Discrete-time switched Lur'e systems: stability analysis, control design, consistency and application to sampled-data Lur'e systems.

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TU/e
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Common work with
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UMR 7039

Outline of the talk

Université de Lorraine, Nancy, CRAN Laboratory

Lur'e systems

Introduction of a new Lyapunov-Lur'e type function

Extension to switched Lur'e systems

About consistency

Application to sampled-data Lur'e systems with nonuniform sampling

Conclusion

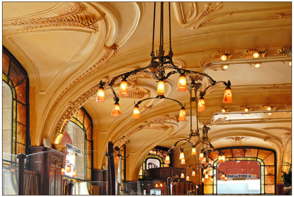
Where is Nancy ?

- The city of Nancy is at the East of Paris (1h30 by direct train) ;
- 2h by car from cities of Strasbourg and Luxembourg ;
- 4h30 by car from Eindhoven.

NANCY TOURISME : An attractive city

- Place Stanislas,
- Nancy Jazz Pulsation,
- St Nicolas,
- Mirabelle, macarons.

Style *Art Nouveau*, Nancy School



The research at Nancy

- New university (january 2012) gathering universities of Nancy, Metz, and INPL ;
- 3700 professors and researchers ;
- 3000 administrative agents ;
- 82 laboratories in all fields ;
- 54200 students (before PhD) ;
- 1700 PhD students

- Centre National de la recherche scientifique
- 11000 researchers ; 1100 units ; all fields.

CRAN Laboratory

- Research Center for Automatic control at Nancy.
- 120 professors and researchers ;
- 80 PhD students

Three departments :

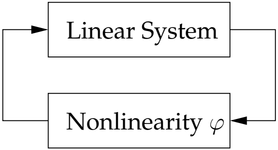
- CID : Control theory, Identification and Diagnostic.
- SBS : Signal Processing for Biology and Health engineering.
- ISET : security and dependability of systems.

Main topics in Control theory : Hybrid systems, switched systems in discrete time, optimal control, generalized Riccati equations, networked control systems, event-triggered approach, observer, multiagent systems, graph and game theory, opinion dynamics ;...

Outline of the talk

Definition of a Lur'e system (i)

A Lur'e system is the interconnection between a linear system and a nonlinearity verifying a cone bounded sector condition [1].

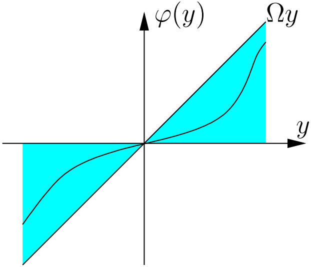


Assumption :

- The nonlinearity $\varphi(\cdot)$ verifies the cone bounded sector condition : $\varphi(\cdot) \in [0, \Omega]$

$$\mathrm{SC}(\varphi(\cdot), y, \Lambda) = \varphi'(y)\Lambda[\varphi(y) - \Omega y] \leq 0, \quad (1)$$

with $\Lambda \in \mathbb{R}^{p \times p}$ diagonal positive definite.



Issue of absolute stability, that is the stability of such a system for any nonlinearity verifying the condition.

1. A. I. Lur'e et V. N. Postnikov. "On the theory of stability of control systems". In : *Applied Mathematics and Mechanics* 8.3 (1944), p. 3–13.

Definition of a Lur'e system (ii) : Continuous-time

Continuous-time Lur'e system :

$$\dot{x}(t) = Ax(t) + F\varphi(y(t)), \quad (2)$$
$$y(t) = Cx(t), \quad (3)$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $(t \in \mathbb{R}^+)$.

Classical Lyapunov functions :[2],[3]

- The quadratic function with respect to the state (circle criterion) :

$$v(x(t)) = x'(t)Px(t); \quad (4)$$

- Lur'e-type Lyapunov function (Popov criterion) (scalar case for clarity) :

$$v(x(t)) = x'(t)Px(t) + 2\eta \int_0^{Cx(t)} \Omega\varphi(s)\mathrm{d}s, \ \alpha > 0, \ \eta \geq 0; \quad (5)$$

  - $\varphi(\cdot)$ must be time-invariant to have : $\int_0^{Cx} \varphi(s)\mathrm{d}s \geq 0$;
  - In *continuous-time* case, $\varphi(Cx)$ appears in the expression of $\dot{v}$, only (1) is needed to conclude $\dot{v} < 0$;

2. A. I. Lur'e et V. N. Postnikov. "On the theory of stability of control systems". In : *Applied Mathematics and Mechanics* 8.3 (1944), p. 3–13.

3. R.E. Kalman. "Lyapunov functions for the problem of Lur'e in automatic control". In : *Proceedings of National Academy of Sciences* 49 (1962), p. 201–205.

Classical Lyapunov function for Lur'e systems

The main idea to ensure $\dot{v}(x(t)) < 0$, thanks to $\varphi(\cdot) \in [0, \Omega]$ via the S-procedure, that is :

$$\dot{v}(x(t)) - 2\mathrm{SC}(\varphi(\cdot), y, \Lambda) < 0, \quad \forall x(t) \neq 0. \tag{6}$$

With $\xi(t) = \begin{pmatrix} x(t) \\ \varphi(y(t)) \end{pmatrix} \neq 0$, (equivalent to $x(t) \neq 0$) :

- Circle criterion :

$$\xi(t)' \left(\begin{bmatrix} A'P + PA & PB \\ \star & 0 \end{bmatrix} + \begin{bmatrix} 0 & C'\Omega\Lambda \\ \star & -2\Lambda \end{bmatrix} \right) \xi(t) < 0. \tag{7}$$

- Popov criterion :

$$\xi(t)' \left(\begin{bmatrix} A'P + PA & PB + \eta A'C'\Omega \\ \star & \eta(\Omega CF + F'C'\Omega) \end{bmatrix} + \begin{bmatrix} 0 & C'\Omega\Lambda \\ \star & -2\Lambda \end{bmatrix} \right) \xi(t) < 0. \tag{8}$$

Links with KYP Lemma, frequency approach...

Definition of a Lur'e system (iii) : Discrete-time

Discrete-time Lur'e system :

$$x_{k+1} = Ax_k + F\varphi(y_k), \quad (9)$$

$$y_k = Cx_k, \quad (10)$$

where $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^p$, $(k \in \mathbb{N})$.

Classical Lyapunov functions : Extensions provided by Tsypkin [4].

- The quadratic function with respect to the state (extension of Circle criterion) :

$$v(x_k) = x_k' P x_k; \quad (11)$$

- Lur'e-type Lyapunov function (extension of Popov criterion) :

$$v(x_k) = x_k' P x_k + 2\eta \int_0^{Cx_k} \Omega\varphi(s)\mathrm{d}s, \ \alpha > 0, \ \eta \geq 0; \quad (12)$$

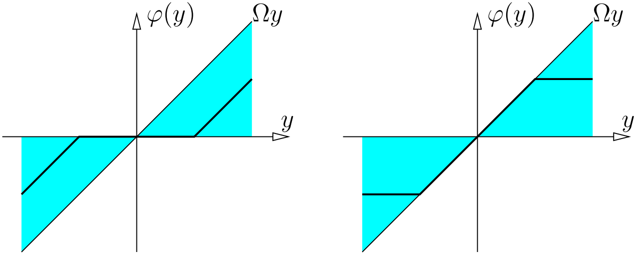
  - $\varphi(\cdot)$ must be time-invariant to have : $\int_0^{C'x} \varphi(s)\mathrm{d}s \geq 0$;
  - $v(\cdot)$ is inspired from the *continuous-time* ;
  - An extra assumption [5] [6]. is necessary to bound $\int_{y_k}^{y_{k+1}} \varphi(s)\mathrm{d}s$. Ex : $\frac{\mathrm{d}\varphi(y)}{\mathrm{d}y} \leq K_{\max}$.

4. Y. Z. Tsypkin. "The absolute stability of large-scale nonlinear sampled-data systems". In : *Doklady Akademii Nauk SSSR* 145 (1962), p. 52–55.

5. J. B. Pearson et J. E. Gibson. "On the Asymptotic Stability of a Class of Saturating Sampled-Data Systems". In : *IEEE Transactions on Industry Applications* AI–83 (1964), p. 81–86.

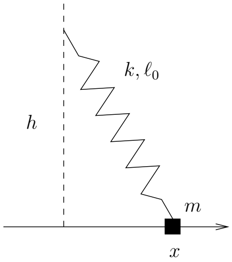
6. G. P. Szegö. "On the Absolute Stability of Sampled-Data Control Systems". In : *Proceedings of National Academy of Sciences* 50 (1963), p. 558–560.

Motivation example (i) : Deadzone and Saturation



$$\varphi'(y)\Lambda\left(\varphi(y) - y\right) \leq 0.$$

Motivation example (ii) : a mechanical system with spring



A mass m is constrained to slide along a straight horizontal wire, with a viscous damping force of coefficient α. A spring of relaxed length ℓ_0 and spring stiffness k is attached to the mass and to the support point a distance h from the wire. The horizontal coordinate of the mass is denoted $x(t)$ and we define $x = 0$ when the spring is vertical. [7].

The nonlinear motion equation of the mass m is given by the Newton's law :

$$\ddot{x}(t) = -\frac{\alpha}{m}\dot{x}(t) - \frac{k}{m}x(t) + \frac{k}{m}\frac{\ell_0}{\sqrt{x^2(t) + h^2}}x(t).$$

$$\varphi(x)\left(\varphi(x) - \Omega x\right) \leq 0, \quad \Omega = \frac{\ell_0}{h}; \quad \varphi_1(x) = \frac{\ell_0}{\sqrt{x^2 + h^2}}x.$$

- If $\ell_0 > h$, the origin is unstable ;
- If $\ell_0 \leq h$, the origin is globally asymptotically stable.

7. S. H. STROGATZ. *Nonlinear Dynamics and Chaos*. Studies in nonlinearity. Perseus books, 1994.

Motivation example (iii) : *Duffing* system

Differential equation

$$m\ddot{\xi} + \gamma\dot{\xi} + \alpha\xi + \beta\xi^3 = F\cos(wt) \tag{13}$$

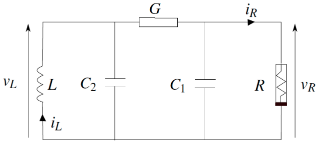
where ξ is the position, m the mass, γ damping coefficient, α stiffness, β return force, F amplitude and w pulsation of input force.

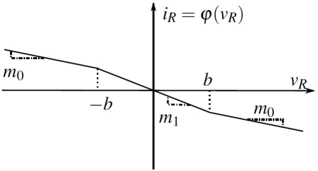
$$\begin{cases} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ \frac{-\alpha}{m} & \frac{\gamma}{m} \end{bmatrix} x(t) - \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \varphi(y(t)) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t), \quad t \in \mathbb{R}^+, \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t), \\ u(t) &= F\cos(wt), \end{cases}$$

with $x(t) = \begin{pmatrix} \dot{\xi}(t) \\ \xi(t) \end{pmatrix}$ and $\varphi(y(t)) = \beta y^3$.

Then $\Omega = +\infty$, that is $y\varphi(y) \geq 0$.

Motivation example (iv) : *Chua*'s Circuit





Let $x(t) = \begin{pmatrix} v_R & v_L & i_L \end{pmatrix}'$, thus Chua's circuit is a Lur'e system :

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \begin{bmatrix} -\frac{G}{C_1} & \frac{G}{C_1} & 0 \\ -\frac{G}{C_2} & \frac{G}{C_2} & \frac{1}{C_2} \\ 0 & \frac{1}{L} & 0 \end{bmatrix} x(t) + \begin{bmatrix} \frac{-1}{C_1} \\ 0 \\ 0 \end{bmatrix} \varphi(y(t)), \quad t \in \mathbb{R}^+, \\ y(t) & = & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t), \end{array} \right.$$

where

$$\varphi(y(t)) = m_0 y(t) + \frac{m_1 - m_0}{2} \left(|y(t) + b| - |y(t) - b| \right),$$

with scalar parameters m_0, m_1 and b. This is a chaotic system.

Motivation example (v) : link with uncertainty

An uncertain system

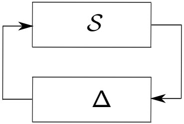
$$\dot{x}(t) = Ax(t) + F\Delta Cx(t), \quad 0 \leq \Delta \leq \Delta_{\max}, \tag{14}$$

can be reformulated into a Lur'e system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + F\varphi(y(t)), \\ y(t) &= Cx(t), \\ \varphi(y) &= \Delta y \end{aligned}$$

and with

$$\varphi(y)(\varphi(y) - \Delta_{\max} y) \leq 0.$$

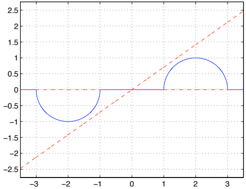


Outline of the talk

Main difficulty in discrete-time case

In discrete-time, extra assumption about the slope of the nonlinearity is required. That introduces a break of analogy with respect to the continuous-time framework.

A counterexample : half-circle allowing vertical tangents.



Aim : Consider a suitable Lur'e-like Lyapunov function in order to

- propose sufficient conditions for the global stability analysis problem (Lur'e problem) ;
- cover a wider range of cone bounded nonlinearities ;
- relax the assumptions of the classical literature of the Lur'e problem.

Taking into account the nonlinearity by avoiding the integral term.

A Lur'e-like Lyapunov function for discrete-time

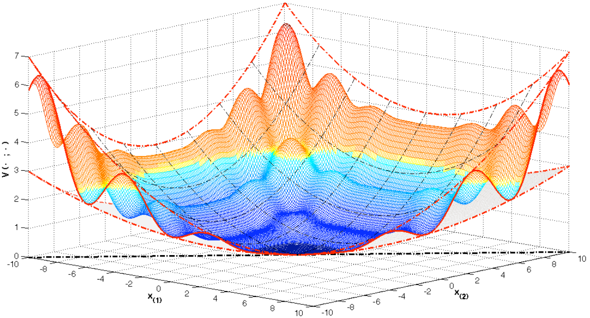
Definitions

$$V : \begin{cases} \mathbb{R}^n \times \mathbb{R}^p & \longrightarrow \quad \mathbb{R}, \\ (x; \varphi(Cx)) & \longmapsto \quad x'Px + 2\varphi(Cx)'\Delta\Omega Cx, \end{cases} \qquad (15)$$

- with $0_n < P = P' \in \mathbb{R}^{n \times n}$ and $0_p \leq \Delta \in \mathbb{R}^{p \times p}$ diagonal.
- Bounding quadratic functions :

$$\underline{V}(x) \leq V(x; \varphi(Cx)) \leq \overline{V}(x). \qquad (16)$$

where $\underline{V}(x) = x'Px$ and $\overline{V}(x) = x'(P + 2C'\Omega'\Delta\Omega C)x$.



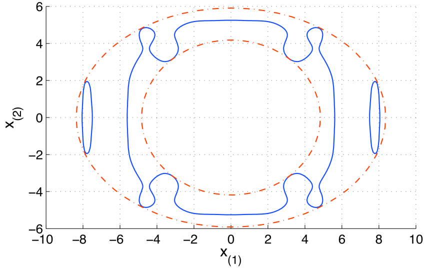
Basic properties

Candidate Lyapunov function :

- $V(x;\varphi(Cx)) \geq 0$ due to $P > 0_n$ and the sector condition (1) of $\varphi(\cdot)$.
- $V(x;\varphi(Cx)) = 0 \Leftrightarrow x = 0$, due to $P > 0_n$.
- Relation (16) implies that function (15) is radially unbounded
- Lyapunov difference : $\delta_k V = V(x_{k+1};\varphi(Cx_{k+1})) - V(x_k;\varphi(Cx_k))$.

The level set of our function (15)

$$L_V(\gamma) = \left\{x \in \mathbb{R}^n;\, V(x;\varphi(Cx)) \leq \gamma\right\}. \tag{17}$$



- The set $L_V(\gamma)$ may be non-convex and disconnected.

Global stability analysis

Theorem

Global Stability Analysis *If there exists a matrix $0_n < P = P' \in \mathbb{R}^{n\times n}$, a diagonal matrix $0_p \leq \Delta \in \mathbb{R}^{p\times p}$ and diagonal matrices $0_p < T, W \in \mathbb{R}^{p\times p}$, such that the LMI*

$$\begin{bmatrix} A' \\ F' \\ 0_{p\times n} \end{bmatrix} P \begin{bmatrix} A' \\ F' \\ 0_{p\times n} \end{bmatrix}' + \begin{bmatrix} -P & C'\Omega\,[T-\Delta] & A'C'\Omega\,[W+\Delta] \\ \star & -2T & F'C'\Omega\,[W+\Delta] \\ \star & \star & -2W \end{bmatrix} < 0_{2n+2p}, \tag{18}$$

is verified, then the function $V(x;\varphi(Cx))$ is a Lyapunov function and the origin of system (9)-(10) is globally asymptotically stable.

Main idea :

$$V(x_{k+1};\varphi(Cx_{k+1})) - V(x_k;\varphi(Cx_k)) \\ - 2\mathrm{SC}(\varphi(\cdot), y_{k+1}, W) - 2\mathrm{SC}(\varphi(\cdot), y_k, T) < 0, \quad \forall x_k \neq 0.$$

No assumption about the variation of $\varphi(\cdot)$.

8. C. A. C. Gonzaga, M. Jungers et J. Daafouz. “Stability analysis of discrete time Lur’e systems”. In : *Automatica* 48 (9 2012), p. 2277–2283

Illustration for global stability analysis

Example 1 : global stability analysis

- Lur'e system with $n = 2$, $p = 1$, $\Omega = \frac{1}{\sqrt{2}}$:

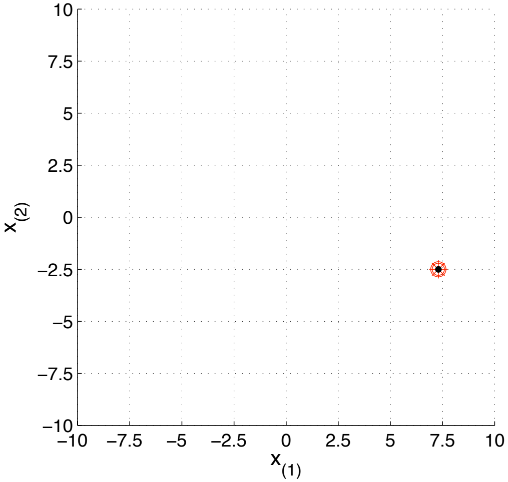
$$A = \begin{bmatrix} 0.5 & 0.1 \\ 0.3 & -0.4 \end{bmatrix} ; F = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} ; C' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ;$$

- $\varphi(y) = 0.5\Omega y(1 + \cos(10y))$ (unbounded derivative on $y \in \mathbb{R}$) ;
- The Lyapunov function (15) exists and applying Theorem 18 leads to :

$$P = \begin{bmatrix} 0.9825 & -0.0846 \\ -0.0846 & 0.9476 \end{bmatrix} ; \ \Delta = 0.7503.$$

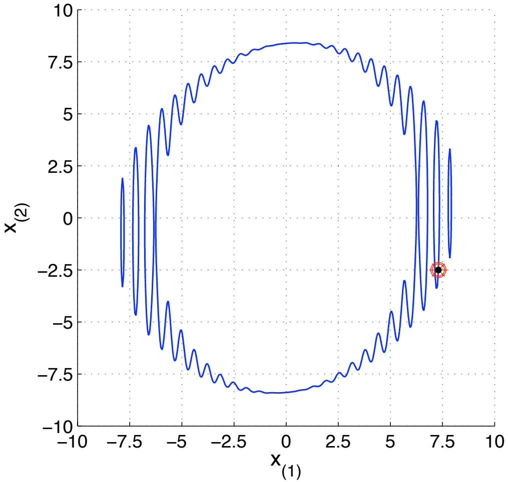
Global stability analysis

One initial condition x_0 $\quad k = 0$



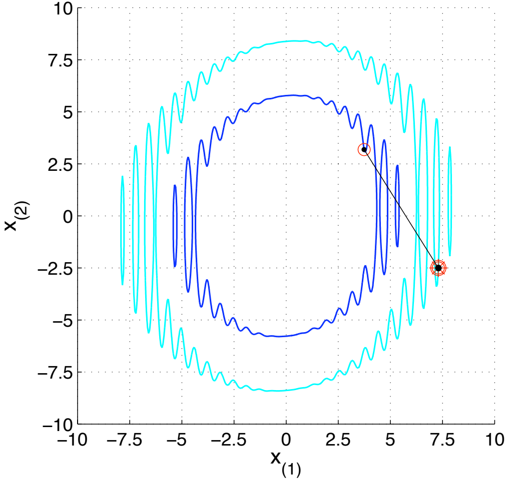
Global stability analysis

Contractivity of the level set $L_V(\gamma = V(x_0, \varphi(y_0)))$; $\quad k = 0$



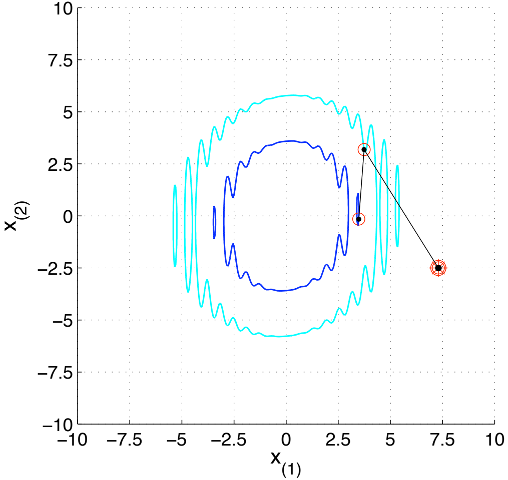
Global stability analysis

$L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1})))$ and $L_V(\gamma = V(x_k, \varphi(y_k)))$; $\quad k = 1$



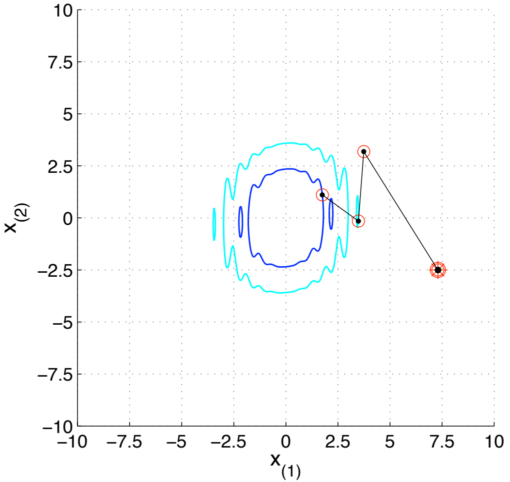
Global stability analysis

$L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1})))$ and $L_V(\gamma = V(x_k, \varphi(y_k)))$; $\qquad k = 2$



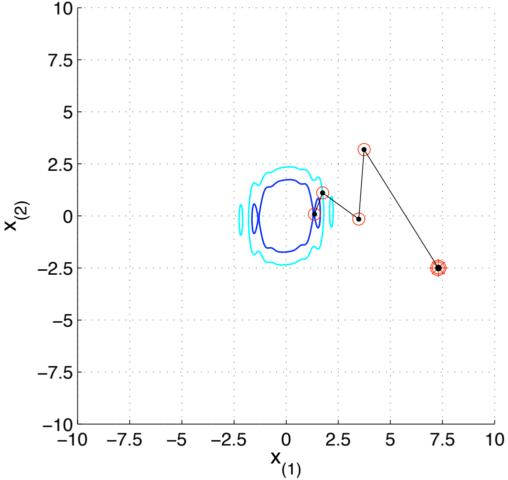
Global stability analysis

$L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1})))$ and $L_V(\gamma = V(x_k, \varphi(y_k)))$; $k = 3$



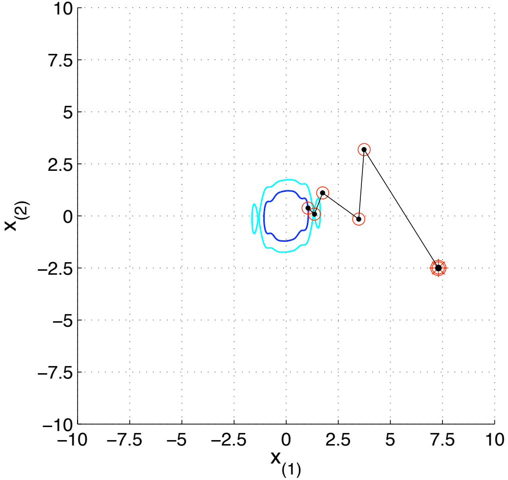
Global stability analysis

$L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1})))$ and $L_V(\gamma = V(x_k, \varphi(y_k)))$; $\qquad k = 4$



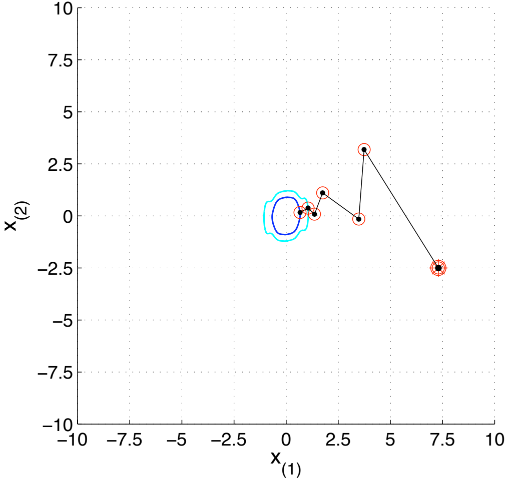
Global stability analysis

$L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1})))$ and $L_V(\gamma = V(x_k, \varphi(y_k)))$; $\qquad k = 5$



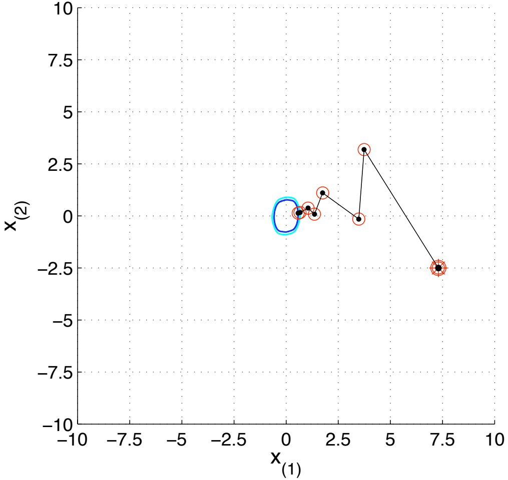
Global stability analysis

$L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1})))$ and $L_V(\gamma = V(x_k, \varphi(y_k)))$; $\quad k = 6$



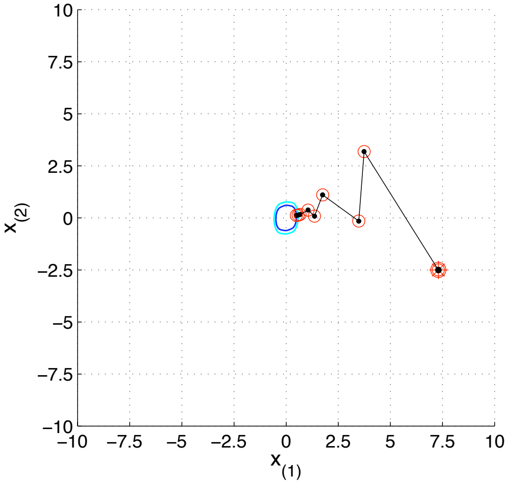
Global stability analysis

$L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1})))$ and $L_V(\gamma = V(x_k, \varphi(y_k)))$; $\qquad k = 7$



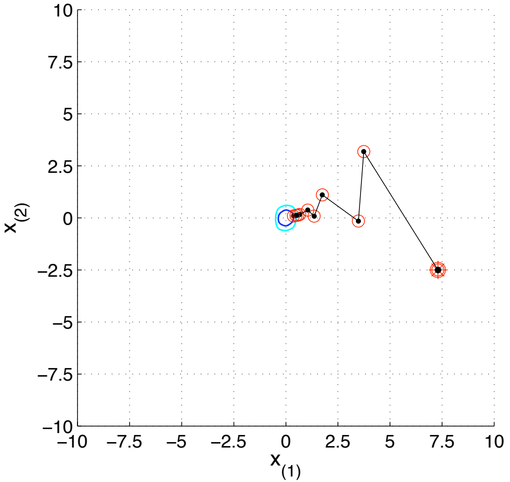
Global stability analysis

$L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1})))$ and $L_V(\gamma = V(x_k, \varphi(y_k)))$; $\qquad k = 8$



Global stability analysis

$L_V(\gamma = V(x_{k-1}, \varphi(y_{k-1})))$ and $L_V(\gamma = V(x_k, \varphi(y_k)))$; $\quad k = 9$

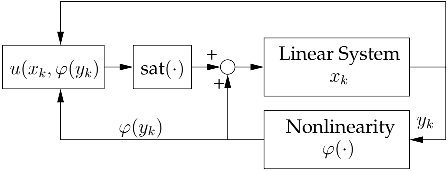


Lur'e system with saturated input

$$x_{k+1} = Ax_k + F\varphi(y_k) + B\mathrm{sat}(u_k), \quad \forall k \in \mathbb{N} \qquad (19)$$

$$y_k = Cx_k \qquad (20)$$

Class of state and nonlinearity feedbacks as controller : $u_k = Kx_k + \Gamma\varphi(y_k)$.



Due to the saturated input in discrete-time :

- Only local stability ;
- The basin of attraction of the origin $\mathcal{B}_0$ may be non-convex and disconnected.

Aims :

- Stability analysis and control synthesis,
- Estimate the basin of attraction $\mathcal{B}_0$ via the level set $L_V(1)$;

Tools :

- The deadzone $\Psi(u_k) = u_k - \mathrm{sat}(u_k)$, is dual to the saturation.
- On the set

$$\mathcal{S}(\hat{K} - \hat{J}, \rho) = \{\theta \in \mathbb{R}^{n+p}; -\rho \leq (\hat{K} - \hat{J})\theta \leq \rho\}, \tag{21}$$

with $\hat{K} = [K \ \Gamma]$ and $\hat{J} = [J_1 \ J_2]$, $\Psi(u_k)$ verifies a generalized LOCAL cone bounded condition :

$$\mathrm{SC}_{u_k} = \Psi'(u_k) U \left[\Psi(u_k) - J_1 x_k - J_2 \varphi(y_k)\right] \leq 0, \tag{22}$$

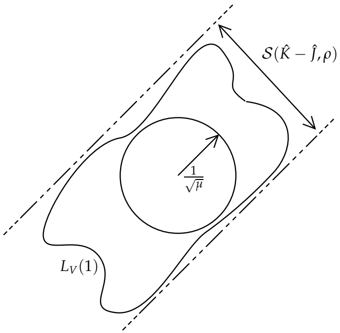
for any diagonal matrix $0_m < U \in \mathbb{R}^{m \times m}$.

Closed-loop system :

$$x_{k+1} = A_{\mathrm{cl}} x_k + F_{\mathrm{cl}} \varphi(y_k) - B\Psi(u_k), \tag{23}$$

where $A_{\mathrm{cl}} = A + BK$ and $F_{\mathrm{cl}} = F + B\Gamma$.

Main idea :



Inclusions as Matrix Inequalities [9]

IM1) Ball of radius $1/\sqrt{\mu}$ included inside $L_V(1)$.

IM2) $L_V(1) \subset \mathcal{S}(\hat{K}-\hat{J},\rho)$ such that $\mathrm{SC}_{u_k} \leq 0$.

IM3) $\delta_k V - 2\mathrm{SC}_{u_k} - 2\mathrm{SC}(\varphi(\cdot), y_{k+1}, W) - 2\mathrm{SC}(\varphi(\cdot), y_k, T) < 0$.

Conclusion : on $L_V(1)$, $\delta_k V < 0, \forall\, x \neq 0$.

9. C. A. C. Gonzaga, M. Jungers et J. Daafouz. "Stability analysis of discrete time Lur'e systems". In : *Automatica* 48 (9 2012), p. 2277–2283.

Inequalities implying the inclusions (i)

- The LMI

$$\begin{bmatrix} \mu I_n - P & -C'\Omega\,[R+\Delta] \\ \star & 2R \end{bmatrix} > 0_{n+p}, \tag{24}$$

leads to

$$\mathcal{E}(I_n, \frac{1}{\mu}) \subset L_V(1). \tag{25}$$

- The LMI

$$\begin{bmatrix} P & C'\Omega\,[\Delta - Q] & (K-J_1)'_{(\ell)} \\ \star & 2Q & (\Gamma - J_2)'_{(\ell)} \\ \star & \star & \rho^2_{(\ell)} \end{bmatrix} > 0_{n+p+1}, \tag{26}$$

yields, with $\hat{K} = [K\ \Gamma]$ and $\hat{J} = [J_1\ J_2]$

$$V(x_k, \varphi(y_k)) + 2\mathrm{SC}(\varphi(\cdot), y_k, Q) \geq \frac{\|(K-J_1)_{(\ell)}x_k + (\Gamma - J_2)\varphi(y_k)\|^2}{\rho^2_{(\ell)}}; \tag{27}$$

and finally

$$L_V(1) \subset \mathcal{S}((\hat{K} - \hat{J}), \rho). \tag{28}$$

Inequalities implying the inclusions (ii)

If the BMI is feasible (LMI by applying the Finsler's Lemma, or setting U),

$$\begin{bmatrix} A'_{\rm cl} \\ F'_{\rm cl} \\ -B' \\ 0_{p\times n} \end{bmatrix} P \begin{bmatrix} A'_{\rm cl} \\ F'_{\rm cl} \\ -B' \\ 0_{p\times n} \end{bmatrix}' + \begin{bmatrix} -P & \Pi_1 & J_1{}'U' & A'_{\rm cl}\Pi_2 \\ \star & -2T & J_2{}'U' & F'_{\rm cl}\Pi_2 \\ \star & \star & -2U & -B'\Pi_2 \\ \star & \star & \star & -2W \end{bmatrix} < 0, \tag{29}$$

with $\Pi_1 = C'\Omega\,[T-\Delta]\,;\,\Pi_2 = C'\Omega\,[W+\Delta]$, then one obtain

$$\delta_k V - 2\mathrm{SC}_{u_k} - 2\mathrm{SC}(\varphi(\cdot), y_{k+1}, W) - 2\mathrm{SC}(\varphi(\cdot), y_k, T) < 0. \tag{30}$$

Inequalities (26) and (29) ensure the asymptotic stability on $x_0 \in L_V(1)$.

Optimization problem for increasing the *size* of $L_V(1)$

Theorem

Local asymptotic stability and best $L_V(1)$ *If there exist matrices $G \in \mathbb{R}^{n\times n}$, $J_1 \in \mathbb{R}^{m\times n}$, $J_2 \in \mathbb{R}^{m\times p}$, matrix $0_n < P = P' \in \mathbb{R}^{n\times n}$; diagonal matrices $0_p \leq \Delta \in \mathbb{R}^{p\times p}$, $0_p < R, Q, T, W \in \mathbb{R}^{p\times p}$, and a scalar μ solutions of the following optimization problem :*

$$\min_{G,\ P,\ J_1,\ J_2,\ Q,\ R,\ T,\ W,\ \Delta,\ \mu} \mu$$

under the constraints (24), (26) and (29)

then an estimate of $\mathcal{B}_0$ is given by the set $L_V(1)$.

Illustration

Example 2 :

- Lur'e system defined by : $n = 2$; $p = m = 1$; $\rho = 1.5$; $\Omega = 0.9$.

$$A = \begin{bmatrix} 0.85 & 0.4 \\ 0.6 & 0.95 \end{bmatrix} ; \; B = \begin{bmatrix} 1.3 \\ 1.2 \end{bmatrix} ; \; F = \begin{bmatrix} 1.3 \\ 1.2 \end{bmatrix} ; \; C = \begin{bmatrix} -0.5 & 0.9 \end{bmatrix}.$$

- With given gains :

$$K = \begin{bmatrix} -0.3324 & -1.0006 \end{bmatrix}$$

- The theorem leads to :

$$P = \begin{bmatrix} 0.0418 & 0.0173 \\ 0.0173 & 0.2305 \end{bmatrix} ; \; \Delta = 0.0381.$$

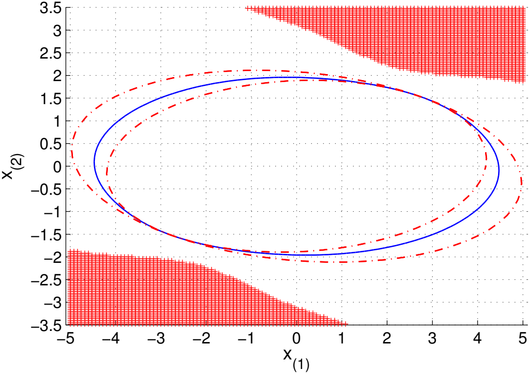
Without knowing $\varphi(y_k)$, the estimate of $\mathcal{B}_0$ is the inner ellipsoid :

$$\mathcal{E}(P + 2C'\Omega\Delta\Omega C)$$

... but with knowing $\varphi(y_k)$...

Illustration

$L_V(1)$ for distinct nonlinearities :
$\varphi(y) = 0.5\Omega y(1 + \exp(-0.5y^2))$.

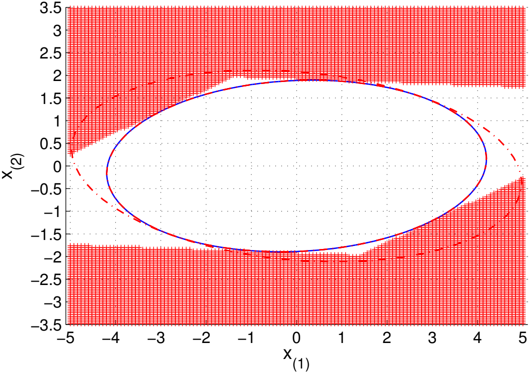


Initial conditions x_0 leading to unstable trajectories

The basin of attraction of the origin $\mathcal{B}_0$ depends on the nonlinearity.

Illustration

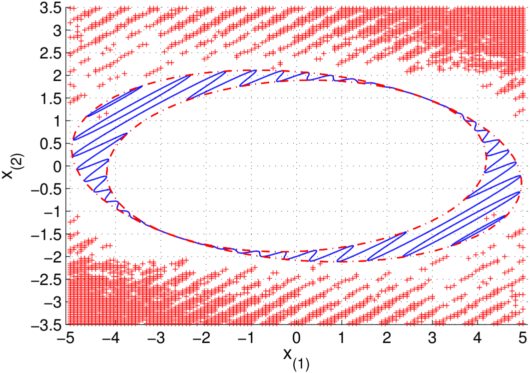
$L_V(1)$ for distinct nonlinearities :
$\varphi(y) = \Omega y$.



Initial conditions x_0 leading to unstable trajectories
The basin of attraction of the origin $\mathcal{B}_0$ depends on the nonlinearity.

Illustration

$L_V(1)$ for distinct nonlinearities :
$\varphi(y) = 0.5\Omega y(1 + \cos(20y))$.



Initial conditions x_0 leading to unstable trajectories

The basin of attraction of the origin $\mathcal{B}_0$ depends on the nonlinearity.

Outline of the talk

Université de Lorraine, Nancy, CRAN Laboratory

Lur'e systems

Introduction of a new Lyapunov-Lur'e type function

Extension to switched Lur'e systems
- Definition
- Global stability analysis
- Global stabilization
- Local stability analysis
- Local stabilization

About consistency

Application to sampled-data Lur'e systems with nonuniform sampling

Conclusion

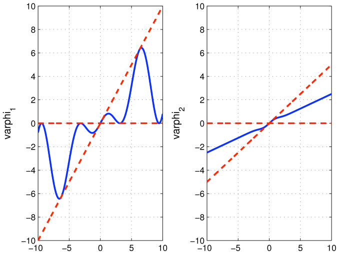
Switched Lur'e system

Discrete-time switched system composed of Lur'e subsystems :

$$x_{k+1} = A_{\sigma(k)}x_k + F_{\sigma(k)}\varphi_{\sigma(k)}(y_k), \quad (31)$$
$$y_k = C_{\sigma(k)}x_k, \quad (32)$$

where $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^p$, $\sigma(\cdot) : \mathbb{N} \to \mathcal{I}_N = \{1, ..., N\}$.

Motivation :



- The active nonlinearity is defined by the switching rule.
- Each mode is associated with a nonlinearity ;
- The sector conditions are mode-dependents, $\forall i \in \mathcal{I}_N$:

$$\mathrm{SC}(\varphi_i(\cdot), y, \Lambda_i) = \varphi_i'(y)\Lambda_i[\varphi_i(y) - \Omega_i y] \leq 0 \quad (33)$$

Tools

Main tool :

- The extension of our function (15) to the switched systems framework [10] :

$$V : \left\{ \begin{array}{lcl} \mathcal{I}_N \times \mathbb{R}^n \times \mathbb{R}^p & \longrightarrow & \mathbb{R}, \\ (i, x, \varphi_i(C_i x)) & \longmapsto & x' P_i x + 2(\varphi_i(C_i x))' \Delta_i \Omega_i C_i x, \end{array} \right. \quad (34)$$

- Consider the function $V_{min}(x_k) = \min_{i \in \mathcal{I}_N} V(i, x_k, \varphi_i(C_i x_k))$
  - inherits all the basic properties of function (34).

Auxiliary notation :

- Extended system matrices and state vector :

$$\mathbb{A}_i = [\ A_i \quad F_i \quad 0_{n \times Np}\ ] \in \mathbb{R}^{n \times (n+(N+1)p)};$$
$$\mathbb{E}_i = [\ 0_{p \times (n+ip)} \quad I_p \quad 0_{p \times (N-i)p}\ ] \in \mathbb{R}^{p \times (n+(N+1)p)};$$
$$z_k' = \left(x_k' \quad \varphi_i'(C_i x_k) \quad \varphi_1'(C_1 x_{k+1}) \quad \ldots \quad \varphi_N'(C_N x_{k+1}) \right)' \in \mathbb{R}^{(n+(N+1)p)}.$$

- Set of Metzler matrices (in discrete time) :
  The matrix $\Pi \in \mathcal{M}_{\mathrm{d}}$, where $\mathcal{M}_{\mathrm{d}}$ is the Metzler matrices set :

$$\mathcal{M}_{\mathrm{d}} = \left\{ \Pi \in \mathbb{R}^{N \times N},\ \pi_{ii} \geq 0,\ \sum_{\ell \in \mathcal{I}_N} \pi_{\ell i} = 1, \forall i \in \mathcal{I}_N \right\}.$$

10. M. Jungers, C. A. C. Gonzaga et J. Daafouz. “Min-Switching Stabilization for Discrete-Time Switching Systems with Nonlinear Modes”. In : *4th IFAC Conference on Analysis and Design of Hybrid Systems, ADHS 2012*. Eindhoven, The Netherlands, 2012, p. 234–239.

Global stability with arbitrary switching law

Analogy with not switching Lur'e systems

Tools	Not switching	Switching
Lyapunov function	$V(x;\varphi(Cx))$	$V(i;x;\varphi_i(C_ix))$
$L_V(\gamma)$	$\{x\in\mathbb{R}^n; V(x;\varphi(Cx))\leq\gamma\}$	$\bigcap_{i\in\mathcal{I}_N}\{x\in\mathbb{R}^n; V(i;x;\varphi_i(C_ix))\leq\gamma\}$
# LMIs	1	N^2
Bounds of L_V	Ellipsoids	Intersections of Ellipsoids

Global stability analysis

Theorem

Global Stability Analysis [11] *If there exists N matrices* $0_n < P_i = P_i' \in \mathbb{R}^{n\times n}$, *N diagonal matrices* $0_p \leq \Delta_i \in \mathbb{R}^{p\times p}$ *and diagonal matrices* $0_p < T_i, W_i \in \mathbb{R}^{p\times p}$, *such that the LMI,* $\forall (i,j) \in \{1,\cdots,N\}^2$

$$\begin{bmatrix} A_i' \\ F_i' \\ 0_{p\times n} \end{bmatrix} P_j \begin{bmatrix} A_i' \\ F_i' \\ 0_{p\times n} \end{bmatrix}' + \begin{bmatrix} -P_i & C_i'\Omega_i\,[T_i - \Delta_i] & A_i' C_j' \Omega_j\,[W_j + \Delta_j] \\ \star & -2T_i & F_i' C_j' \Omega_j\,[W_j + \Delta_j] \\ \star & \star & -2W_j \end{bmatrix} < 0_{2n+2p}, \tag{35}$$

is verified, then the function $V(\sigma_k; x_k; \varphi_{\sigma(k)}(C_{\sigma(k)}x_k))$ *is a Lyapunov function and the origin of system (9)-(10) is globally asymptotically stable.*

Main idea :

$$V(\sigma(k+1), x_{k+1}; \varphi(Cx_{k+1})) - V(\sigma(k), x_k; \varphi(Cx_k)) \\ - 2\mathrm{SC}(\varphi_{\sigma(k+1)}(\cdot), y_{k+1}, W_{\sigma(k+1)}) - 2\mathrm{SC}(\varphi_{\sigma(k)}(\cdot), y_k, T_{\sigma(k)}) < 0, \quad \forall x_k \neq 0.$$

No assumption about the variation of $\varphi_{\sigma(k)}(\cdot)$ and $\varphi_{\sigma(k+1)}(\cdot)$.

11. C. A. C. Gonzaga, M. Jungers et J. Daafouz. "Stability analysis and stabilisation of switched nonlinear systems". In : *International Journal of Control* 85.7 (2012), p. 822–829.

Global stabilization : Min-switching strategy

Theorem : Min-switching strategy based on $V(i, x_k, \varphi_i(C_i x_k))$ [12]

Assume there exist a matrix $\Pi \in \mathcal{M}_{\mathrm{d}}$; matrices $0_n < P_i = P_i' \in \mathbb{R}^{n\times n}$ and diagonal matrices $0_p < T_i$, W_i, $0_p \leq \Delta_i \in \mathbb{R}^{p\times p}$, $(i \in \mathcal{I}_N)$, such that the Lyapunov-Metzler inequalities are satisfied $\forall i \in \mathcal{I}_N$

$$\mathbb{A}_i'(P)_{p,i}\mathbb{A}_i + \mathrm{He}(\mathbb{A}_i'(C'\Omega\Delta\mathbb{E})_{p,i}) - \sum_{q\in\mathcal{I}_N}\Big(2\mathbb{E}_q' W_q \mathbb{E}_q - \mathrm{He}(\mathbb{E}_q' W_q \Omega_q C_q \mathbb{A}_i)\Big) - \begin{bmatrix} P_i & \star & \star \\ (\Delta_i - T_i)\Omega_i C_i & 2T_i & \star \\ 0_{Np\times n} & 0_{Np\times p} & 0_{Np} \end{bmatrix} < 0_{n+(N+1)p}, \quad (36)$$

where $(P)_{p,i} = \sum_{\ell\in\mathcal{I}_N} \pi_{\ell i} P_\ell$, then the min-switching strategy

$$\sigma(k) = u(x_k) = \arg\min_{i\in\mathcal{I}_N} V(i, x_k, \varphi_i(C_i x_k)) \quad (37)$$

globally asymptotically stabilizes the system (31)-(32).

12. M. Jungers, C. A. C. Gonzaga et J. Daafouz. "Min-Switching Stabilization for Discrete-Time Switching Systems with Nonlinear Modes". In : *4th IFAC Conference on Analysis and Design of Hybrid Systems, ADHS 2012*. Eindhoven, The Netherlands, 2012, p. 234–239.

Sketch of the proof

The matrix inequalities (36) are formulated in order to :

- Consider the sum of :
  - the sector condition at time $k+1$:
  $$\varphi_q'(C_q x_{k+1}) W_q [\varphi_q(C_q x_{k+1}) - \Omega_q C_q x_{k+1}] \leq 0, \quad (38)$$
  - written in the equivalent form :
  $$-z_k' \left(2\mathbb{E}_q' W_q \mathbb{E}_q - \mathrm{He}(\mathbb{E}_q' W_q \Omega_q C_q \mathbb{A}_i)\right) z_k \geq 0,$$

  with $0_p < W_q \in \mathbb{R}^{p \times p}$ diagonal.

- Upper-bound the function $V_{min}(x_{k+1}) = \min_{j \in \mathcal{I}_N} V(j, x_{k+1}, \varphi_j(C_j x_{k+1}))$ by the aid of these sector conditions ;

- Guarantee, due to properties of the Metzler matrix $\Pi \in \mathcal{M}_{\mathrm{d}}$, that $V_{min}(x_{k+1}) - V_{min}(x_k) < 2SC(\varphi_{\sigma(k}(\cdot), y_k, T_{\sigma(k)}) \leq 0.$

State space partition

State space partition :

- Let the sets $\mathcal{S}_i$ allowing to activate the mode $i \in \mathcal{I}_N$:
$$\mathcal{S}_i = \left\{x \in \mathbb{R}^n,\ V_{min}(x) = V(i, x, \varphi_i(C_i x))\right\}, \quad \forall i \in \mathcal{I}_N. \tag{39}$$
- $0 \in \mathcal{S}_i$, $\forall i \in \mathcal{I}_N$;
- $\cup_{i \in \mathcal{I}_N} \mathcal{S}_i = \mathbb{R}^n$, at least one mode reaches the minimum of our function ;
- the sets $\mathcal{S}_i$ are not necessarily disjoint.

Remark : Feasibility of Inequalities (36) implies inclusions $\pi_{ii}^{\frac{1}{2}} A_i$ and $\pi_{ii}^{\frac{1}{2}}(A_i + B_i \Omega_i C_i)$ stable, $\forall i \in \mathcal{I}_N$.

Illustration

Example : global stabilization

- Switched Lur'e system with $N = n = 2$, $p = 1$, $\Omega_1 = 0.6\,; \Omega_2 = 0.4$:

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 1.08 & 0 \\ 0 & -0.72 \end{bmatrix}; F_1 = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}; C_1' = \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}; \\
A_2 &= \begin{bmatrix} -0.48 & 0.8 \\ 0 & 0.8 \end{bmatrix}; F_2 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}; C_2' = \begin{bmatrix} 0.4 \\ 1 \end{bmatrix}.
\end{aligned}
$$

- The nonlinearities are : $\varphi_1(y) = 0.5\Omega_1 y(1 + cos(2y))$ and $\varphi_2(y) = 0.5\Omega_2 y(1 - \sin(2.5y))$.
- The numerical results are obtained :

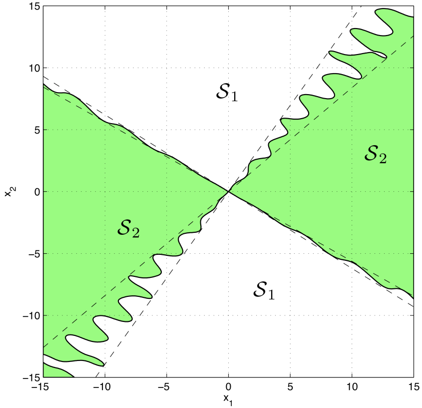
$$
P_1 = \begin{bmatrix} 1.1490 & -0.0832 \\ -0.0832 & 1.9764 \end{bmatrix};\; P_2 = \begin{bmatrix} 0.3508 & -0.4489 \\ -0.4489 & 3.1440 \end{bmatrix};
$$

$$
\Delta_1 = 0.2585;\; \Delta_2 = 1.0509;\; \text{with the Meztler matrix } \Pi = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}
$$

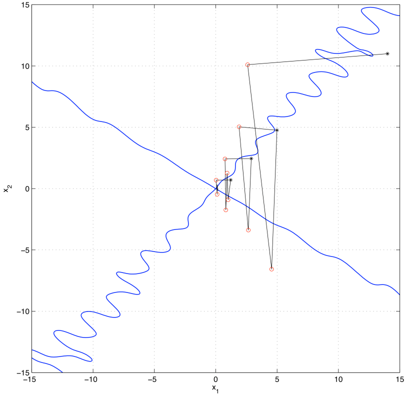
Illustration

State space partition and a trajectory for $x_0 = (14; 11)'$

Set $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$ and bounding cones $\mathcal{C}_1$; $\mathcal{C}_2$.



Trajectory x_k and the modes selected at each instant k.



With $\Delta_i \neq 0_p$, the state partition exhibits ripples.

Switched Lur'e system with input saturation

Discrete-time switched Lur'e systems with control saturation :

$$x_{k+1} = A_{\sigma(k)} x_k + F_{\sigma(k)} \varphi_{\sigma(k)}(y_k) + B_{\sigma(k)} \mathrm{sat}(u_k), \quad (40)$$

$$y_k = C_{\sigma(k)} x_k, \quad (41)$$

where $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^p$ and $u_k \in \mathbb{R}^m$.

Assumptions :

- The state and the modal nonlinearities are available in real time ;
- The switched feedback control law is considered :

$$u_k = K_{\sigma(k)} x_k + \Gamma_{\sigma(k)} \varphi_{\sigma(k)}(y_k).$$

Input saturation :

- Only local stability can be assured ;
- The basin of attraction $\mathcal{B}_0$ may be non-convex and disconnected.

Tools

Main tools :

- Consider the function $V_{min}(x) = \min_{i \in \mathcal{I}_N} V(i, x, \varphi_i(C_i x))$ as candidate Lyapunov function,
- whose the level sets are given by :

$$\begin{aligned} L_{V_{\min}}(\gamma) &= \left\{x \in \mathbb{R}^n; V_{min}(x) \leq \gamma\right\} \\ &= \bigcup_{j \in \mathcal{I}_N} \left\{x \in \mathbb{R}^n; V(j; x; \varphi_j(C_j x)) \leq \gamma\right\}. \end{aligned}$$

and the set $L_{V_{\min}}(1)$ will be considered as an estimate of $\mathcal{B}_0$.

The approach is similar to the previous one [13].

13. M. Jungers, C. A. C. Gonzaga et J. Daafouz. "Min-Switching Local Stabilization for Discrete-Time Switching Systems with Nonlinear Modes". In : *Nonlinear Analysis : Hybrid Systems* 9 (2013), p. 18–26.

Illustration : Local stability analysis

Exemple :

- Lur'e system defined by $N = n = 2$; $p = m = 1$; $\rho = 1.5$, $C_1 = \begin{bmatrix} 0.9 & 0.5 \end{bmatrix}$; $C_2 = \begin{bmatrix} 1 & -0.7 \end{bmatrix}$; $\Omega_1 = 0.7$; $\Omega_2 = 1.3$.
- $\varphi_1(y) = 0.5\Omega_1 y(1 + \sin(30y))$; $\varphi_2(y) = 0.5\Omega_2 y(1 + \cos(\frac{100y}{3}))$

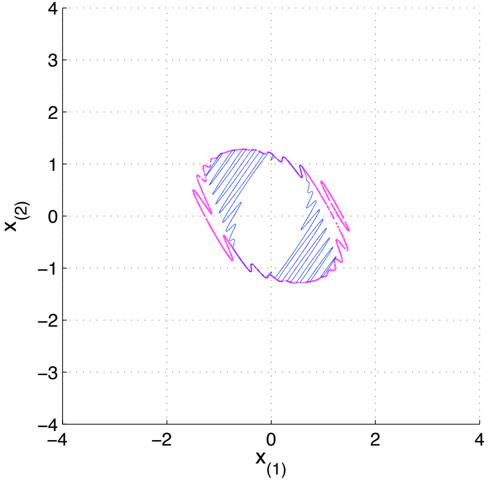
$$A_1 = \begin{bmatrix} 0.4 & 0.4 \\ 0.2 & 1 \end{bmatrix} ; B_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} ; F_1 = \begin{bmatrix} 1 \\ 1.2 \end{bmatrix} ;$$
$$A_2 = \begin{bmatrix} 1.1 & 0.6 \\ 0.3 & 0.4 \end{bmatrix} ; B_2 = \begin{bmatrix} 0.7 \\ 0.5 \end{bmatrix} ; F_2 = \begin{bmatrix} 1.2 \\ 1 \end{bmatrix} .$$

The switched gains are given as follows :

$$K_1 = \begin{bmatrix} -0.72 & -1.01 \end{bmatrix} ; \Gamma_1 = -1.2636;$$
$$K_2 = \begin{bmatrix} -1.27 & -0.74 \end{bmatrix} ; \Gamma_2 = -1.4744.$$

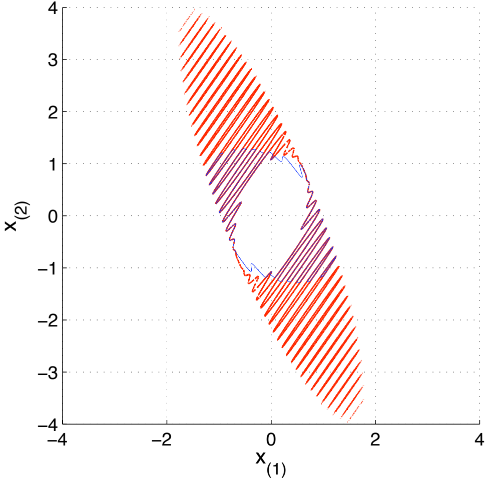
Illustration

$$\{x \in \mathbb{R}^n; V(1; x; \varphi_1(C_1 x) \leq 1\}.$$



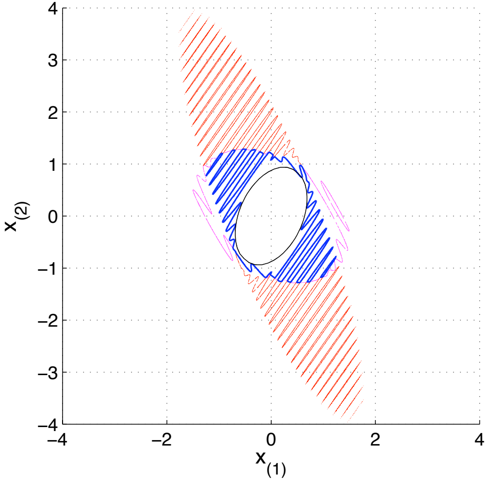
Illustration

$$\{x \in \mathbb{R}^n;\ V(2; x; \varphi_2(C_2 x) \leq 1\}.$$



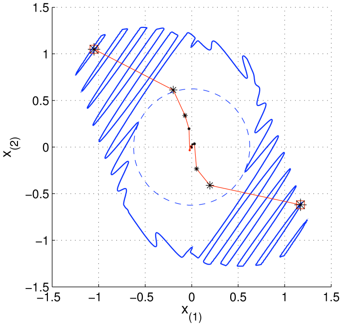
Illustration

$L_V(1)$ and the best estimate with the quadratic Lyapunov approach.



Illustration

Two trajectories with different arbitrary switching laws.



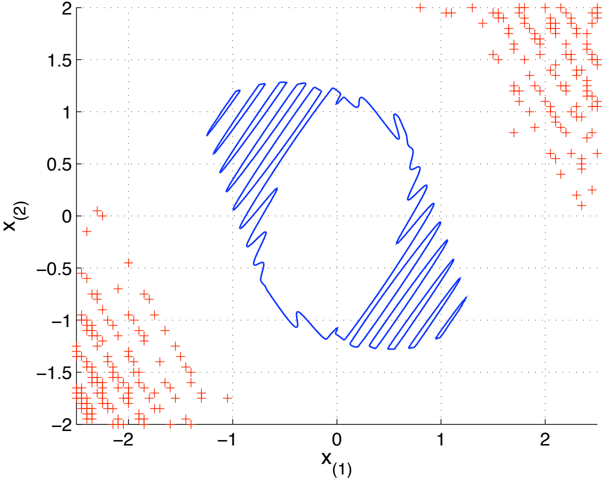
Question : what about the gap between $L_V(1)$ and $\mathcal{B}_0$?

Four (constant and periodic) switching laws are considered.

- $\sigma_a(2k) = 1$; $\sigma_a(2k+1) = 2$ $\forall k \in \mathbb{N}$;
- $\sigma_c(2k) = 2$; $\sigma_c(2k+1) = 1$ $\forall k \in \mathbb{N}$;
- $\sigma_b(k) = 1$; $\forall k \in \mathbb{N}$;
- $\sigma_d(k) = 2$; $\forall k \in \mathbb{N}$.

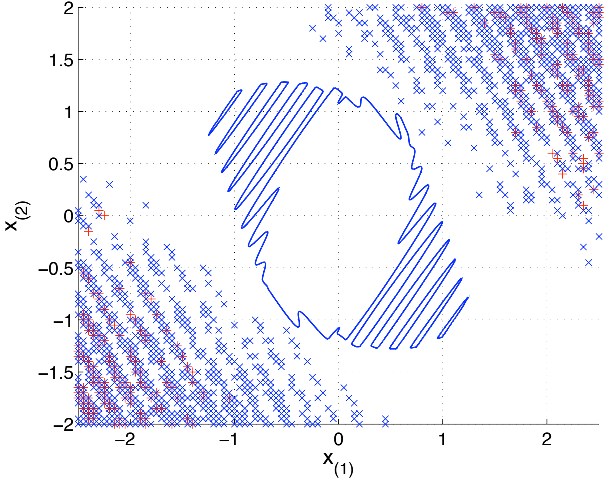
Illustration

$x_0 \notin L_V(1)$ leads to unstable trajectories with $\sigma_a(k)$.



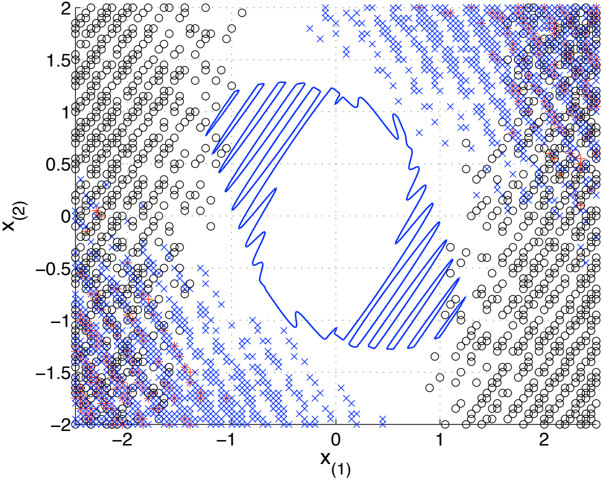
Illustration

$x_0 \notin L_V(1)$ leads to unstable trajectories with $\sigma_a(k)$, $\sigma_b(k)$.



Illustration

$x_0 \notin L_V(1)$ leads to unstable trajectories with $\sigma_a(k)$, $\sigma_b(k)$, $\sigma_c(k)$.



Illustration

$x_0 \notin L_V(1)$ leads to unstable trajectories with $\sigma_a(k)$, $\sigma_b(k)$, $\sigma_c(k)$, $\sigma_d(k)$.

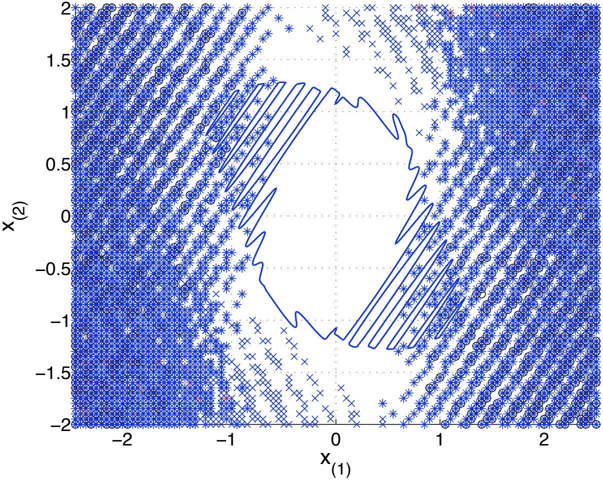


Illustration : local stabilization

Example :

- Switched Lur'e system with input saturation with $N = n = 2$, $p = 1$, $\rho = 5$; $\Omega_1 = 0.7$; $\Omega_2 = 0.5$:

$$\begin{aligned} A_1 &= \begin{bmatrix} 1.4 & 0.4 \\ 0.2 & 1 \end{bmatrix}; F_1 = \begin{bmatrix} 1 \\ 1.2 \end{bmatrix}; B_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} C_1' = \begin{bmatrix} 0.9 \\ 0.5 \end{bmatrix}; \\ A_2 &= \begin{bmatrix} 1.1 & 0.6 \\ 0.3 & 1.5 \end{bmatrix}; F_2 = \begin{bmatrix} 1.2 \\ 1 \end{bmatrix}; B_2 = \begin{bmatrix} 0.7 \\ 0.5 \end{bmatrix} C_2' = \begin{bmatrix} 1 \\ 0.7 \end{bmatrix}. \end{aligned}$$

- The nonlinearities $\varphi_i(y)$ are defined by, $\forall y \in \mathbb{R}$:

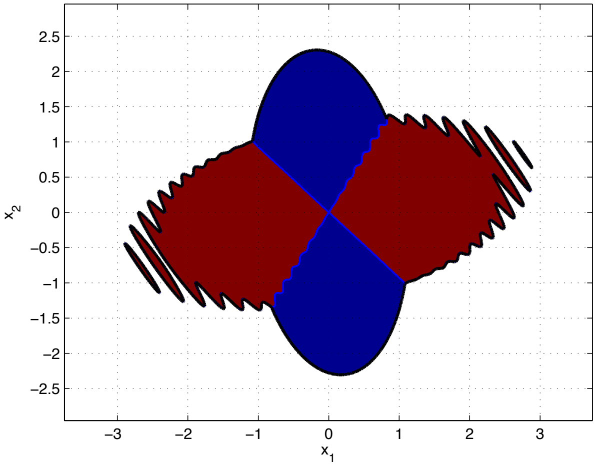
$$\varphi_1(y) = 0.5\Omega_1 y\,(1 + \cos(20y))\,;\,\varphi_2(y) = 0.5\Omega_2 y\,(1 - \sin(25y))\,.$$

- The control gains are given by :

$$\begin{aligned} K_1 &= \begin{bmatrix} -0.7168 & -1.0136 \end{bmatrix};\ \Gamma_1 = -1.2923; \\ K_2 &= \begin{bmatrix} -1.2581 & -0.7326 \end{bmatrix};\ \Gamma_2 = -1.4650; \end{aligned}$$

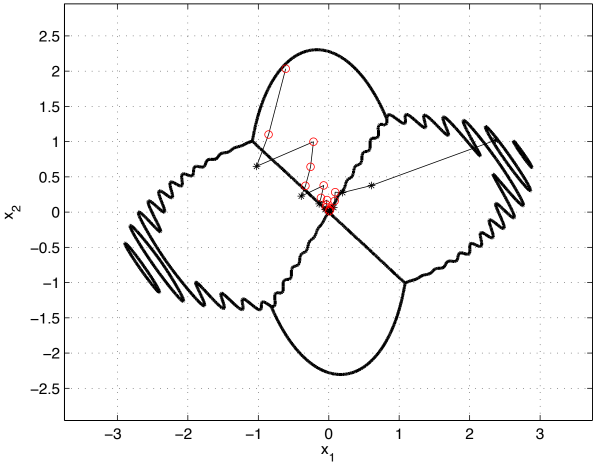
Illustrations

State-space partition inside $L_{V_{\min}}(1)$
mode 1 is the blue region and mode 2 is the red region.



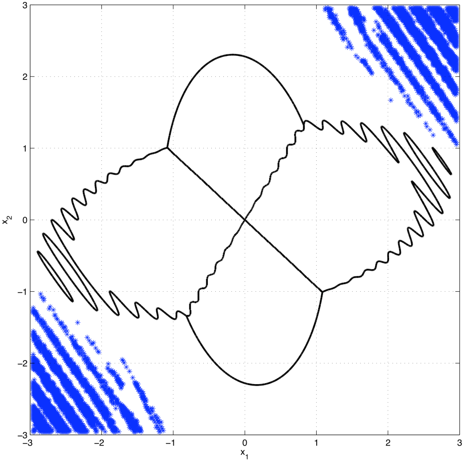
Illustrations

2 trajectories, one from x_0 settled in the disconnected $L_{V_{\min}}(1)$.
Red circle (resp. a black star) means the mode 1 is active (resp. mode 2).



Illustrations

Mapped x_0 leading to unstable trajectories.



Our estimate is adapted to the shape of $\mathcal{B}_0$.

Outline of the talk

Université de Lorraine, Nancy, CRAN Laboratory

Lur'e systems

Introduction of a new Lyapunov-Lur'e type function

Extension to switched Lur'e systems

About consistency

- Reminder of the consistency for switched linear systems
- What about consistency for switched Lur'e systems

Application to sampled-data Lur'e systems with nonuniform sampling

Conclusion

Closed-loop performance for linear switched systems

Let us consider here the following switched linear systems

$$x_{k+1} = A_{\sigma(k)} x_k, \quad \mathcal{J}_\sigma(x_0) = \sum_{k \in \mathbb{N}} x_k' Q_{\sigma(k)} x_k. \tag{42}$$

Theorem

If there exist matrices $P_i > 0$, $\forall i \in \mathcal{I}_N$ and $\Pi \in \mathcal{M}$ solution of the optimization problem

$$\min_{P_i, \Pi} \left(\min_{i \in \mathcal{I}_N} trace(P_i) \right), \tag{43}$$

subject to

$$A_i' \left(\sum_{\ell \in \mathcal{I}_N} \pi_{\ell i} P_\ell \right) A_i - P_i + Q_i < 0, \quad \forall i \in \mathcal{I}_N \tag{44}$$

then the state feedback switching strategy $\sigma(k) = \arg \min_{i \in \mathcal{I}_N} x_k' P_i x_k$, called min-switching strategy, ensures that the origin x = 0 is globally asymptotically stable and

$$\mathcal{J}_\sigma(x_0) \leq \min_{i \in \mathcal{I}_N} x_0' P_i x_0 = V_{\min}(x_0). \tag{45}$$

Consistency for switched linear systems

Definition

Consistent switching law for linear switched systems [14] Consider the class of switched discrete-time linear systems, where $\sigma : \mathbb{N} \to \mathcal{I}_N$ is the switching law. A particular switching strategy $\sigma_s(\cdot)$ is consistent, with respect to the performance $\mathcal{J}_\sigma(\cdot)$, if it improves the performance when compared to the performances of each isolated subsystem supposed to be asymptotically stable.

$$\mathcal{J}_{\sigma_s}(x_0) \leq \min_{i \in \mathcal{I}_N} \mathcal{J}_{\sigma=i}(x_0). \tag{46}$$

Theorem

The min-switching strategy $\sigma_s(k) = \arg\min_{i \in \mathcal{I}_N} x_k' P_i x_k$, where P_i are solution of Optimization Problem (43) is consistent.

Idea of the proof : The inequality $A_i' P_i A_i - P_i + Q_i < 0$ is a particular case of the constraints (44).

14. J.C. Geromel, G.S. Deaecto et J. Daafouz. "Suboptimal Switching State Feedback Control Consistency Analysis for Switched Linear Systems". In : *18th IFAC World Congress*. 2011, p. 5849–5854.

Closed-loop performance for switched Lur'e systems

Theorem

If there exist matrices $P_i > 0$, $\forall i \in \mathcal{I}_N$ and $\Pi \in \mathcal{M}$ solution of the optimization problem, with $(P)_{p,i} = \sum_{\ell \in \mathcal{I}_N} \pi_{\ell i} P_\ell$,

$$\min_{P_i,\Pi}\Big(\min_{i\in\mathcal{I}_N} trace(P_i)\Big), \tag{47}$$

subject to

$$\begin{bmatrix} A_i'(P)_{p,i}A_i - P_i + Q_i & \star \\ B_i'(P)_{p,i}A_i + S_i\Omega_i C_i & B_i'(P)_{p,i}B_i - 2S_i \end{bmatrix} < 0, \tag{48}$$

then the state feedback switching strategy $\sigma(k) = arg \min_{i\in\mathcal{I}_N} x_k' P_i x_k$ ensures that the origin x = 0 is globally asymptotically stable and

$$\mathcal{J}_\sigma(x_0) \leq \min_{i\in\mathcal{I}_N} x_0' P_i x_0 = V_{\min}(x_0). \tag{49}$$

$$\mathcal{J}_{\sigma_s}(x_0) \leq V_{\min}(x_0) \overset{?}{\leq} \min_{i\in\mathcal{I}_2} \mathcal{J}_{\sigma=i}(x_0).$$

The answer is NO ! This is due to the dependency of $\mathcal{J}_\sigma(x_0)$ with respect to the nonlinearity $\varphi_\sigma(\cdot)$ [15]

15. J. Louis, M. Jungers et J. Daafouz. "Switching control consistency of switched Lur'e systems with application to digital control design with non uniform sampling". In : *14th annual European Control Conference, ECC 2015*. Linz, Austria, 2015, p. 1748–1753.

Extension of Consistency concept

Definition

Consider switched Lur'e systems, a particular switching strategy $\sigma_s(\cdot)$ is consistent, with respect to the performance $\mathcal{J}_{\sigma_s}$, if it improves the upper bound of the performance when compared to the upper bounds of performances of each isolated subsystem.

$$\mathcal{J}_{\sigma_s}(x_0) \leq V_{\min}(x_0) \leq \min_{i \in \mathcal{I}_N} \overline{\mathcal{J}_{\sigma=i}}(x_0), \tag{50}$$

Theorem

The min-switching strategy $\sigma(k) = \arg\min_{i \in \mathcal{I}_N} x_k' P_i x_k$*, given by last theorem is consistent according this revised definition.*

See [16]

16. J. Louis, M. Jungers et J. Daafouz. "Switching control consistency of switched Lur'e systems with application to digital control design with non uniform sampling". In : *14th annual European Control Conference, ECC 2015*. Linz, Austria, 2015, p. 1748–1753.

Illustration : consistency for switched Lur'e systems

Consider a switched Lur'e system defined by

$$A_1 = \begin{bmatrix} 0.9 & 0 \\ 0.4 & -0.72 \end{bmatrix}, \; A_2 = \begin{bmatrix} -0.58 & -0.8 \\ 0 & -0.8 \end{bmatrix}, \; B_1 = -\begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.6 & 0.24 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0.4 & 1.1 \end{bmatrix}, \varphi_1(y_k) = \frac{\Omega_1 y_k}{2}(1+\cos(2y_k)),$$

$$\varphi_2(y_k) = \frac{\Omega_2 y_k}{2}(1-\sin(5.5y_k)), \; \Omega_1 = 0.6, \; \Omega_2 = 1.2, \; x_0 = \begin{pmatrix} -4 \\ 5 \end{pmatrix}.$$

$Q_i = q_i I_n$ with $i \in \mathcal{I}_2$

q_1	q_2	$\mathcal{J}_{\sigma_s}$	$V_{\min}(x_0)$	$\overline{\mathcal{J}_1}$	$\overline{\mathcal{J}_2}$	$\mathcal{J}_1$	$\mathcal{J}_2$
1	1	52	96	175	231	121	59
4	1	76	168	782	231	484	59
1	4	121	175	175	927	121	238

Outline of the talk

Sampled-data Lur'e system with nonuniform sampling

Sampled-data Lur'e system :

$$\mathcal{S}_c : \begin{cases} \dot{x}(t) &= Ax(t) + B\varphi(y(t)) + F\tilde{u}(t), \quad t \in \mathbb{R}^+, \\ y(t) &= Cx(t), \\ \tilde{u}(t) &= u(t_k) = K_{t_k}x(t_k) + \Gamma_{t_k}\varphi(y(t_k)), \quad [t_k; t_{k+1}[\,, \end{cases} \tag{51}$$

where

- $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ the output $\tilde{u}(t) \in \mathbb{R}^r$ the control input.
- $\varphi(\cdot)$ is a nonlinearity verifying the cone bounded sector condition

$$\varphi(0) = 0; \qquad \varphi(y)'\Lambda(\varphi(y) - \Omega y) \leq 0. \tag{52}$$

  with $\Lambda \in \mathbb{R}^{p\times p}$ any diagonal positive definite.
- The sampling times $\{t_k\}_{k\in\mathbb{N}}$ verify

$$t_{k+1} - t_k \in \{T_i\}_{i\in\{1;\cdots;N\}}\,, \quad \forall k \in \mathbb{N}. \tag{53}$$

Issue 1 : Design jointly a control law $\tilde{u}(t)$ and a sequence of (nonuniform) sampling periods, ensuring that the origin $x = 0$ is globally asymptotically stable.

Remark : uniform sampling consists in assuming $\{T_i\}_{i\in\{1;\cdots;N\}} = \{T_1\}$.

Stability of a samped-data system with nonuniform sampling

Theorem

Consider $\mathcal{S}_c$ with a finite family of sampling period $\{T_i\}_{i\in\{1;\cdots;N\}}$, and a given control law

- *(A1) If there exits a function $\beta \in \mathcal{KL}$ such that $\forall k \geq k_0 \geq 0$,*

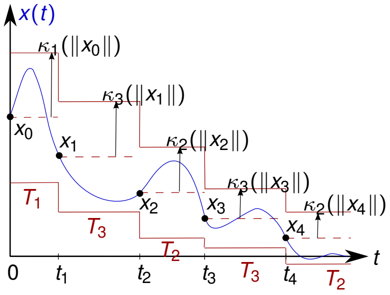
$$\|x_k\| \leq \beta\left(\|x_{k_0}\|, k - k_0\right),$$

- *(A2) If there exist N $\kappa_i \in \mathcal{K}_\infty$, satisfying $\forall i \in \{1; \cdots; N\}$, $\forall t \in [t_{init}; t_{init} + T_i]$,*

$$\|x(t)\| \leq \kappa_i\left(\|x(t_{init})\|\right),$$

then the sampled-data system $\mathcal{S}_c$ is globally uniformly asympotically stable and there exists $\overline{\beta} \in \mathcal{KL}$, such that $\forall t \geq t_{init} \geq 0$

$$\|x(t)\| \leq \overline{\beta}\left(\|x(t_{init})\|, t - t_{init}\right).$$



17. D. S. Laila, D. Nešić et A. Astolfi. "Advanced topics in control systems theory II". In : sous la dir. d'A. Loria, F. Lamnabhi-Lagarrigue et E. Panteley. T. 328. Lecture notes from FAP 2006. Springer, 2005. Chap. Sampled-Data Control of Nonlinear Systems, p. 91–137.

18. J. Louis, M. Jungers et J. Daafouz. "Stabilization of sampled-data Lur'e systems with nonuniform sampling". In : *proceedings of the 54th IEEE Conference on Decision and Control*. Osaka, Japan, 2015, p. 2881–2886.

First consequence and reformulation of Problem 1

Guideline :

- (A2) is always satisfied for Lur'e systems here.
- Problem 1 reduces to verify (A1).
  $\Rightarrow$ introduction of the exact discretized system

$$F^{e}_{T_i}(x_k) = x_k + \int_{t_k}^{t_k+T_i} \Big(Ax(\tau) + B\varphi(y(\tau)) + F\tilde{u}(t_k) \Big) \,\mathrm{d}\tau, \quad \forall k \in \mathbb{N}. \qquad (54)$$

Reformulation 1 of Problem 1 : Determine jointly a control law and a switching law stabilizing the nonlinear switching system :

$$x_{k+1} = F^{e}_{T_{\sigma(k)}}(x_k), \quad k \in \mathbb{N}, \qquad (55)$$

where the switching law $\sigma : \mathbb{N} \to \{1; \cdots ; N\}$ select the active sampling period in $\{T_i\}_{i\in\{1;\cdots;N\}}$.

Further discussion

Among all the solutions, it may be interesting to add to Problem 1 a criterion and to consider an optimization problem.
Performance criterion : Degree of freedom to select the nonuniform sampling time

$$\mathcal{J}_\sigma(x_0) = \sum_{k \in \mathbb{N}} x_k' Q_{\sigma(k)} x_k. \tag{56}$$

For instance, $Q_i \# \frac{1}{T_i}, \forall i \in \{1, \cdots, N\}$.

Difficulty : due to the presence of the non-linearity $\varphi(\cdot)$:

- It is not possible to obtain an analytical value of the function $F^{e}_{T_i}(\cdot)$;
- $F^{e}_{T_i}(\cdot)$ is not of Lur'e type structure.

Question :

How to handle (easily) the function $F^{e}_{T_i}(\cdot)$?

Reformulation of the issue :

Reformulation 2 of Problem 1 : Design jointly the switching gains (K_i, Γ_i) and the switching law ensuring that the discrete-time Lur'e system with norm bounded uncertainties written as $\exists \Delta_{1,i}, \Delta_{2,i}$, such that

$$\begin{cases} x_{k+1} &= \left(A^{\mathrm{d}}_{\sigma(k)} + \Delta_{2,\sigma(k)}\right) x_k + B^{\mathrm{d}}_{\sigma(k)} \varphi(Cx_k) + \left(I_n + \Delta_{1,\sigma(k)}\right) F^{\mathrm{d}}_{\sigma(k)} u_k, \\ \Delta'_{1,i}\Delta_{1,i} &\leq r_1(T_i)^2 I_n, \\ \Delta'_{2,i}\Delta_{2,i} &\leq r_2(T_i)^2 I_n, \\ u_k &= K_{\sigma(k)} x_k + \Gamma_{\sigma(k)} \varphi(Cx_k) \end{cases}$$

is globally asymptotically stable and that minimize the cost $\mathcal{J}_\sigma(\cdot)$.
Solution given by the optimization problem

$$\min\big(\min_{i \in \{1;\cdots;N\}} -\mathrm{trace}(P_i^{-1})\big), \tag{57}$$

under LMI constraints provided in [19]. Then the switching law $\sigma(k) = \mathrm{argmin}\,(x_k' P_i x_k)$, leads to

$$\mathcal{J}_{\sigma(k)}(x_0) \leq \overline{\mathcal{J}}(x_0) = \min_{i \in \{1;\cdots;N\}} \left(x_0' P_i x_0\right); \tag{58}$$

and is consistent to the quadratic upper bound taking into account all the nonlinearities and all the uncertainties.

19. J. Louis, M. Jungers et J. Daafouz. "Stabilization of sampled-data Lur'e systems with nonuniform sampling". In : *proceedings of the 54th IEEE Conference on Decision and Control.* Osaka, Japan, 2015, p. 2881–2886.

Numerical example

Let

$$A = \begin{bmatrix} 0 & 1,6 \\ -0,8 & -0,1 \end{bmatrix},\ B = \begin{bmatrix} 0,25 \\ 0,25 \end{bmatrix},\ F = \begin{bmatrix} 0 \\ 0,20 \end{bmatrix},\ C = \begin{bmatrix} 0,1 & -0,15 \end{bmatrix},$$

$$\varphi(y[k]) = \frac{\Omega_c y[k]}{2}(1 + \cos(6y[k] + 0,1y^2[k])),$$

$$\Omega = \frac{\sqrt{2}}{2},\ x_0 = \begin{pmatrix} 6 \\ 3 \end{pmatrix},\ T_1 = 0,1\ T_2 = 0,3,$$

and $R_1 = 3$, $R_2 = 1$, $Q_1 = 3I_2$ et $Q_2 = I_2$.

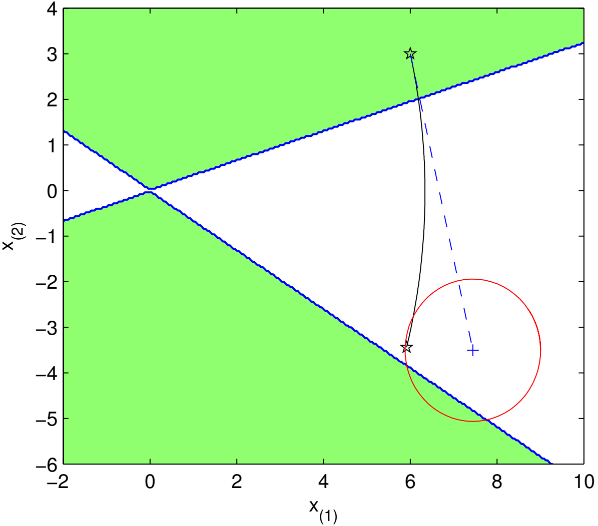
Then the optimization problem leads to

$$P_1 = \begin{bmatrix} 358,42 & 280,49 \\ 280,49 & 671,26 \end{bmatrix},\ P_2 = \begin{bmatrix} 383,91 & 260,67 \\ 260,67 & 548,82 \end{bmatrix},\ \gamma_1 = 0,3,\ \gamma_2 = 0,$$

$$K_1 = \begin{bmatrix} -1,46 & -4,04 \end{bmatrix}, K_2 = \begin{bmatrix} -4,50 & -18,57 \end{bmatrix},\ \Gamma_1 = -0,14,\ \Gamma_2 = -1,74.$$

Numerical example : the trajectory

State partition for the choice of the sampling period



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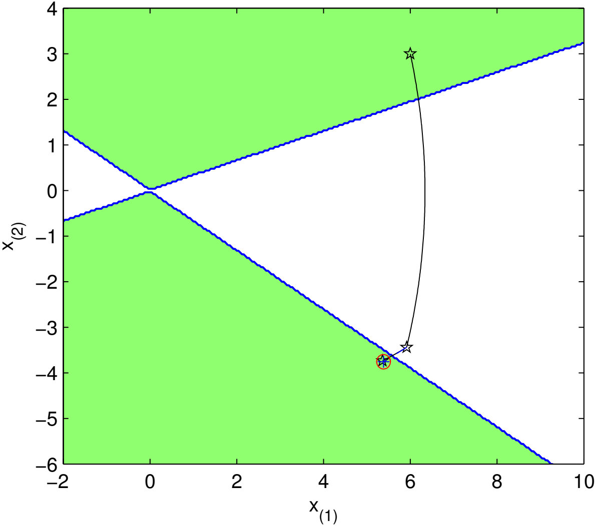

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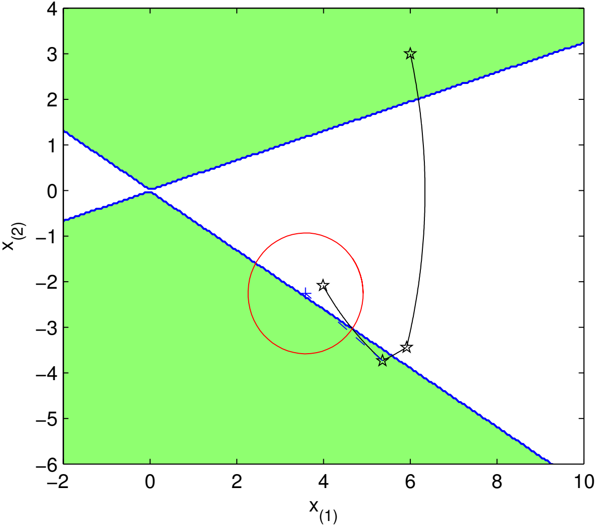

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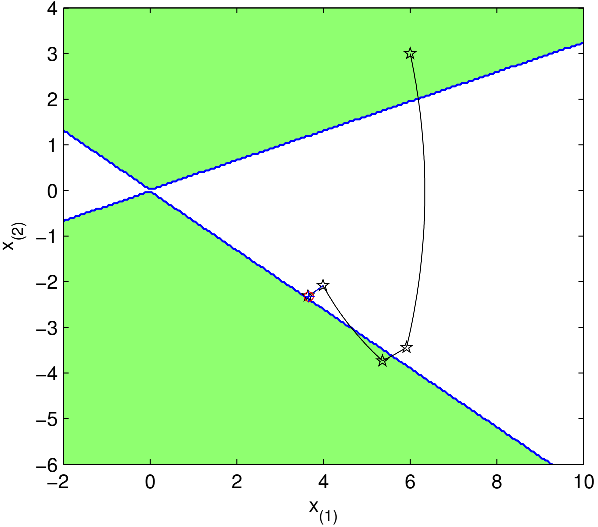

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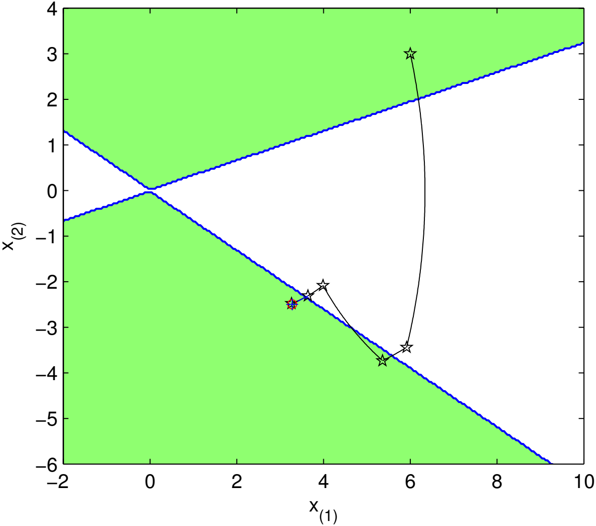

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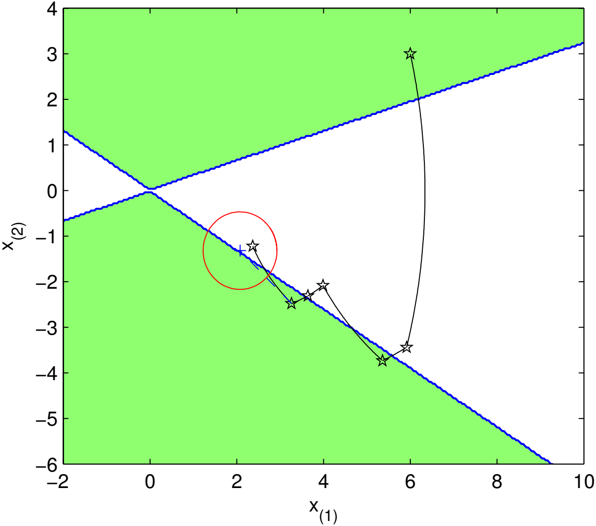
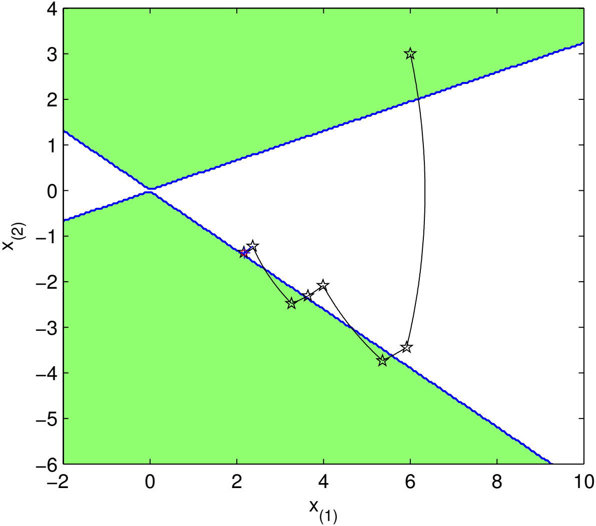


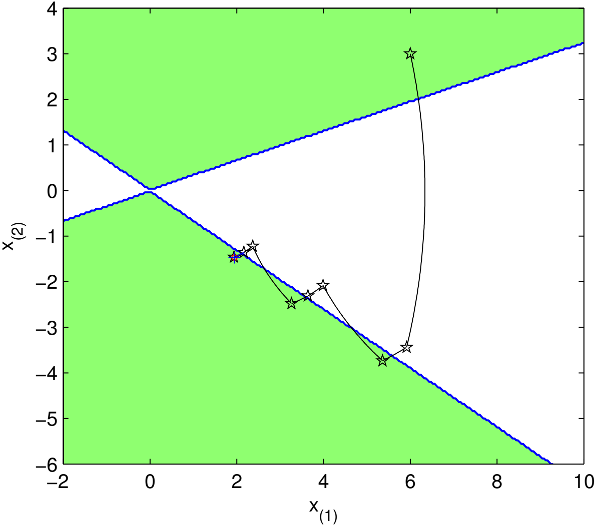
Numerical example : the trajectory

State partition for the choice of the sampling period



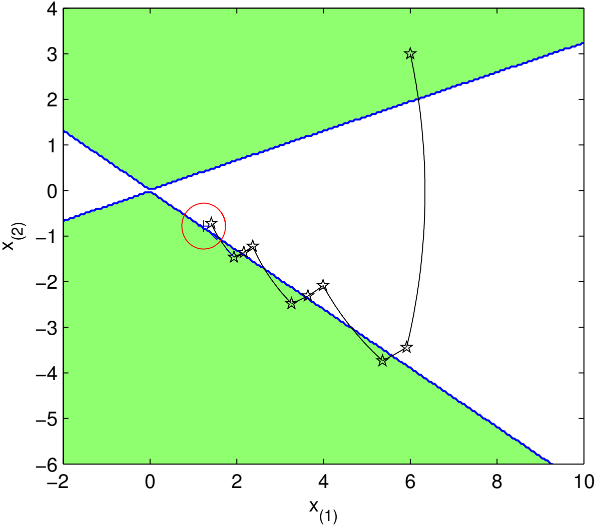
Numerical example : the trajectory

State partition for the choice of the sampling period



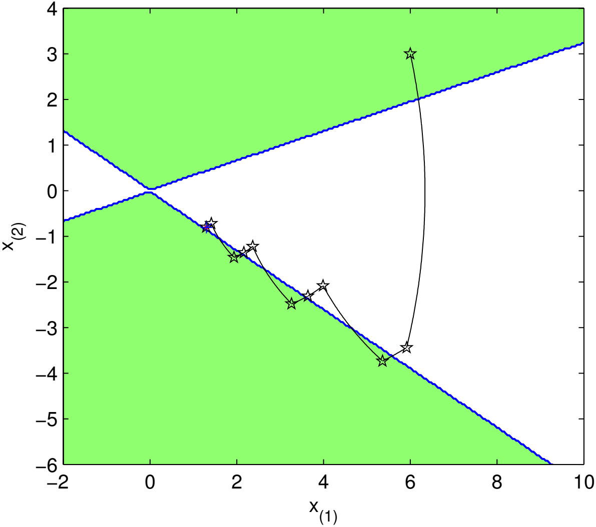
Numerical example : the trajectory

State partition for the choice of the sampling period



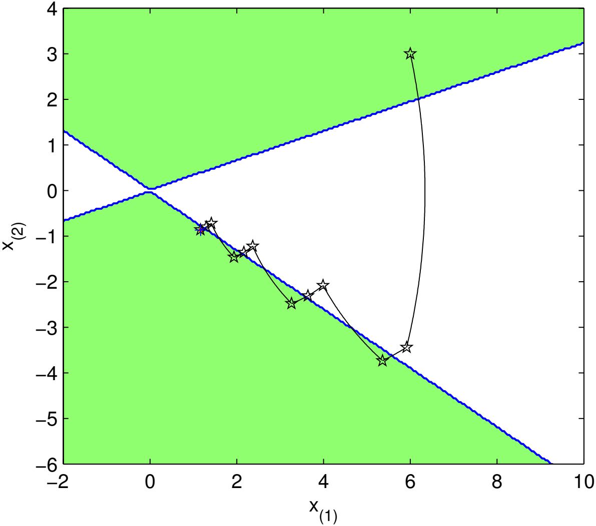
Numerical example : the trajectory

State partition for the choice of the sampling period



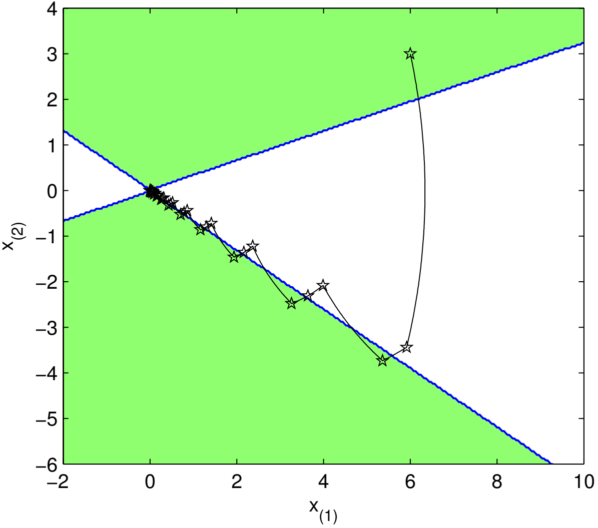
Numerical example : the trajectory

State partition for the choice of the sampling period



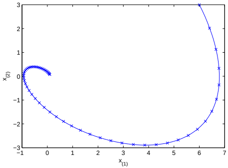
Numerical example : the trajectory

State partition for the choice of the sampling period

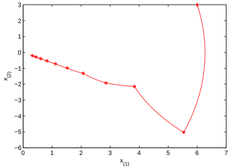


Numerical example : the performance

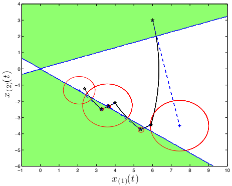
Uniform sampling T_1,
$\mathcal{J}_1(x_0) = 13247$



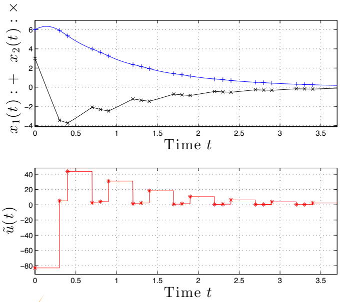
Uniform sampling T_2,
$\mathcal{J}_2(x_0) = 17363$



Nonuniform sampling,
$\mathcal{J}_\sigma(x_0) = 10895$



Nonuniform sampling, $\mathcal{J}_\sigma(x_0) = 10895$



Improvement

$$\frac{\mathcal{J}_1(x_0) - \mathcal{J}_\sigma(x_0)}{\mathcal{J}_1(x_0)} = 17,8\%.$$

Conclusion

Discrete-time Lur'e system have been studied :

- A new discrete-time Lyapunov-Lur'e function suitable has been provided ;
- Global stability analysis and Global stabilization ;
- Local stability analysis and local stabilization ;
- Revision of the notion of consistency taking into account all the nonlinearities ;
- Application to sampled-data Lur'e systems.

Thank you very much for your attention !

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