Design of Observers for Takagi-Sugeno Systems with unmeasurable Premise Variables: an $\mathscr{L}_2$ Approach

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17th IFAC World Congress
July 6-11, 2008, Seoul, Korea

Outline

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1. Multiple model approach
   - Basics on multiple model approach
   - Objective and motivations

2. State estimation
   - Observer structure

3. Simulation example

4. Conclusions and future works

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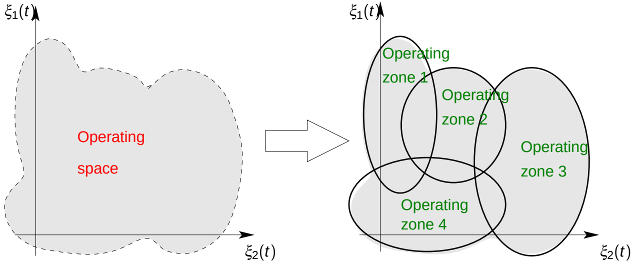
Multiple model Approach

- Decomposition of the operating space into operating zones
- Modelling each zone by a single submodel
- The contribution of each submodel is quantified by a weighting function



Nonlinear system | Multiple model representation

Multiple model = an association of a set of submodels blended by an interpolation mechanism

Basics on multiple model approach

Interest of multiple models

- Intuitive and simple way to represent a complex system.
- Any nonlinear system can be approximated with a given precision by a multiple model.
- Some of the results obtained for linear systems can be generalized to nonlinear systems.

Definition. Takagi-Sugeno multiple model

$$\begin{cases} \dot{x}(t) & = & \{\sum_{i=1}^{r} \mu_i(\xi(t))A_i\}\, x(t) + \{\sum_{i=1}^{r} \mu_i(\xi(t))B_i\}\, u(t)\ , \\ y(t) & = & \{\sum_{i=1}^{r} \mu_i(\xi(t))C_i\}\, x(t)\ , \end{cases}$$

$$\sum_{i=1}^{L} \mu_i(\xi(t)) = 1 \text{ and } 0 \le \mu_i(\xi(t)) \le 1, \forall t, \forall i \in \{1, ..., r\}$$

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- The contribution of each sub-model is quantified by $\mu_i(\xi(t))$.
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Objective of the work

- $$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases}$$

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Interest

- Exact representation of nonlinear models : $\dot{x}(t) = f(x, u)$
- Diagnosis of sensor and actuator faults (observer banks) using the same multiple model compared to the case where $\xi(t)$ is measurable (u or y).

State estimation

Proportional Observer

$$\begin{cases} \dot{\hat{x}} = A_0\hat{x} + \sum_{i=1}^{r} \mu_i(\hat{x}) \left(\overline{A}_i\hat{x} + B_iu + G_i(y - \hat{y})\right) \\ \hat{y} = C\hat{x} \end{cases}$$

with

$$A_0 = \frac{1}{r}\sum_{i=1}^{r} A_i \quad \text{and} \quad \overline{A}_i = A_i - A_0$$

- $\hat{x}(t)$ denotes the estimation of the state variable.
- The gains G_i must be determined in order that the state estimation error asymptotically converges to 0.

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Observer design

- The evolution of the state estimation error $e(t)$ is described by

$$\dot{e}(t) = \sum_{i=1}^{r} \mu_i(\hat{x})(A_0 - G_i C)e(t) + \bar{A}_i \delta_i(t) + B_i \Delta_i(t)$$

with

$$\delta_i(t) = \mu_i(x)x - \mu_i(\hat{x})\hat{x} \quad \text{and} \quad \Delta_i = (\mu_i(x) - \mu_i(\hat{x}))u$$

- It is well known that a system is asymptotically stable if there exists a Lyapunov function $V(e,t)$ verifying

$$\dot{V}(e,t) < 0$$

- The convergence of the estimation error is studied by defining the following Lyapunov function

$$V(e,t) = e(t)^T P e(t), \quad \text{with} \quad P = P^T > 0$$

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  - **A1.** The weighting functions $\mu_i(x)$ are Lipschitz:
    $$|\mu_i(x) - \mu_i(\hat{x})| < N_i\,|x - \hat{x}|$$
  - **A2.** The functions $\mu_i(x)x$ are Lipschitz:
    $$|\mu_i(x)x - \mu_i(\hat{x})\hat{x}| < M_i\,|x - \hat{x}|$$
  - **A3.** The input $u(t)$ of the system is bounded:
    $$|u(t)| \leq \beta_1$$
- and with some basic linear algebra, the following sufficient conditions for
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1st observer design

theorem

The state estimation error is asymptotically stable if there exists matrices $P = P^T > 0$, $Q = Q^T > 0$, K_i and positive scalars γ, λ_1 et λ_2 such that:

$$A_0^T P + PA_0 - C^T K_i^T - K_i C < -Q$$

$$\begin{bmatrix} -Q + \lambda_1 M_i^2 I & P\overline{A}_i & PB_i & N_i \gamma I \\ \overline{A}_i^T P & -\lambda_1 I & 0 & 0 \\ B_i^T P & 0 & -\lambda_2 I & 0 \\ N_i \gamma I & 0 & 0 & -\lambda_2 I \end{bmatrix} < 0$$

$$\gamma - \beta_1 \lambda_2 \geq 0$$

The gains G_i are derived from $G_i = P^{-1} K_i$.

Remark

The weighting functions must be Lipschitz.

Goal

Synthesize an observer while relaxing the assumption **A1**, **A2** and **A3**.

- The evolution of $e(t)$ can be written as

$$\dot{e}(t) = \sum_{i=1}^{r} \mu_i(\hat{x})(A_0 - G_i C)e(t) + H_i \omega(t)$$

where the signals $\delta_i(t)$ and $\Delta_i(t)$ are considered as a disturbance $\omega(t)$

$$\omega_i^T(t) = \begin{bmatrix} \delta_i^T(t) & \Delta_i(t)u^T(t) \end{bmatrix} \quad \text{and} \quad \omega^T(t) = \begin{bmatrix} \omega_1^T(t) & \ldots & \omega_r^T(t) \end{bmatrix}$$

- It is well known that the $\mathscr{L}_2$-gain from $\omega(t)$ to $e(t)$ is bounded if there exists a Lyapunov function $V(e,t)$ verifying

$$\dot{V}(e,t) + e^T(t)e(t) - \gamma^2 \omega^T(t)\omega(t) < 0$$

- Defining the following Lyapunov function $V(e,t) = e(t)^T Pe(t)$ with $P = P^T > 0$ and after some manipulations the following conditions are obtained . . .

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Generalization of the method

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Theorem 2

The optimal proportional observer for the system is obtained by minimizing $\tilde{\gamma} > 0$ under the constraints

$$P = P^T > 0 \begin{bmatrix} \frac{S_i}{N} & PH_j \\ H_j^T P & -\frac{\tilde{\gamma}}{N} \end{bmatrix} < 0, \quad \forall i, j = 1, \ldots, N$$

where

$$S_i = A_0^T P + PA_0 - K_i C - C^T K_i^T + I$$

The observer gains are given by $G_i = P^{-1} K_i$ and the $\mathscr{L}_2$-gain from $\omega(t)$ to $e(t)$ is $\gamma = \sqrt{\tilde{\gamma}}$.

Eigenvalue assignment

- Pole-clustering is used to improve the temporal response of $e(t)$
- The gains G_i are determined in order that the poles of the system generating $e(t)$ should lie in $S(\alpha,\beta)$, defined by

$$S(\alpha,\beta) = \{z \in \mathbb{C} | Re(z) < -\alpha,\ |z| < \beta\}$$

- The eigenvalues of M lie in $S(\alpha,\beta)$ if $\exists P = P^T > 0$ such that

$$\begin{bmatrix} \beta P & PM \\ (PM)^T & \beta P \end{bmatrix} > 0$$
$$M^T P + PM + 2\alpha P < 0$$

- The previous result is slightly modified in order to add the eigenvalue assignment constraints

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Theorem 3

The optimal observer for the multiple model, satisfying the pole clustering in $S(\alpha,\beta)$, is obtained by minimizing $\tilde{\gamma} > 0$ under the following constraints:

$$P = P^T > 0$$

$$\begin{bmatrix} \beta P & P(A_0 - G_i C) \\ (A_0 - G_i C)^T P & \beta P \end{bmatrix} > 0$$

$$A_0^T P + P A_0 - C^T K_i^T - K_i C + 2\alpha P < 0$$

$$\begin{bmatrix} \frac{S_i}{N} & P H_j \\ H_j^T P & -\frac{\tilde{\gamma}}{N} \end{bmatrix} < 0, \quad \forall i,j = 1,\ldots,N$$

where:

$$S_i = P A_0 + A_0^T P - K_i C - C^T K_i + I$$

The observer gains are given by $G_i = P^{-1} K_i$, and the $\mathscr{L}_2$-gain is $\gamma = \sqrt{\tilde{\gamma}}$.

Example

Simulation example

- Let us consider the system defined by:

$$A_1 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 1 & -6 \end{pmatrix} \quad A_2 = \begin{pmatrix} -3 & 2 & -2 \\ 5 & -3 & 0 \\ 0.5 & 0.5 & -4 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 \\ 0.5 \\ 0.5 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0.5 \\ 1 \\ 0.25 \end{pmatrix} \quad C = B_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

- The weighting functions are

$$\begin{cases} \mu_1(x) = \frac{1-\tanh(x_1)}{2} \\ \mu_2(x) = 1 - \mu_1(x) = \frac{1+\tanh(x_1)}{2} \end{cases}$$

- The eigenvalues are clustered in the region $S(\alpha, \beta)$ defined by $\beta = 15$ and $\alpha = 5$

- After solving the optimization problem in theorem 2, we obtain :

$$P = \begin{pmatrix} 0.10 & 0.04 & 0.12 \\ 0.04 & 0.18 & 0.15 \\ 0.12 & 0.15 & 0.40 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} 9.04 & 5.08 \\ 10.24 & -7.58 \\ -5.60 & 1.63 \end{pmatrix} \quad G_2 = \begin{pmatrix} 8.41 & 5.68 \\ 10.87 & -8.06 \\ -5.30 & 0.73 \end{pmatrix}$$

- The minimal value of the attenuation of the perturbation terms is

$$\gamma = 0.46$$

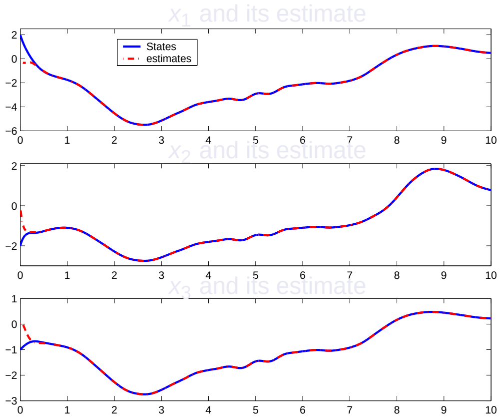

Figure: State estimation

Conclusions and perspectives

Conclusions

- State estimation of nonlinear systems modeled by a multiple model is achieved with a P observer
- The decision variable is assumed to be not measurable (useful in the framework of system diagnosis).
- Sufficient conditions for asymptotic convergence of the state estimation error are proposed in LMI formulation.
- The method is generalized for all types of weighting functions, and the performances of the observer are improved by eigenvalues assignment.

Perspectives

- Application to the diagnosis of complex systems.
- Reduction of the conservatism of the conditions by using other types of Lyapunov functions (Polytopic functions, etc.).

Thank you for your attention!