end{pmatrix} = \tilde{f}(z(t), u(t)) = \begin{pmatrix} f(x(t), u(t), t) \\ L(x(t), u(t), t) \end{pmatrix}.$$

In other words $x^0(t) = \int_{t_0}^{t_f} L(x(t), u(t), t)\mathrm{d}t$, the current value of the criterion.



If the control u associated with the system in x is optimal for the criterion to optimize J, then it is singular for the augmented system in z.

Pontryaguin Minimum Principle (weak version)

Theorem :
If the control u associated with the system (1) is optimal for the cost (3), then there exists an application $p(\cdot)$ absolutely continuous on $[t_0, t_f]$, $\mathbb{R}^n$-valued, called costate vector and a real $p^0 \leq 0$, such that $(p(t_f), p^0)$ is not trivial and verifies :

$$\dot{x}(t) = \frac{\partial \mathcal{H}}{\partial p}(x(t), u(t), p(t), p^0), \quad (23)$$

$$\dot{p}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x(t), u(t), p(t), p^0), \quad (24)$$

$$0 = \frac{\partial \mathcal{H}}{\partial u}(x(t), u(t), p(t), p^0), \quad (25)$$

and $p(t_f) = p^0 \dfrac{\partial K_f}{\partial x}(x(t_f), t_f)$, where

$$\mathcal{H}(x, u, p, p^0) = p' f(x, u, t) + p^0 L(x, u, t). \quad (26)$$

Pontryaguin Minimum Principle : sketch of proof

$$\varphi' \mathrm{d}E_{z_0,t_f}(u).\delta u = 0$$

implies that

$$\varphi' M(t_f)M^{-1}(s)B(s) = 0 \quad \text{almost everywhere on } [t_0, t_f].$$

Let define $\begin{pmatrix} p(t) \\ p^0(t) \end{pmatrix}' = \varphi' M(t_f)M^{-1}(t)$. Then

- $\begin{pmatrix} \dot{p}(t_f) \\ \dot{p}^0(t_f) \end{pmatrix} = \varphi$ not trivial !
- $(\dot{p}'(t);\ \dot{p}^0) = -(p'(t);\ p^0) \begin{pmatrix} \frac{\partial f}{\partial x} & 0 \\ \frac{\partial L}{\partial x} & 0 \end{pmatrix}$.
- $\begin{pmatrix} p(t) \\ p^0(t) \end{pmatrix}' B(t) = 0 = \begin{pmatrix} p(t) \\ p^0(t) \end{pmatrix}' \begin{pmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial L}{\partial u} \end{pmatrix}$.

Pontryaguin Minimum Principle (strong version)

Theorem :
If the control u $\mathcal{U}$-valued associated with the system (1) is optimal for the cost (3), then there exists an application $p(\cdot)$ absolutely continuous on $[t_0, t_f]$, $\mathbb{R}^n$-valued, called costate vector and a real $p^0 \leq 0$, such that $(p(t_f), p^0)$ is not trivial and verifies :

$$\dot{x}(t) = \frac{\partial \mathcal{H}}{\partial p}(x(t), u(t), p(t), p^0), \quad (27)$$

$$\dot{p}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x(t), u(t), p(t), p^0), \quad (28)$$

$$\Rightarrow \quad u^*(t) = \text{argmin}_{u \in \mathcal{U}} \mathcal{H}(x(t), u(t), p(t), p^0), \quad (29)$$

and $p(t_f) = p^0 \dfrac{\partial K_f}{\partial x}(x(t_f), t_f)$, where

$$\mathcal{H}(x, u, p, p^0) = p' f(x, u, t) + p^0 L(x, u, t). \quad (30)$$

Pontryaguin Minimum Principle (strong version)

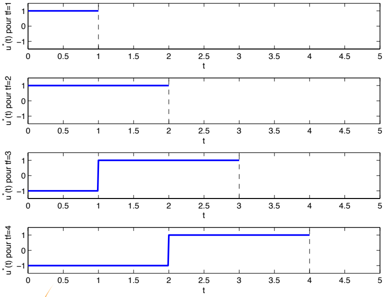
Interpretation :

More difficult to prove. A larger set of admissible controls (u not anymore continuous) is considered. PMP and Euler-Lagrange Theorem differ on the fact that the PMP tells us that the Hamiltonian is minimized at the optimal trajectory and that it is also applicable when the minimum is attained at the boundary of $\mathcal{U}$. See also that p^0 cannot be null.

Numerical example

Let us consider the system $\dot{x} = u$, with the criterion $\int_{t_0}^{t_f} x(t)\mathrm{d}t - 2x(t_f)$ and the constraint $\|u\| \leq 1$. We have the Hamiltonian $\mathcal{H} = x + pu$, leading to $\dot{p}(t) = -1$, $p(t_f) = -2$. One gets $p(t) = t_f - t - 2$. Furthermore

$$u^*(t) = \mathrm{argmin}_{u(t)\in[-1,1]}(x(t) + p(t)u(t)) = \begin{cases} 1 \text{ if } p(t) < 0 \\ \text{undefined if } p(t) = 0 \\ -1 \text{ if } p(t) > 0. \end{cases}$$



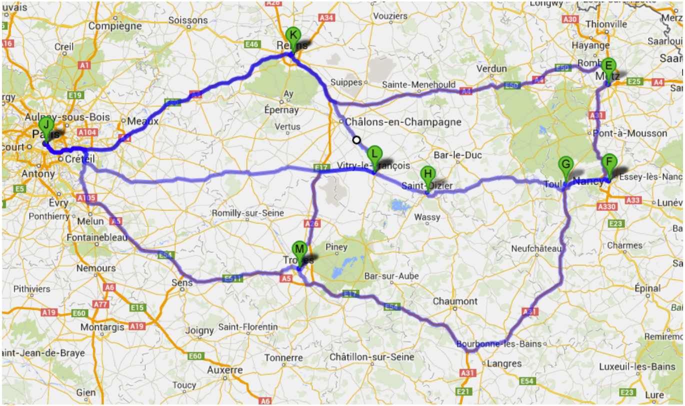
If $t_f \leq 2$: $u^*(t) = 1$.

If $t_f \geq 2$:

$$u^*(t) = \begin{cases} 1 \text{ if } t_f - 2 < t \leq t_f, \\ -1 \text{ if } 0 \leq t < t_f - 2. \end{cases}$$

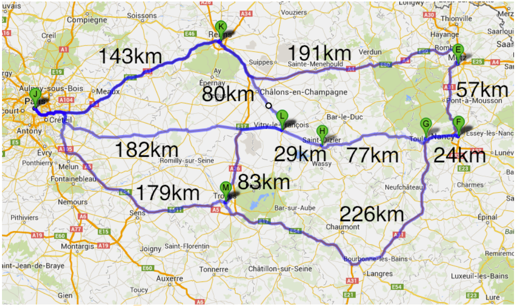
Bellman's Principle : an intuition

How to reach optimally Paris from Nancy ?



Bellman's Principle : an intuition

How to reach optimally Paris from Nancy (in terms of distance) ?

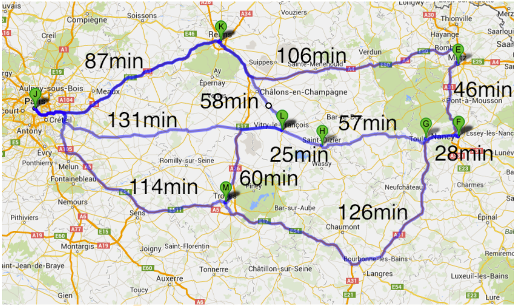


Nancy
↓ 130km
Vitry-le-François
↓ 182km
Paris

Vitry-le-François
↓ optimally ?
Paris
223km ? 182km ? 179km ?

Bellman's Principle : an intuition

How to reach optimally Paris from Nancy (in terms of time) ?

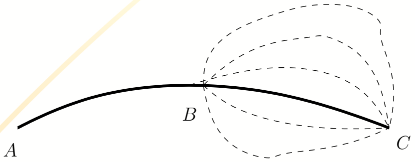


Nancy
↓
Metz
↓
Reims
↓
Paris

Bellman's Principle : sufficient conditions of optimality

Dynamic Programming (DP) is a commonly used method of optimally solving complex problems by breaking them down into simpler problems.

Principle : An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.



"it's better to be smart from the beginning, than to be stupid for a time and then become smart"

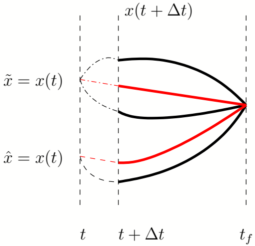
Value function (I)

Let us define the value function as the optimal criterion on the restricted time-horizon : $[t, t_f]$, always ending at t_f and starting at time t with the initial condition $x(t)$:

$$V(t, x(t)) = J^*_{|[t,t_f]} = \min_{u_{|[t,t_f]}} \left(K_f(x(t_f), t_f) + \int_t^{t_f} L(x(s), u(s), s)\mathrm{d}s \right). \quad (31)$$

We have (with the minimum taken under $\dot{x}(t) = f(x(t), u(t), t)$)

$$V(t, x(t)) = \min_{u_{|[t,t_f]}} \Big(\underbrace{\int_t^{t+\Delta t} L(x(s), u(s), s)\mathrm{d}s}_{\text{Cost between } [t,\, t+\Delta t]} + \underbrace{V(t+\Delta t, x(t+\Delta t))}_{\text{Optimal between } [t+\Delta t,\, t_f] \text{ from } x(t+\Delta t)} \Big).$$



Value function (II)

When $\Delta t \to 0$, we have the Hamiltonian Jacobi Bellmann equation

$$-\frac{\partial V}{\partial t}(t, x(t)) = \min_{u \in \mathcal{U}} \left(L(x(t), u(t), t) + \frac{\partial V}{\partial x}(t, x(t)) f(x(t), u(t), t) \right) \quad (32)$$

and

$$V(t_f, x(t_f)) = K_f(x(t_f), t_f). \quad (33)$$

Dynamic Programming

Let $V(t,x)$ defined by (31), and assume that both partial derivatives of $V(t,x)$ exist, $\frac{\partial V}{\partial x}$ is continuous and moreover $\frac{\mathrm{d}}{\mathrm{d}t}V(t,x(t))$ exists, then

$$-\frac{\partial V}{\partial t}(t,x(t)) = \min_{u\in\mathcal{U}} \left(L(x(t),u(t),t) + \frac{\partial V}{\partial x}(t,x(t)) f(x(t),u(t),t) \right); \quad (34)$$

$$V(t_f,x(t_f)) = K_f(x(t_f),t_f) \quad (35)$$

and

$$u^*(t) = \mathrm{argmin}_{u\in\mathcal{U}} \left(L(x(t),u(t),t) + \frac{\partial V}{\partial x}(t,x(t)) f(x(t),u(t),t) \right) \quad (36)$$

Numerical example

Let consider $\dot{x}(t) = x(t)u(t)$, with $u \in [-1, 1]$ and $J = x(t_f)$. If $x_0 > 0$, then $u = -1$ and if $x_0 < 0$, then $u = 1$. If $x_0 = 0$, $J = 0$.

$$V(t,x) = \begin{cases} e^{-(t_f - t)} \text{ if } x > 0, \\ e^{(t_f - t)} \text{ if } x < 0, \\ 0 \text{ if } x = 0, \end{cases}$$

No $\mathcal{C}^1$ solution to HJB equation. Viscosity Solution for HJB equation is required.

Linear Quadratic control problem

System :

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^r. \\ x(t_0) &= x_0. \end{aligned}$$

Criterion to minimize :

$$J = x'(t_f)K_f x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left(x'(t)Qx(t) + u'(t)Ru(t)\right) \mathrm{d}t$$

Assumptions :

$$Q = Q' \geq 0_n, \qquad R = R' > 0_r.$$

Necessary Conditions

Hamiltonian :

$$\mathcal{H} = \frac{1}{2}\left(x'Qx + u'Ru\right) + p'\left(Ax + Bu\right).$$

The necessary conditions are given by the Pontryagin Minimum Principle

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}t} &= -\frac{\partial \mathcal{H}}{\partial x} = -Qx(t) - A'p(t), \\ \frac{\partial \mathcal{H}}{\partial u} &= 0, \quad \text{(first order).} \Rightarrow u^*(t) = -R^{-1}B'p(t). \end{aligned}$$

The final state being free, the condition of transversality is

$$p(t_f) = K_f x_f.$$

Two Boundary Problem

Resolve

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = H \begin{pmatrix} x \\ p \end{pmatrix} = \begin{bmatrix} A & -S \\ -Q & -A' \end{bmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

where $S = BR^{-1}B'$, with the two boundary conditions

$$x(t_0) = x_0, \qquad p(t_f) = K_f x(t_f).$$

How to solve this kind of problem ?

Differential Riccati Equation (DRE)

The necessary conditions are linear and the conditions of transversality are linear with respect to the initial state x_0. The solutions are

$$p(t) = K(t)\xi(t),$$

with $K(t) \in \mathbb{R}^{n\times n}$ and $\xi(t)$ the predetermined state of the system. The controls are

$$u^*(t) = -R^{-1}B'K(t)\xi(t),$$

$$\dot{\xi}(t) = \big(A - SK(t)\big)\xi(t), \quad \xi(t_0) = x_0.$$

The matrice $K(t)$ should verify the Differential Riccati Equation

$$\dot{K}(t) = -A'K(t) - K(t)A - Q + K(t)SK(t),$$

with

$$K(t_f) = K_f.$$

Solving Differential Riccati Equation

Set the Hamiltonian matrix

$$H = \begin{bmatrix} A & -S \\ -Q & -A' \end{bmatrix}, \qquad \text{where } S = BR^{-1}B'.$$

Set the transition matrix associated to H

$$\phi(t,t_f) = \begin{pmatrix} \phi_{11}(t,t_f) & \phi_{12}(t,t_f) \\ \phi_{21}(t,t_f) & \phi_{22}(t,t_f) \end{pmatrix}, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}t}(t,t_f) = H\phi(t,t_f), \quad \phi(t_f,t_f) = I_{2n}.$$

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \phi(t,t_f) \begin{pmatrix} I_n \\ K_f \end{pmatrix}$$

Then if $X(t)$ is invertible for all $t \in [t_0, t_f]$

$$K(t) = Y(t)X^{-1}(t) = \left(\phi_{21}(t,t_f) + \phi_{22}(t,t_f)K_f\right)\left(\phi_{11}(t,t_f) + \phi_{12}(t,t_f)K_f\right)^{-1}.$$

Example

Minimize

$$J = K_f x^2(1) + \int_0^{t_f} \big(x^2(t) + u^2(t)\big)\mathrm{d}t,$$

under the constraint

$$\dot{x}(t) = u(t), \quad x(0) = x_0 = 1.$$

$$H = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \Rightarrow \phi(t, t_f) = \frac{1}{2}\begin{bmatrix} e^{-(t-t_f)} + e^{(t-t_f)} & e^{-(t-t_f)} - e^{(t-t_f)} \\ e^{-(t-t_f)} - e^{(t-t_f)} & e^{-(t-t_f)} + e^{(t-t_f)} \end{bmatrix}$$

So we obtain

$$K(t, t_f) = \frac{(1 - e^{2(t-t_f)}) + K_f(1 + e^{2(t-t_f)})}{(1 + e^{2(t-t_f)}) + K_f(1 - e^{2(t-t_f)})}$$

Dynamic Programming

Let us define $V(t, x(t)) = \frac{1}{2}x'(t)K(t)x(t)$ with respect to the quadratic form of the criterion to optimize. Then

$$-\frac{\partial V}{\partial t}(t, x(t)) = \min_{u \in \mathcal{U}} \left(L(x(t), u(t), t) + \frac{\partial V}{\partial x}(t, x(t)) f(x(t), u(t), t) \right) \quad (37)$$

becomes

$$-x'(t)\dot{K}(t)x(t) = \min_{u \in \mathcal{U}} (x'(t)Qx(t) + u'(t)Ru(t) + 2x'(t)K(t)(Ax(t) + Bu(t))) \quad (38)$$

and

$$u^*(t) = -R^{-1}B'K(t)x(t) \quad (39)$$

with the Riccati equation :

$$-\dot{K} = Q + A'K + KA - KSK; \quad K(t_f) = K_f. \quad (40)$$

In this case, the PMP and the DP coincide.

Explosion in finite time

The Riccati equation may have no solution on the interval $[t_0, t_f]$, due to explosion in finite time (which are characteristic of nonlinear equations).

For example

$$\dot{k}(t) = 1 + k^2(t), \qquad k(0) = 0 \tag{41}$$

has a solution $t \mapsto \tan(t)$, which exploses in finite time at $t = \dfrac{\pi}{2}$.

Asymptotic Behaviour

For different applications it is assumed that

$$t_f \to +\infty$$

The terminal cost $x'(t_f)K_f x(t_f)$ has no sense anymore.

$$J = \int_0^{+\infty} \big(x'(t)Qx(t) + u'(t)Ru(t)\big)\mathrm{d}t.$$

Idea :
The limit (if it exists) $K(-\infty, t_f)$ is a solution of the Algebraic Riccati standard Equation (ARE)

$$0_n = -A'K - KA - Q + KSK.$$

How to solve the ARE

Integration method :

$$\lim_{t \to -\infty} K(t, t_f)$$

This solution depends on the initial condition K_f.

Algebraic method :
All the solutions K of ARE, there exists matrices Y, X and J (X invertible, J Jordan canonical form) such that

$$K = YX^{-1}, \qquad H \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} J.$$

In the other hand, if

$$H \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} J,$$

with X invertible, then $K = YX^{-1}$ is a solution of ARE.

How to solve the ARE

The matrix H is Hamiltonian

$$\mathrm{Re}(\lambda_1) \leq \cdots \leq \mathrm{Re}(\lambda_n) \leq 0 \leq \mathrm{Re}(\lambda_{n+1}) \leq \cdots \leq \mathrm{Re}(\lambda_{2n}).$$

Select the eigenvalues $\lambda_1, \cdots, \lambda_n$ and construct by the (generalized) eigenvectors associated with the invariant subspace

$$H \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} J.$$

If X is invertible :

$$K = YX^{-1}$$

Note that when K is a solution of ARE,

$$\lambda(A - SK) \in \lambda(H).$$

Example

integration method

$$\lim_{t_f \to +\infty} K(t, t_f) = +1$$

This limit does not depend on K_f.

algebraic method

$$H = VDV^{-1}, \qquad V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The only one stabilizing solution is given by the eigenvalue -1 and the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$K = YX^{-1} = 1 \qquad A - SK = -1.$$

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Thank you very much !