Discrete-time switched Lur’e systems: stability analysis, control design, consistency and application to sampled-data Lur’e systems.

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Common work with Jamal Daafouz, Carlos A. C. Gonzaga and Julien Louis
Outline of the talk

Université de Lorraine, Nancy, CRAN Laboratory

Lur’e systems

Introduction of a new Lyapunov-Lur’e type function

Extension to switched Lur’e systems

About consistency

Application to sampled-data Lur’e systems with nonuniform sampling

Conclusion
Where is Nancy?

- The city of Nancy is at the East of Paris (1h30 by direct train);
- 2h by car from cities of Strasbourg and Luxembourg;
- 4h30 by car from Eindhoven.
An attractive city

- Place Stanislas,
- Nancy Jazz Pulsation,
- St Nicolas,
- Mirabelle, macarons.

Style Art Nouveau, Nancy School
The research at Nancy

- New university (January 2012) gathering universities of Nancy, Metz, and INPL;
- 3700 professors and researchers;
- 3000 administrative agents;
- 82 laboratories in all fields;
- 54200 students (before PhD);
- 1700 PhD students

- Centre National de la recherche scientifique
- 11000 researchers; 1100 units; all fields.
CRAN Laboratory

- Research Center for Automatic control at Nancy.
- 120 professors and researchers;
- 80 PhD students

Three departments:
- **CID**: Control theory, Identification and Diagnostic.
- **SBS**: Signal Processing for Biology and Health engineering.
- **ISET**: security and dependability of systems.

Main topics in Control theory: Hybrid systems, switched systems in discrete time, optimal control, generalized Riccati equations, networked control systems, event-triggered approach, observer, multiagent systems, graph and game theory, opinion dynamics;...
Outline of the talk

Université de Lorraine, Nancy, CRAN Laboratory

Lur’è systems
  Definitions
  Motivation examples

Introduction of a new Lyapunov-Lur’è type function
  Global stability analysis
  Local stability analysis

Extension to switched Lur’è systems
  Definition
  Global stability analysis
  Global stabilization
  Local stability analysis
  Local stabilization

About consistency
  Reminder of the consistency for switched linear systems
  What about consistency for switched Lur’è systems

Application to sampled-data Lur’è systems with nonuniform sampling

Conclusion
Definition of a Lur’e system (i)

A Lur’e system is the interconnection between a linear system and a nonlinearity verifying a cone bounded sector condition\(^1\).

Assumption:

- The nonlinearity \( \varphi(\cdot) \) verifies the cone bounded sector condition: \( \varphi(\cdot) \in [0, \Omega] \)

\[
SC(\varphi(\cdot), y, \Lambda) = \varphi'(y)\Lambda[\varphi(y) - \Omega y] \leq 0, \quad (1)
\]

with \( \Lambda \in \mathbb{R}^{p \times p} \) diagonal positive definite.

Issue of absolute stability, that is the stability of such a system for any nonlinearity verifying the condition.

---

Definition of a Lur’ë system (ii) : Continuous-time

Continuous-time Lur’ë system :

\[
\dot{x}(t) = Ax(t) + F\varphi(y(t)),
\]
where \(x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^p, (t \in \mathbb{R}^+).\)

Classical Lyapunov functions :\(^2,^3\)

- The quadratic function with respect to the state (circle criterion) :

\[
\nu(x(t)) = x'(t)Px(t);
\]

- Lur’ë-type Lyapunov function (Popov criterion) (scalar case for clarity) :

\[
\nu(x(t)) = x'(t)Px(t) + 2\eta\int_0^{Cx(t)} \Omega\varphi(s)ds, \quad \alpha > 0, \quad \eta \geq 0;
\]

\(\varphi(\cdot)\) must be \textit{time-invariant} to have : \(\int_0^{Cx} \varphi(s)ds \geq 0;\)

In \textit{continuous-time} case, \(\varphi(Cx)\) appears in the expression of \(\dot{\nu}\), \textit{only (1) is needed to conclude \(\dot{\nu} < 0;\)}


Classical Lyapunov function for Lur’e systems

The main idea to ensure $\dot{v}(x(t)) < 0$, thanks to $\varphi(\cdot) \in [0, \Omega]$ via the S-procedure, that is:

$$
\dot{v}(x(t)) - 2SC(\varphi(\cdot), y, \Lambda) < 0, \quad \forall x(t) \neq 0.
$$

(6)

With $\xi(t) = \left( \begin{array}{c} x(t) \\ \varphi(y(t)) \end{array} \right) \neq 0$, (equivalent to $x(t) \neq 0$):

- **Circle criterion**:

$$
\xi(t)' \left( \begin{array}{cc} A'P + PA & PB \\ \star & 0 \end{array} \right) + \left( \begin{array}{c} 0 \\ C'\Omega\Lambda \end{array} \right) \xi(t) < 0.
$$

(7)

- **Popov criterion**:

$$
\xi(t)' \left( \begin{array}{cc} A'P + PA & PB + \eta A' C' \Omega \\ \star & \eta(\Omega CF + F' C' \Omega) \end{array} \right) + \left( \begin{array}{c} 0 \\ C'\Omega\Lambda \end{array} \right) \xi(t) < 0.
$$

(8)

Links with KYP Lemma, frequency approach...
Definition of a Lur’e system (iii) : Discrete-time

Discrete-time Lur’e system:

\[ x_{k+1} = Ax_k + F \varphi(y_k), \]  
\[ y_k = Cx_k, \]  
\[ (9) \]  
\[ (10) \]

where \( x_k \in \mathbb{R}^n, y_k \in \mathbb{R}^p, (k \in \mathbb{N}). \)

Classical Lyapunov functions : Extensions provided by Tsypkin\(^4\).

- The quadratic function with respect to the state (extension of Circle criterion) :
  \[ v(x_k) = x_k'Px_k; \]  
  \[ (11) \]

- Lur’e-type Lyapunov function (extension of Popov criterion) :
  \[ v(x_k) = x_k'Px_k + 2\eta \int_0^{Cx_k} \Omega \varphi(s)ds, \alpha > 0, \eta \geq 0; \]  
  \[ (12) \]

  - \( \varphi(\cdot) \) must be time-invariant to have  \( \int_0^{C} \varphi(s)ds \geq 0; \)
  - \( v(\cdot) \) is inspired from the continuous-time
  - An extra assumption\(^5\)\(^6\). is necessary to bound  \( \int_{y_k}^{y_{k+1}} \varphi(s)ds. \) Ex : \( \frac{d\varphi(y)}{dy} \leq K_{max}. \)

---


Motivation example (i) : Deadzone and Saturation

\[ \varphi'(y) \wedge (\varphi(y) - y) \leq 0. \]
Motivation example (ii): a mechanical system with spring

A mass $m$ is constrained to slide along a straight horizontal wire, with a viscous damping force of coefficient $\alpha$. A spring of relaxed length $\ell_0$ and spring stiffness $k$ is attached to the mass and to the support point a distance $h$ from the wire. The horizontal coordinate of the mass is denoted $x(t)$ and we define $x = 0$ when the spring is vertical.

The nonlinear motion equation of the mass $m$ is given by the Newton’s law:

$$\ddot{x}(t) = -\frac{\alpha}{m}\dot{x}(t) - \frac{k}{m}x(t) + \frac{k}{m} \frac{\ell_0}{\sqrt{x^2(t) + h^2}} x(t).$$

$$\varphi(x)(\varphi(x) - \Omega x) \leq 0, \quad \Omega = \frac{\ell_0}{h}; \quad \varphi_1(x) = \frac{\ell_0}{\sqrt{x^2 + h^2}} x.$$

- If $\ell_0 > h$, the origin is unstable;
- If $\ell_0 \leq h$, the origin is globally asymptotically stable.

---

Motivation example (iii) : Duffing system

Differential equation

\[ m\ddot{\xi} + \gamma \dot{\xi} + \alpha \xi + \beta \xi^3 = F \cos(wt) \]  

(13)

where \( \xi \) is the position, \( m \) the mass, \( \gamma \) damping coefficient, \( \alpha \) stiffness, \( \beta \) return force, \( F \) amplitude and \( w \) pulsation of input force.

\[
\begin{align*}
\dot{x}(t) &= \left[ \begin{array}{cc} 0 & 1 \\ -\alpha & \gamma \\ \frac{1}{m} & \frac{1}{m} \end{array} \right] x(t) - \left[ \begin{array}{c} 0 \\ 1 \\ \frac{1}{m} \end{array} \right] \varphi(y(t)) + \left[ \begin{array}{c} 0 \\ 1 \\ \frac{1}{m} \end{array} \right] u(t), \quad t \in \mathbb{R}^+, \\
\varphi(y(t)) &= \beta y^3.
\end{align*}
\]

Then \( \Omega = +\infty \), that is \( y \varphi(y) \geq 0 \).
Motivation example (iv) : Chua's Circuit

Let \( x(t) = (v_R \ v_L \ i_L)' \), thus Chua's circuit is a Lur'e system:

\[
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix}
-\frac{G}{C_1} & \frac{G}{C_1} & 0 \\
-\frac{G}{C_2} & \frac{1}{C_2} & 0 \\
0 & 0 & 1 \\
\end{bmatrix} x(t) + \begin{bmatrix}
-\frac{1}{C_1} \\
0 \\
0 \\
\end{bmatrix} \varphi(y(t)), \quad t \in \mathbb{R}^+,
\end{aligned}
\]

\[y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t),\]

where

\[
\varphi(y(t)) = m_0 y(t) + \frac{m_1 - m_0}{2} (|y(t) + b| - |y(t) - b|),
\]

with scalar parameters \( m_0, m_1 \) and \( b \). This is a chaotic system.
Motivation example (v) : link with uncertainty

An uncertain system

\[ \dot{x}(t) = Ax(t) + F\Delta Cx(t), \quad 0 \leq \Delta \leq \Delta_{\text{max}}, \quad (14) \]

can be reformulated into a Lur’re system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + F\varphi(y(t)), \\
y(t) &= Cx(t), \\
\varphi(y) &= \Delta y
\end{align*}
\]

and with

\[ \varphi(y)(\varphi(y) - \Delta_{\text{max}}y) \leq 0. \]
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Conclusion
Main difficulty in discrete-time case

In discrete-time, extra assumption about the slope of the nonlinearity is required. That introduces a break of analogy with respect to the continuous-time framework.

A counterexample: half-circle allowing vertical tangents.

Aim: Consider a suitable Lur’é-like Lyapunov function in order to

• propose sufficient conditions for the global stability analysis problem (Lur’é problem);
• cover a wider range of cone bounded nonlinearities;
• relax the assumptions of the classical literature of the Lur’é problem.

Taking into account the nonlinearity by avoiding the integral term.
A Lur’e-like Lyapunov function for discrete-time

Definitions

\[ V : \begin{cases} \mathbb{R}^n \times \mathbb{R}^p & \longrightarrow & \mathbb{R}, \\ (x; \varphi(Cx)) & \longmapsto & x'Px + 2\varphi(Cx)'\Delta\Omega Cx, \end{cases} \] (15)

- with \( 0_n < P = P' \in \mathbb{R}^{n \times n} \) and \( 0_p \leq \Delta \in \mathbb{R}^{p \times p} \) diagonal.
- Bounding quadratic functions:

\[ V(x) \leq V(x; \varphi(Cx)) \leq \bar{V}(x). \] (16)

where \( V(x) = x'Px \) and \( \bar{V}(x) = x'(P + 2C'\Omega'\Delta\Omega C)x \).
Basic properties

Candidate Lyapunov function:

- \( V(x; \varphi(Cx)) \geq 0 \) due to \( P > 0_n \) and the sector condition (1) of \( \varphi(\cdot) \).
- \( V(x; \varphi(Cx)) = 0 \iff x = 0 \), due to \( P > 0_n \).
- Relation (16) implies that function (15) is radially unbounded.
- Lyapunov difference: \( \delta_k V = V(x_{k+1}; \varphi(Cx_{k+1})) - V(x_k; \varphi(Cx_k)) \).

The level set of our function (15)

\[ L_V(\gamma) = \{ x \in \mathbb{R}^n; V(x; \varphi(Cx)) \leq \gamma \} . \quad (17) \]

- The set \( L_V(\gamma) \) may be non-convex and disconnected.
Global stability analysis

Theorem

Global Stability Analysis If there exists a matrix $0_n < P = P' \in \mathbb{R}^{n \times n}$, a diagonal matrix $0_p \leq \Delta \in \mathbb{R}^{p \times p}$ and diagonal matrices $0_p < T, W \in \mathbb{R}^{p \times p}$, such that the LMI

$$
\begin{bmatrix}
A' & A' \\
F' & F'
\end{bmatrix}
\begin{bmatrix}
P & C\Omega [T - \Delta] & A'C\Omega [W + \Delta] \\
* & -2T & F'C\Omega [W + \Delta] \\
* & * & -2W
\end{bmatrix}
< 0_{2n+2p},
$$

(18)

is verified, then the function $V(x; \varphi(Cx))$ is a Lyapunov function and the origin of system (9)-(10) is globally asymptotically stable.

Main idea:

$$
V(x_{k+1}; \varphi(Cx_{k+1})) - V(x_k; \varphi(Cx_k)) - 2SC(\varphi(\cdot), y_{k+1}, W) - 2SC(\varphi(\cdot), y_k, T) < 0, \quad \forall x_k \neq 0.
$$

No assumption about the variation of $\varphi(\cdot)$.

Illustration for global stability analysis

Example 1: global stability analysis

- Lur’e system with $n = 2$, $p = 1$, $\Omega = \frac{1}{\sqrt{2}}$:

  \[
  A = \begin{bmatrix}
  0.5 & 0.1 \\
  0.3 & -0.4
  \end{bmatrix};
  F = \begin{bmatrix}
  0.5 \\
  0
  \end{bmatrix};
  C' = \begin{bmatrix}
  1 \\
  0
  \end{bmatrix};
  \]

- $\varphi(y) = 0.5\Omega y(1 + \cos(10y))$ (unbounded derivative on $y \in \mathbb{R}$);

- The Lyapunov function (15) exists and applying Theorem 18 leads to:

  \[
  P = \begin{bmatrix}
  0.9825 & -0.0846 \\
  -0.0846 & 0.9476
  \end{bmatrix};
  \Delta = 0.7503.
  \]
Global stability analysis

One initial condition $x_0 \quad k = 0$
Global stability analysis

Contractivity of the level set $L_V(\gamma = V(x_0, \varphi(y_0)))$; $k = 0$
Global stability analysis

$L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1})))$ and $L_V(\gamma = V(x_k, \varphi(y_k)))$; \hspace{1cm} k = 1
Global stability analysis

\[ L_V(\gamma = V(x_k, \varphi(y_k))) \] and \[ L_V(\gamma = V(x_k, \varphi(y_k))) \]; \quad k = 2
Global stability analysis

\[ L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1})) \text{ and } L_V(\gamma = V(x_k, \varphi(y_k))) ; \quad k = 3 \]
Global stability analysis

\[ L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1}))) \text{ and } L_V(\gamma = V(x_k, \varphi(y_k))); \quad k = 4 \]
Global stability analysis

\[ L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1}))) \text{ and } L_V(\gamma = V(x_k, \varphi(y_k))); \quad k = 5 \]
Global stability analysis

\[ L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1}))) \] and \[ L_V(\gamma = V(x_k, \varphi(y_k))) ; \quad k = 6 \]
Global stability analysis

\[ L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1}))) \text{ and } L_V(\gamma = V(x_k, \varphi(y_k))); \quad k = 7 \]
Global stability analysis

\[ L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1}))) \] and \[ L_V(\gamma = V(x_k, \varphi(y_k))) ; \quad k = 8 \]
Global stability analysis

\[ L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1})) \text{ and } L_V(\gamma = V(x_k, \varphi(y_k)) ; \quad k = 9 \]
Lur’e system with saturated input

\[ x_{k+1} = Ax_k + F \varphi(y_k) + B \text{sat}(u_k), \quad \forall k \in \mathbb{N} \]  

(19)

\[ y_k = Cx_k \]  

(20)

Class of state and nonlinearity feedbacks as controller: \( u_k = Kx_k + \Gamma \varphi(y_k) \).

Due to the saturated input in discrete-time:

- Only local stability;
- The basin of attraction of the origin \( B_0 \) may be non-convex and disconnected.

Aims:

- Stability analysis and control synthesis,
- Estimate the basin of attraction \( B_0 \) via the level set \( L_V(1) \);
Tools:

- The **deadzone** $\psi(u_k) = u_k - \text{sat}(u_k)$, is dual to the saturation.
- On the set

$$S(\hat{K} - \hat{J}, \rho) = \{ \theta \in \mathbb{R}^{n+p}; -\rho \leq (\hat{K} - \hat{J})\theta \leq \rho \}, \quad (21)$$

with $\hat{K} = [K \ \Gamma]$ and $\hat{J} = [J_1 \ J_2]$, $\psi(u_k)$ verifies a generalized **LOCAL cone bounded condition**:

$$SC_{u_k} = \psi'(u_k)U[\psi(u_k) - J_1x_k - J_2\varphi(y_k)] \leq 0, \quad (22)$$

for any diagonal matrix $0_m < U \in \mathbb{R}^{m \times m}$.

Closed-loop system:

$$x_{k+1} = A_{cl}x_k + F_{cl}\varphi(y_k) - B\psi(u_k), \quad (23)$$

where $A_{cl} = A + BK$ and $F_{cl} = F + B\Gamma$. 
Main idea:

\[ S(\hat{K} - \hat{J}, \rho) \]

Inclusions as Matrix Inequalities

IM1) Ball of radius \(1/\sqrt{\mu}\) included inside \(L_V(1)\).

IM2) \(L_V(1) \subset S(\hat{K} - \hat{J}, \rho)\) such that \(SC_{u_k} \leq 0\).

IM3) \(\delta_k V - 2SC_{u_k} - 2SC(\varphi(\cdot), y_{k+1}, W) - 2SC(\varphi(\cdot), y_k, T) < 0\).

Conclusion: on \(L_V(1)\), \(\delta_k V < 0, \forall x \neq 0\).

Inequalities implying the inclusions (i)

- The LMI

\[
\begin{bmatrix}
\mu I_n - P & -C'\Omega [R + \Delta] \\
\ast & 2R
\end{bmatrix} > 0_{n+p},
\]

leads to

\[
\mathcal{E}(I_n, \frac{1}{\mu}) \subset L_V(1).
\]

- The LMI

\[
\begin{bmatrix}
P & C'\Omega [\Delta - Q] & (K - J_1)'(\ell) \\
\ast & 2Q & (\Gamma - J_2)'(\ell) \\
\ast & \ast & \rho^2(\ell)
\end{bmatrix} > 0_{n+p+1},
\]

yields, with \(\hat{K} = [K \Gamma]\) and \(\hat{J} = [J_1 J_2]\)

\[
V(x_k, \varphi(y_k)) + 2SC(\varphi(\cdot), y_k, Q) \geq \frac{\| (K - J_1)'(\ell)x_k + (\Gamma - J_2)\varphi(y_k) \|^2}{\rho^2(\ell)} ;
\]

and finally

\[
L_V(1) \subset S((\hat{K} - \hat{J}), \rho).
\]
Inequalities implying the inclusions (ii)

If the BMI is feasible (LMI by applying the Finsler’s Lemma, or setting $U$),

$$
\begin{bmatrix}
A_{\text{cl}}^t \\
F^t_{\text{cl}} \\
-B^t \\
0_{p \times n}
\end{bmatrix}
\begin{bmatrix}
P & \Pi_1 & J_1'U' & A_{\text{cl}}^t\Pi_2 \\
0 & -2T & J_2'U' & F_{\text{cl}}^t\Pi_2
\end{bmatrix}
$$

with $\Pi_1 = C'\Omega[T - \Delta]$; $\Pi_2 = C'\Omega [W + \Delta]$, then one obtain

$$
\delta_k V - 2SC_{u_k} - 2SC(\varphi(\cdot), y_{k+1}, W) - 2SC(\varphi(\cdot), y_k, T) < 0.
$$

Inequalities (26) and (29) ensure the asymptotic stability on $x_0 \in L_V(1)$. 

Optimization problem for increasing the size of $L_V(1)$

**Theorem**

*Local asymptotic stability and best $L_V(1)$* If there exist matrices $G \in \mathbb{R}^{n \times n}$, $J_1 \in \mathbb{R}^{m \times n}$, $J_2 \in \mathbb{R}^{m \times p}$, matrix $0_n < P = P' \in \mathbb{R}^{n \times n}$; diagonal matrices $0_p \leq \Delta \in \mathbb{R}^{p \times p}$, $0_p < R, Q, T, W \in \mathbb{R}^{p \times p}$, and a scalar $\mu$ solutions of the following optimization problem:

$$\min_{G, P, J_1, J_2, Q, R, T, W, \Delta, \mu} \mu$$

under the constraints (24), (26) and (29)

then an estimate of $\mathcal{B}_0$ is given by the set $L_V(1)$. 

Illustration

Example 2:

• Lur’e system defined by: $n = 2$; $p = m = 1$; $\rho = 1.5$; $\Omega = 0.9$.

$$A = \begin{bmatrix} 0.85 & 0.4 \\ 0.6 & 0.95 \end{bmatrix}; \quad B = \begin{bmatrix} 1.3 \\ 1.2 \end{bmatrix}; \quad F = \begin{bmatrix} 1.3 \\ 1.2 \end{bmatrix}; \quad C = \begin{bmatrix} -0.5 & 0.9 \end{bmatrix}.$$  

• With given gains:

$$K = \begin{bmatrix} -0.3324 & -1.0006 \end{bmatrix}$$

• The theorem leads to:

$$P = \begin{bmatrix} 0.0418 & 0.0173 \\ 0.0173 & 0.2305 \end{bmatrix}; \quad \Delta = 0.0381.$$  

Without knowing $\varphi(y_k)$, the estimate of $B_0$ is the inner ellipsoid:

$$\mathcal{E}(P + 2C'\Omega\Delta\Omega C)$$

... but with knowing $\varphi(y_k)$...
Illustration

$LV(1)$ for distinct nonlinearities:

$\varphi(y) = 0.5\Omega y(1 + \exp(-0.5y^2))$.

Initial conditions $x_0$ leading to unstable trajectories

The basin of attraction of the origin $B_0$ depends on the nonlinearity.
Illustration

$L_V(1)$ for distinct nonlinearities:

$\phi(y) = \Omega y$.

Initial conditions $x_0$ leading to unstable trajectories

The basin of attraction of the origin $B_0$ depends on the nonlinearity.
Illustration

$L_V(1)$ for distinct nonlinearities:

$\varphi(y) = 0.5\Omega y(1 + \cos(20y))$.

Initial conditions $x_0$ leading to unstable trajectories.

The basin of attraction of the origin $B_0$ depends on the nonlinearity.
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Switched Lur’e system

Discrete-time switched system composed of Lur’e subsystems:

\[ x_{k+1} = A_{\sigma(k)}x_k + F_{\sigma(k)}\varphi_{\sigma(k)}(y_k), \quad (31) \]
\[ y_k = C_{\sigma(k)}x_k, \quad (32) \]

where \( x_k \in \mathbb{R}^n, y_k \in \mathbb{R}^p, \sigma(\cdot) : \mathbb{N} \to \mathcal{I}_N = \{1, \ldots, N\}. \)

Motivation:

- The active nonlinearity is defined by the switching rule.
- Each mode is associated with a nonlinearity;
- The sector conditions are mode-dependents, \( \forall i \in \mathcal{I}_N : \)

\[ \text{SC}(\varphi_i(\cdot), y, \Lambda_i) = \varphi_i'(y)\Lambda_i[\varphi_i(y) - \Omega_iy] \leq 0 \quad (33) \]
Main tool:

• The extension of our function (15) to the switched systems framework\(^\text{10}\):

\[
V : \left\{ \begin{array}{c}
I_N \times \mathbb{R}^n \times \mathbb{R}^p \\
(i, x, \varphi_i(C_i x))
\end{array} \right\} \rightarrow \mathbb{R},
\]

\[x' P_i x + 2(\varphi_i(C_i x))' \Delta_i \Omega_i C_i x, \quad (34)\]

• Consider the function \(V_{\text{min}}(x_k) = \min_{i \in I_N} V(i, x_k, \varphi_i(C_i x_k))\)
  
  ◦ inherits all the basic properties of function (34).

Auxiliary notation:

• Extended system matrices and state vector:

\[
A_i = \begin{bmatrix} A_i & F_i & 0_{n \times Np} \end{bmatrix} \in \mathbb{R}^{n \times (n+(N+1)p)};
\]

\[
E_i = \begin{bmatrix} 0_{p \times (n+ip)} & l_p & 0_{p \times (N-i)p} \end{bmatrix} \in \mathbb{R}^{p \times (n+(N+1)p)};
\]

\[
z'_k = (x'_k \quad \varphi'_i(C_i x_k) \quad \varphi'_1(C_1 x_{k+1}) \quad \ldots \quad \varphi'_N(C_N x_{k+1}))' \in \mathbb{R}^{(n+(N+1)p)}.
\]

• Set of Metzler matrices (in discrete time):

The matrix \(\Pi \in \mathcal{M}_d\), where \(\mathcal{M}_d\) is the Metzler matrices set:

\[
\mathcal{M}_d = \left\{ \Pi \in \mathbb{R}^{N \times N}, \quad \pi_{ii} \geq 0, \quad \sum_{\ell \in I_N} \pi_{\ell i} = 1, \quad \forall i \in I_N \right\}.
\]

Global stability with arbitrary switching law

Analogy with not switching Lur’e systems

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<td>Lyapunov function</td>
<td>$V(x; \varphi(Cx))$</td>
<td>$V(i; x; \varphi_i(C_i x))$</td>
</tr>
<tr>
<td>$L_V(\gamma)$</td>
<td>${x \in \mathbb{R}^n; V(x; \varphi(Cx)) \leq \gamma}$</td>
<td>$\bigcap_{i \in I_N} {x \in \mathbb{R}^n; V(i; x; \varphi_i(C_i x)) \leq \gamma}$</td>
</tr>
<tr>
<td># LMIs</td>
<td>1</td>
<td>$N^2$</td>
</tr>
<tr>
<td>Bounds of $L_V$</td>
<td>Ellipsoids</td>
<td>Intersections of Ellipsoids</td>
</tr>
</tbody>
</table>
Global stability analysis

**Theorem**

**Global Stability Analysis**\(^{11}\) If there exists \(N\) matrices \(0_n < P_i = P_i' \in \mathbb{R}^{n \times n}\), \(N\) diagonal matrices \(0_p \leq \Delta_i \in \mathbb{R}^{p \times p}\) and diagonal matrices \(0_p < T_i, W_i \in \mathbb{R}^{p \times p}\), such that the LMI, \(\forall (i, j) \in \{1, \cdots, N\}\)

\[
\begin{bmatrix}
A'_{i} \\
F'_{i} \\
0_{p \times n}
\end{bmatrix}
\begin{bmatrix}
P_j \\
A'_{j} \\
F'_{j} \\
0_{p \times n}
\end{bmatrix}' + \begin{bmatrix}
-P_i & C_i' \Omega_i [T_i - \Delta_i] & A_i' C_j' \Omega_j [W_j + \Delta_j] \\
* & -2T_i & F_i' C_j' \Omega_j [W_j + \Delta_j] \\
* & * & -2W_j
\end{bmatrix} < 0_{2n+2p},
\]

(35)

is verified, then the function \(V(\sigma_k; x_k; \varphi_{\sigma(k)}(C_{\sigma(k)}x_k))\) is a Lyapunov function and the origin of system (9)-(10) is globally asymptotically stable.

**Main idea**:

\[
V(\sigma(k + 1), x_{k+1}; \varphi(Cx_{k+1})) - V(\sigma(k), x_k; \varphi(Cx_k)) - 2SC(\varphi_{\sigma(k+1)}(\cdot), y_{k+1}, W_{\sigma(k+1)}) - 2SC(\varphi_{\sigma(k)}(\cdot), y_k, T_{\sigma(k)}) < 0, \quad \forall x_k \neq 0.
\]

No assumption about the variation of \(\varphi_{\sigma(k)}(\cdot)\) and \(\varphi_{\sigma(k+1)}(\cdot)\).

---

Theorem: Min-switching strategy based on $V(i, x_k, \varphi_i(C_i x_k))$  

Assume there exist a matrix $\Pi \in \mathcal{M}_d$; matrices $0_n < P_i = P'_i \in \mathbb{R}^{n \times n}$ and diagonal matrices $0_p < T_i, W_i, 0_p \leq \Delta_i \in \mathbb{R}^{p \times p}$, $(i \in \mathcal{I}_N)$, such that the Lyapunov-Metzler inequalities are satisfied $\forall i \in \mathcal{I}_N$

$$
\begin{align*}
&A'_i(P)p_iA_i + \text{He}(A'_i(C'\Omega\Delta E)p_i) - \sum_{q \in \mathcal{I}_N} \left(2E'_qW_qE_q - \text{He}(E'_qW_q\Omega_qC_qA_i)\right) \\
&= \\
&- \begin{bmatrix}
P_i & * & * \\
(\Delta_i - T_i)\Omega_i C_i & 2T_i & * \\
0_{Np \times n} & 0_{Np \times p} & 0_{Np}
\end{bmatrix} < 0_{n+(N+1)p}, \quad (36)
\end{align*}
$$

where $(P)p_i = \sum_{\ell \in \mathcal{I}_N} \pi_{\ell i} P_{\ell}$, then the min-switching strategy

$$
\sigma(k) = u(x_k) = \arg \min_{i \in \mathcal{I}_N} V(i, x_k, \varphi_i(C_i x_k)) \quad (37)
$$

globally asymptotically stabilizes the system (31)-(32).
Sketch of the proof

The matrix inequalities (36) are formulated in order to:

- Consider the sum of:
  - the sector condition at time $k + 1$:
    $$\varphi'_q(C_qx_{k+1})W_q[\varphi_q(C_qx_{k+1}) - \Omega_qC_qx_{k+1}] \leq 0,$$
    (38)
  - written in the equivalent form:
    $$-z_k'(2E_q'W_qE_q - \text{He}(E_q'W_q\Omega_qC_q\Lambda_i))z_k \geq 0,$$
    with $0_p < W_q \in \mathbb{R}^{p \times p}$ diagonal.

- Upper-bound the function $V_{\text{min}}(x_{k+1}) = \min_{j \in \mathcal{I}_N} V(j, x_{k+1}, \varphi_j(C_jx_{k+1}))$ by the aid of these sector conditions;

- Guarantee, due to properties of the Metzler matrix $\Pi \in \mathcal{M}_d$, that
  $$V_{\text{min}}(x_{k+1}) - V_{\text{min}}(x_k) < 2SC(\varphi_{\sigma(k)}(\cdot), y_k, T_{\sigma(k)}) \leq 0.$$
State space partition

State space partition:

- Let the sets $S_i$ allowing to activate the mode $i \in \mathcal{I}_N$:
  \[ S_i = \{ x \in \mathbb{R}^n, \ V_{\min}(x) = V(i, x, \varphi_i(C_ix)) \} , \quad \forall i \in \mathcal{I}_N. \]  
  \[ (39) \]

- $0 \in S_i, \forall i \in \mathcal{I}_N$;
- $\bigcup_{i \in \mathcal{I}_N} S_i = \mathbb{R}^n$, at least one mode reaches the minimum of our function;
- the sets $S_i$ are not necessarily disjoint.

Remark: Feasibility of Inequalities (36) implies inclusions $\pi_{ji}^{1/2} A_i$ and $\pi_{ji}^{1/2} (A_i + B_i \Omega_i C_i)$ stable, $\forall i \in \mathcal{I}_N$. 

Illustration

Example: global stabilization

- Switched Lur’e system with $N = n = 2$, $p = 1$, $\Omega_1 = 0.6$; $\Omega_2 = 0.4$:

\[
A_1 = \begin{bmatrix} 1.08 & 0 \\ 0 & -0.72 \end{bmatrix}; \\ F_1 = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}; \\ C_1' = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}; \\ A_2 = \begin{bmatrix} -0.48 & 0.8 \\ 0 & 0.8 \end{bmatrix}; \\ F_2 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}; \\ C_2' = \begin{bmatrix} 0.4 \\ 1 \end{bmatrix}.
\]

- The nonlinearities are: $\varphi_1(y) = 0.5\Omega_1y(1 + \cos(2y))$ and $\varphi_2(y) = 0.5\Omega_2y(1 - \sin(2.5y))$.

- The numerical results are obtained:

\[
P_1 = \begin{bmatrix} 1.1490 & -0.0832 \\ -0.0832 & 1.9764 \end{bmatrix}; \\ P_2 = \begin{bmatrix} 0.3508 & -0.4489 \\ -0.4489 & 3.1440 \end{bmatrix}; \\ \Delta_1 = 0.2585; \\ \Delta_2 = 1.0509; \text{ with the Meztler matrix } \Pi = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}.
State space partition and a trajectory for $x_0 = (14; 11)'$

Set $S = S_1 \cap S_2$ and bounding cones $C_1, C_2$.

Trajectory $x_k$ and the modes selected at each instant $k$.

With $\Delta_i \neq 0_p$, the state partition exhibits ripples.
Switched Lur’e system with input saturation

Discrete-time switched Lur’e systems with control saturation:

\[
\begin{align*}
x_{k+1} &= A_{\sigma(k)} x_k + F_{\sigma(k)} \varphi_{\sigma(k)}(y_k) + B_{\sigma(k)} \text{sat}(u_k), \\
y_k &= C_{\sigma(k)} x_k,
\end{align*}
\]  
(40) (41)

where \( x_k \in \mathbb{R}^n, y_k \in \mathbb{R}^p \) and \( u_k \in \mathbb{R}^m \).

Assumptions:

- The state and the modal nonlinearities are available in real time;
- The switched feedback control law is considered:

\[
u_k = K_{\sigma(k)} x_k + \Gamma_{\sigma(k)} \varphi_{\sigma(k)}(y_k).
\]

Input saturation:

- Only local stability can be assured;
- The basin of attraction \( \mathcal{B}_0 \) may be non-convex and disconnected.
Main tools:

- Consider the function $V_{\min}(x) = \min_{i \in I_N} V(i, x, \varphi_i(C_i x))$ as candidate Lyapunov function,

- whose the level sets are given by:

$$L_{V_{\min}}(\gamma) = \{ x \in \mathbb{R}^n; V_{\min}(x) \leq \gamma \} = \bigcup_{j \in I_N} \{ x \in \mathbb{R}^n; V(j; x; \varphi_j(C_j x)) \leq \gamma \}.$$ 

and the set $L_{V_{\min}}(1)$ will be considered as an estimate of $B_0$.

The approach is similar to the previous one\textsuperscript{13}.

---

Illustration : Local stability analysis

Exemple :

- Lur’e system defined by $N = n = 2 ; p = m = 1 ; \rho = 1.5$, $C_1 = \begin{bmatrix} 0.9 & 0.5 \end{bmatrix} ; C_2 = \begin{bmatrix} 1 & -0.7 \end{bmatrix} ; \Omega_1 = 0.7 ; \Omega_2 = 1.3$.

- $\varphi_1(y) = 0.5\Omega_1 y(1 + \sin(30y))$; $\varphi_2(y) = 0.5\Omega_2 y(1 + \cos(\frac{100y}{3}))$

\[
A_1 = \begin{bmatrix} 0.4 & 0.4 \\ 0.2 & 1 \end{bmatrix} ; B_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} ; F_1 = \begin{bmatrix} 1 \\ 1.2 \end{bmatrix} ;
\]
\[
A_2 = \begin{bmatrix} 1.1 & 0.6 \\ 0.3 & 0.4 \end{bmatrix} ; B_2 = \begin{bmatrix} 0.7 \\ 0.5 \end{bmatrix} ; F_2 = \begin{bmatrix} 1.2 \\ 1 \end{bmatrix} .
\]

The switched gains are given as follows :

\[
K_1 = \begin{bmatrix} -0.72 & -1.01 \end{bmatrix} ; \Gamma_1 = -1.2636;
\]
\[
K_2 = \begin{bmatrix} -1.27 & -0.74 \end{bmatrix} ; \Gamma_2 = -1.4744.
\]
Illustration

\( \{ x \in \mathbb{R}^n; V(1; x; \varphi_1(C_1 x) \leq 1 \} \).
\[ \{ x \in \mathbb{R}^n; V(2; x; \varphi_2(C_2x) \leq 1 \}. \]
Illustration

$L_V(1)$ and the best estimate with the quadratic Lyapunov approach.
Illustration

Two trajectories with different arbitrary switching laws.

Question: what about the gap between $L_V(1)$ and $B_0$?

Four (constant and periodic) switching laws are considered.

- $\sigma_a(2k) = 1; \sigma_a(2k + 1) = 2 \forall k \in \mathbb{N}$;
- $\sigma_c(2k) = 2; \sigma_c(2k + 1) = 1 \forall k \in \mathbb{N}$;
- $\sigma_b(k) = 1; \forall k \in \mathbb{N}$;
- $\sigma_d(k) = 2; \forall k \in \mathbb{N}$. 
Illustration

$x_0 \notin L_V(1)$ leads to unstable trajectories with $\sigma_a(k)$. 

$x_0 \notin L_V(1)$ leads to unstable trajectories with $\sigma_a(k), \sigma_b(k)$. 

---

Illustration

$x_0 \notin L_V(1)$ leads to unstable trajectories with $\sigma_a(k), \sigma_b(k)$. 

---

Discrete-time switched Lur'e systems

M. Jungers
Illustration

$x_0 \notin L_V(1)$ leads to unstable trajectories with $\sigma_a(k)$, $\sigma_b(k)$, $\sigma_c(k)$. 

![Diagram showing unstable trajectories](image)
Illustration

$x_0 \notin L_V(1)$ leads to unstable trajectories with $\sigma_a(k), \sigma_b(k), \sigma_c(k), \sigma_d(k)$. 
Illustration : local stabilization

Example :

• Switched Lur’e system with input saturation with $N = n = 2$, $p = 1$, $\rho = 5$; $\Omega_1 = 0.7$; $\Omega_2 = 0.5$ :

$$A_1 = \begin{bmatrix} 1.4 & 0.4 \\ 0.2 & 1 \end{bmatrix} ; F_1 = \begin{bmatrix} 1 \\ 1.2 \end{bmatrix} ; B_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} ; C'_1 = \begin{bmatrix} 0.9 \\ 0.5 \end{bmatrix} ;$$

$$A_2 = \begin{bmatrix} 1.1 & 0.6 \\ 0.3 & 1.5 \end{bmatrix} ; F_2 = \begin{bmatrix} 1.2 \\ 1 \end{bmatrix} ; B_2 = \begin{bmatrix} 0.7 \\ 0.5 \end{bmatrix} ; C'_2 = \begin{bmatrix} 1 \\ 0.7 \end{bmatrix} .$$

• The nonlinearities $\varphi_i(y)$ are defined by, $\forall y \in \mathbb{R}$ :

$$\varphi_1(y) = 0.5\Omega_1 y (1 + \cos(20y)) ; \varphi_2(y) = 0.5\Omega_2 y (1 - \sin(25y)).$$

• The control gains are given by :

$$K_1 = \begin{bmatrix} -0.7168 \\ -1.0136 \end{bmatrix} ; \Gamma_1 = -1.2923 ;$$

$$K_2 = \begin{bmatrix} -1.2581 \\ -0.7326 \end{bmatrix} ; \Gamma_2 = -1.4650 ;$$
Illustrations

State-space partition inside $L_{V_{\min}}(1)$
mode 1 is the blue region and mode 2 is the red region.
Illustrations

2 trajectories, one from $x_0$ settled in the disconnected $L_{V_{\min}}(1)$. Red circle (resp. a black star) means the mode 1 is active (resp. mode 2).
Illustrations

Mapped $x_0$ leading to unstable trajectories.

Our estimate is adapted to the shape of $B_0$. 
Outline of the talk

Université de Lorraine, Nancy, CRAN Laboratory

Lur’e systems

Introduction of a new Lyapunov-Lur’e type function

Extension to switched Lur’e systems

About consistency
  Reminder of the consistency for switched linear systems
  What about consistency for switched Lur’e systems

Application to sampled-data Lur’e systems with nonuniform sampling

Conclusion
Closed-loop performance for linear switched systems

Let us consider here the following switched linear systems

\[ x_{k+1} = A_{\sigma(k)} x_k, \quad J_{\sigma}(x_0) = \sum_{k \in \mathbb{N}} x_k' Q_{\sigma(k)} x_k. \]  

(42)

**Theorem**

If there exist matrices \( P_i > 0, \forall i \in \mathcal{I}_N \) and \( \Pi \in \mathcal{M} \) solution of the optimization problem

\[ \min_{P_i, \Pi} \left( \min_{i \in \mathcal{I}_N} \text{trace}(P_i) \right), \]  

(43)

subject to

\[ A_i' \left( \sum_{\ell \in \mathcal{I}_N} \pi_{i\ell} P_\ell \right) A_i - P_i + Q_i < 0, \quad \forall i \in \mathcal{I}_N \]  

(44)

then the state feedback switching strategy \( \sigma(k) = \arg \min_{i \in \mathcal{I}_N} x_k' P_i x_k \), called **min-switching** strategy, ensures that the origin \( x = 0 \) is globally asymptotically stable and

\[ J_{\sigma}(x_0) \leq \min_{i \in \mathcal{I}_N} x_0' P_i x_0 = V_{\min}(x_0). \]  

(45)
Consistency for switched linear systems

**Definition**

Consistent switching law for linear switched systems

Consider the class of switched discrete-time linear systems, where \( \sigma : \mathbb{N} \rightarrow \mathbb{I}_N \) is the switching law. A particular switching strategy \( \sigma_s(\cdot) \) is consistent, with respect to the performance \( J_\sigma(\cdot) \), if it improves the performance when compared to the performances of each isolated subsystem supposed to be asymptotically stable.

\[
J_{\sigma_s}(x_0) \leq \min_{i \in \mathbb{I}_N} J_{\sigma=i}(x_0). \tag{46}
\]

**Theorem**

The *min-switching strategy* \( \sigma_s(k) = \arg \min_{i \in \mathbb{I}_N} x_k' P_i x_k \), where \( P_i \) are solution of Optimization Problem (43) is consistent.

**Idea of the proof**:

The inequality \( A_i' P_i A_i - P_i + Q_i < 0 \) is a particular case of the constraints (44).

---

Closed-loop performance for switched Lur’e systems

**Theorem**

If there exist matrices $P_i > 0$, $\forall i \in \mathcal{I}_N$ and $\Pi \in \mathcal{M}$ solution of the optimization problem, with $(P)_{p,i} = \sum_{\ell \in \mathcal{I}_N} \pi_{\ell i} P_{\ell}$,

$$
\min_{P_i, \Pi} \left( \min_{i \in \mathcal{I}_N} \text{trace}(P_i) \right),
$$

subject to

$$
\begin{bmatrix}
A_i'(P)_{p,i} A_i - P_i + Q_i & * \\
B_i'(P)_{p,i} A_i + S_i \Omega_i C_i & B_i'(P)_{p,i} B_i - 2S_i
\end{bmatrix} < 0,
$$

then the state feedback switching strategy $\sigma(k) = \arg \min_{i \in \mathcal{I}_N} x_k' P_i x_k$ ensures that the origin $x = 0$ is globally asymptotically stable and

$$
J_\sigma(x_0) \leq \min_{i \in \mathcal{I}_N} x_0' P_i x_0 = V_{\min}(x_0).
$$

The answer is NO! This is due to the dependency of $J_\sigma(x_0)$ with respect to the nonlinearity $\varphi_\sigma(\cdot)$.\(^{15}\)

---

\(^{15}\) J. Louis, M. Jungers et J. Daafouz. “Switching control consistency of switched Lur’e systems with application to digital control design with non-uniform sampling”. In : 14th annual European Control Conference, ECC 2015. Linz, Austria, 2015, p. 1748–1753.
Extension of Consistency concept

**Definition**

Consider switched Lur’e systems, a particular switching strategy $\sigma_s(\cdot)$ is consistent, with respect to the performance $J_{\sigma_s}$, if it improves the upper bound of the performance when compared to the upper bounds of performances of each isolated subsystem.

\[
J_{\sigma_s}(x_0) \leq V_{\min}(x_0) \leq \min_{i \in \mathcal{I}_N} J_{\sigma= i}(x_0),
\]

**(50)**

**Theorem**

The min-switching strategy $\sigma(k) = \arg \min_{i \in \mathcal{I}_N} x_k' P_i x_k$, given by last theorem is consistent according this revised definition.

See 16

---

Consider a switched Lur’e system defined by

\[
A_1 = \begin{bmatrix} 0.9 & 0 \\ 0.4 & -0.72 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.58 & -0.8 \\ 0 & -0.8 \end{bmatrix}, \quad B_1 = -\begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} 0.6 \\ 0.24 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.4 \\ 1.1 \end{bmatrix}, \quad \varphi_1(y_k) = \frac{\Omega_1 y_k}{2} (1 + \cos(2y_k)),
\]

\[
\varphi_2(y_k) = \frac{\Omega_2 y_k}{2} (1 - \sin(5.5y_k)), \quad \Omega_1 = 0.6, \quad \Omega_2 = 1.2, \quad x_0 = \begin{pmatrix} -4 \\ 5 \end{pmatrix}.
\]

\[Q_i = q_i l_n \text{ with } i \in \mathcal{I}_2\]

<table>
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<tr>
<th>(q_1)</th>
<th>(q_2)</th>
<th>(\mathcal{I}_{\sigma_s})</th>
<th>(V_{\min}(x_0))</th>
<th>(\overline{\mathcal{J}}_1)</th>
<th>(\overline{\mathcal{J}}_2)</th>
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Conclusion
Sampled-data Lur’e system with nonuniform sampling

Sampled-data Lur’e system:

\[
S_c : \begin{cases}
\dot{x}(t) = Ax(t) + B\varphi(y(t)) + F\tilde{u}(t), & t \in \mathbb{R}^+,
\end{cases}
\]

\[
y(t) = Cx(t),
\]

\[
\tilde{u}(t) = u(t_k) = K_{tk}x(t_k) + \Gamma_{tk}\varphi(y(t_k)), \quad [t_k; t_{k+1}],
\]

where

- \(x(t) \in \mathbb{R}^n\) is the state, \(y(t) \in \mathbb{R}^p\) the output \(\tilde{u}(t) \in \mathbb{R}^r\) the control input.
- \(\varphi(\cdot)\) is a nonlinearity verifying the cone bounded sector condition
  \[
  \varphi(0) = 0; \quad \varphi(y)\Lambda(\varphi(y) - \Omega y) \leq 0.
  \]

with \(\Lambda \in \mathbb{R}^{p \times p}\) any diagonal positive definite.
- The sampling times \(\{t_k\}_{k \in \mathbb{N}}\) verify
  \[
  t_{k+1} - t_k \in \{T_i\}_{i \in \{1; \ldots; N\}}, \quad \forall k \in \mathbb{N}.
  \]

**Issue 1**: Design jointly a control law \(\tilde{u}(t)\) and a sequence of (nonuniform) sampling periods, ensuring that the origin \(x = 0\) is **globally asymptotically stable**.

**Remark**: uniform sampling consists in assuming \(\{T_i\}_{i \in \{1; \ldots; N\}} = \{T_1\}\).
Stability of a sampled-data system with nonuniform sampling

**Theorem**

Consider $S_c$ with a finite family of sampling period $\{T_i\}_{i \in \{1; \ldots ; N\}}$, and a given control law

- **(A1)** If there exits a function $\beta \in \mathcal{KL}$ such that $\forall k \geq k_0 \geq 0$,
  $$\|x_k\| \leq \beta (\|x_{k_0}\|, k - k_0),$$

- **(A2)** If there exist $N \kappa_i \in \mathcal{K}_\infty$, satisfying $\forall i \in \{1; \ldots ; N\}, \forall t \in [t_{init}; t_{init} + T_i]$, 
  $$\|x(t)\| \leq \kappa_i (\|x(t_{init})\|),$$

then the sampled-data system $S_c$ is **globally uniformly asymptotically stable** and there exists $\overline{\beta} \in \mathcal{KL}$, such that $\forall t \geq t_{init} \geq 0$

  $$\|x(t)\| \leq \overline{\beta} (\|x(t_{init})\|, t - t_{init}).$$

---


First consequence and reformulation of Problem 1

Guideline:
- (A2) is always satisfied for Lur’e systems here.
- Problem 1 reduces to verify (A1).
  \[ \Rightarrow \text{introduction of the exact discretized system} \]

\[
F_{T_i}^e(x_k) = x_k + \int_{t_k}^{t_k+T_i} \left( Ax(\tau) + B \varphi(y(\tau)) + F \tilde{u}(t_k) \right) d\tau, \quad \forall k \in \mathbb{N}. \tag{54}
\]

Reformulation 1 of Problem 1: Determine jointly a control law and a switching law stabilizing the nonlinear switching system:

\[
x_{k+1} = F_{T_{\sigma(k)}}^e(x_k), \quad k \in \mathbb{N}, \tag{55}
\]

where the switching law \( \sigma : \mathbb{N} \rightarrow \{1; \cdots; N\} \) select the active sampling period in \( \{T_i\}_{i \in \{1; \cdots; N\}} \).
Further discussion

Among all the solutions, it may be interesting to add to Problem 1 a criterion and to consider an optimization problem.

**Performance criterion**: Degree of freedom to select the nonuniform sampling time

\[
J_\sigma(x_0) = \sum_{k \in \mathbb{N}} x'_k Q_{\sigma(k)} x_k.
\]  (56)

For instance, \( Q_i \neq \frac{1}{T_i}, \forall i \in \{1, \ldots, N\} \).

**Difficulty**: due to the presence of the non-linearity \( \varphi(\cdot) \):

- It is not possible to obtain an analytical value of the function \( F_{T_i}^e(\cdot) \);
- \( F_{T_i}^e(\cdot) \) is not of Lur’ë type structure.

**Question**:

How to handle (easily) the function \( F_{T_i}^e(\cdot) \)?
Reformulation of the issue:

Reformulation 2 of Problem 1: Design jointly the switching gains \((K_i, \Gamma_i)\) and the switching law ensuring that the discrete-time Lur’e system with norm bounded uncertainties written as \(\exists \Delta_{1,i}, \Delta_{2,i}, \text{ such that} \)

\[
\begin{align*}
x_{k+1} &= \left( A_{\sigma(k)}^d + \Delta_{2,\sigma(k)}^d \right) x_k + B_{\sigma(k)}^d \varphi(Cx_k) + (I_n + \Delta_{1,\sigma(k)}) F_{\sigma(k)}^d u_k, \\
\Delta_{1,i}^i \Delta_{1,i} &\leq r_1 (T_i)^2 I_n, \\
\Delta_{2,i}^i \Delta_{2,i} &\leq r_2 (T_i)^2 I_n, \\
u_k &= K_{\sigma(k)} x_k + \Gamma_{\sigma(k)} \varphi(Cx_k)
\end{align*}
\]

is globally asymptotically stable and that minimize the cost \(J_\sigma(\cdot)\).

Solution given by the optimization problem

\[
\min_{i \in \{1; \ldots ; N\}} -\text{trace}(P_i^{-1}) , \tag{57}
\]

under LMI constraints provided in \(^{19}\). Then the switching law \(\sigma(k) = \arg\min (x_k'P_ix_k)\), leads to

\[
J_{\sigma(k)}(x_0) \leq \overline{J}(x_0) = \min_{i \in \{1; \ldots ; N\}} (x_0'P_ix_0) ; \tag{58}
\]

and is consistent to the quadratic upper bound taking into account all the nonlinearities and all the uncertainties.

Numerical example

Let

\[
A = \begin{bmatrix} 0 & 1,6 \\ -0,8 & -0,1 \end{bmatrix}, \quad B = \begin{bmatrix} 0,25 \\ 0,25 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 0,20 \end{bmatrix}, \quad C = \begin{bmatrix} 0,1 & -0,15 \end{bmatrix},
\]

\[
\phi(y[k]) = \frac{\Omega_c y[k]}{2} (1 + \cos(6y[k] + 0,1y^2[k])),
\]

\[
\Omega = \frac{\sqrt{2}}{2}, \quad x_0 = \begin{pmatrix} 6 \\ 3 \end{pmatrix}, \quad T_1 = 0,1 \quad T_2 = 0,3,
\]

and \( R_1 = 3, \quad R_2 = 1, \quad Q_1 = 3I_2 \) et \( Q_2 = I_2 \).

Then the optimization problem leads to

\[
P_1 = \begin{bmatrix} 358,42 & 280,49 \\ 280,49 & 671,26 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 383,91 & 260,67 \\ 260,67 & 548,82 \end{bmatrix}, \quad \gamma_1 = 0,3, \quad \gamma_2 = 0,
\]

\[
K_1 = \begin{bmatrix} -1,46 & -4,04 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -4,50 & -18,57 \end{bmatrix}, \quad \Gamma_1 = -0,14, \quad \Gamma_2 = -1,74.
\]
Numerical example: the trajectory

State partition for the choice of the sampling period
Numerical example: the trajectory

State partition for the choice of the sampling period
Numerical example: the trajectory

State partition for the choice of the sampling period
Numerical example: the trajectory

State partition for the choice of the sampling period

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Numerical example: the trajectory

State partition for the choice of the sampling period
Numerical example: the trajectory

State partition for the choice of the sampling period
Numerical example: the trajectory

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Numerical example: the trajectory

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M. Jungers
Numerical example: the trajectory

State partition for the choice of the sampling period
Numerical example: the trajectory

State partition for the choice of the sampling period
Numerical example: the trajectory

State partition for the choice of the sampling period
Numerical example: the performance

Uniform sampling $T_1$, 
$\mathcal{J}_1(x_0) = 13247$

Uniform sampling $T_2$, 
$\mathcal{J}_2(x_0) = 17363$

Nonuniform sampling, 
$\mathcal{J}_\sigma(x_0) = 10895$

Nonuniform sampling, $\mathcal{J}_\sigma(x_0) = 10895$

Improvement

\[
\frac{\mathcal{J}_1(x_0) - \mathcal{J}_\sigma(x_0)}{\mathcal{J}_1(x_0)} = 17.8\%.
\]
Conclusion

Discrete-time Lur’€e system have been studied :

- A new discrete-time Lyapunov-Lur’€e function suitable has been provided ;
- Global stability analysis and Global stabilization ;
- Local stability analysis and local stabilization ;
- Revision of the notion of consistency taking into account all the nonlinearities ;
- Application to sampled-data Lur’€e systems.
Thank you very much for your attention!

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