Design of observers for Takagi-Sugeno Discrete-time systems with unmeasurable premise variables

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Outline

1. Introduction
   - Multiple Model Structure
   - Problem statement and objective
2. State estimation
3. Extension of the method: $\mathscr{L}_2$ approach
4. Simulation results
   - Comparison between the two methods
   - Example
5. Conclusion and future works

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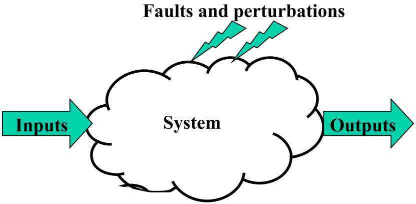
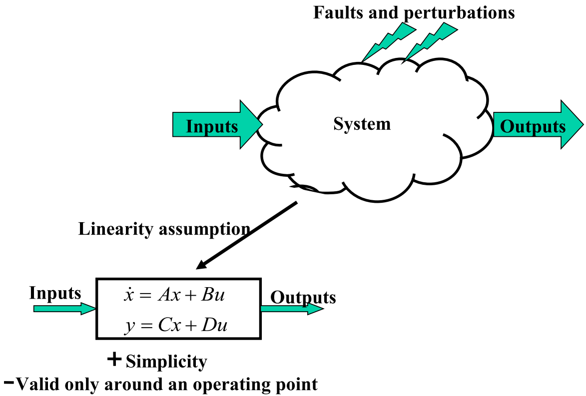
Introduction


Faults and perturbations
Inputs
System
Outputs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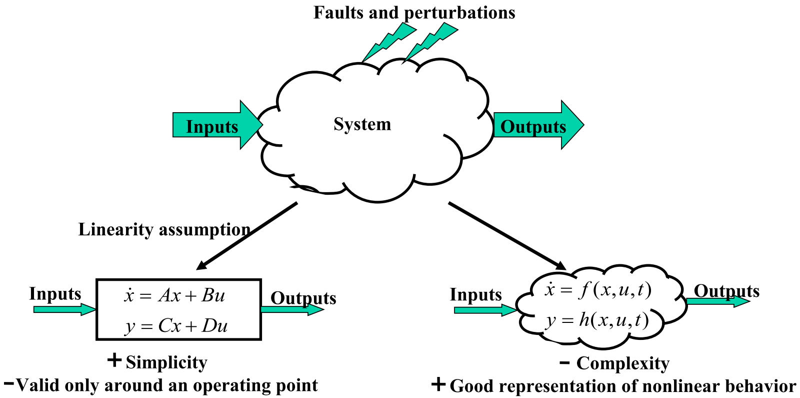
$\dot{x} = Ax + Bu$
$y = Cx + Du$
Outputs
+ Simplicity
−Valid only around an operating point



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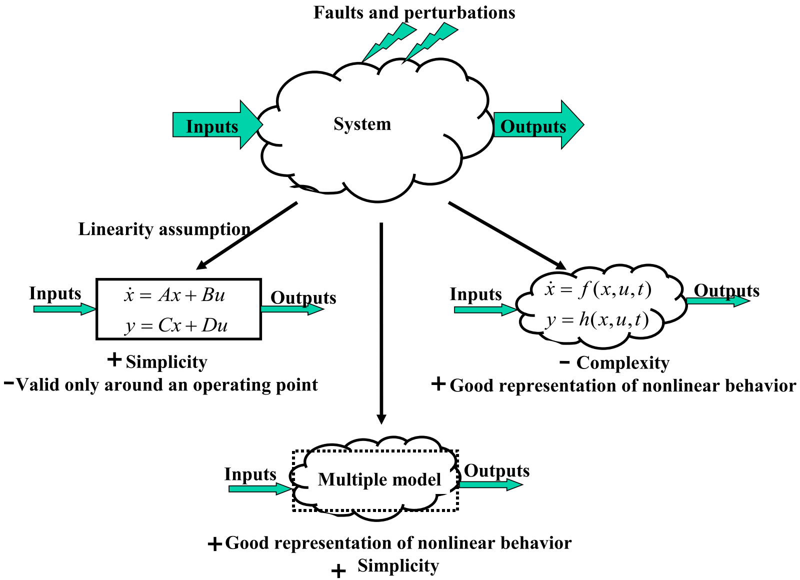
$\dot{x} = f(x,u,t)$
$y = h(x,u,t)$
Outputs
− Complexity
+ Good representation of nonlinear behavior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Multiple model
Outputs
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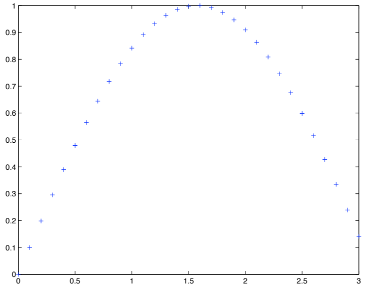
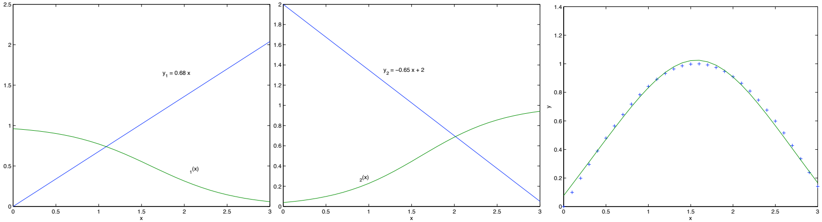
+ Simplicity

Introduction: static case example

$$y = \sin(x)$$



Introduction: static case example



$$\hat{y} = {}_1(x)y_1 + {}_2(x)y_2$$

$$ {}_1(x) + {}_2(x) = 1$$

$$0 \leq {}_i(x) \leq 1$$

$$\begin{cases} x(k+1) = \sum_{i=1}^{r} {}_i(\xi(k))\left(A_i x(k) + B_i u(k)\right) \\ y(k) = \sum_{i=1}^{r} {}_i(\xi(k))\left(C_i x(k) + D_i u(k)\right) \end{cases}$$

${}_i(\xi(k))$: weighting functions.
$\xi(k)$: decision variable.

Multiple Model Structure

$$\begin{cases} x(k+1) = \sum_{i=1}^{r} {}_i(\xi(k))(A_i x(k) + B_i u(k)) \\ y(k) = \sum_{i=1}^{r} {}_i(\xi(k))(C_i x(k) + D_i u(k)) \end{cases}$$

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$$0 \leq {}_i(\xi(k)) \leq 1, \forall k, i = 1, ..., r$$

$$\sum_{i=1}^{r} {}_i(\xi(k)) = 1, \forall k$$

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Advantages of Multiple Models

- Universal approximator.
- Simplicity because this structure is inspired by the linear systems.
- Ability to generalize the tools developed for linear systems to nonlinear systems.

Problem statement and objective

Observer Design

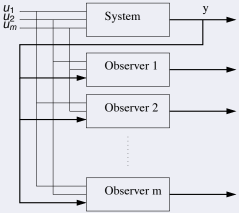
$$\begin{cases} \hat{x}(k+1) = \sum_{i=1}^{r} {}_i(\xi(k))(A_i\hat{x}(k) + B_i u(k) + G_i(y(k) - \hat{y}(k))) \\ \hat{y}(k) = \sum_{i=1}^{r} {}_i(\xi(k))(C_i\hat{x}(k) + D_i u(k)) \end{cases}$$

Objective of the study

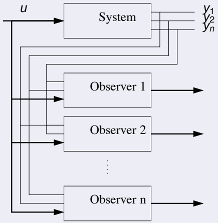
Synthesize observers in order compare the real behavior of the system to its healthy behavior.

$$r(k) = y(k) - \hat{y}(k)$$

actuator faults: $\xi(k) = y(k)$



sensor faults: $\xi(k) = u(k)$



$$x(k+1) = \sum_{i=1}^{r} \;{}_i(\xi(k))(A_i x(k) + B_i u(k))$$

$$y(k) = \sum_{i=1}^{r} \;{}_i(\xi(k))(C_i x(k) + D_i u(k))$$

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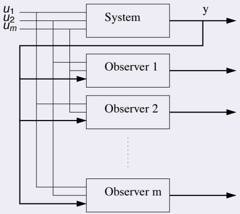

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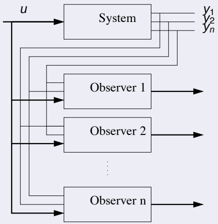

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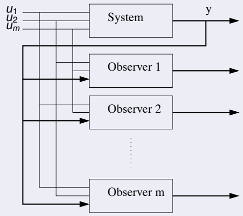

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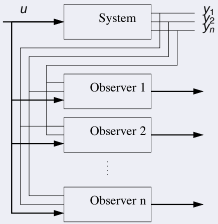

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Problem statement and objective

$$x(k+1) = \sum_{i=1}^{r} {}_i(x(k))\,(A_i x(k) + B_i u(k))$$
$$y(k) = \sum_{i=1}^{r} {}_i(x(k))\,(C_i x(k) + D_i u(k))$$

Advantages of multiple models with unmeasurable premise variables

- Representation of a wider class of nonlinear systems.
- One multiple model is sufficient for the design of observer banks for the detection and isolation of sensors and actuators faults.

Main objective

- Design of nonlinear observers for nonlinear discrete-time systems described under the multiple model form with unmeasurable premise variables i.e. : $\xi(k) = x(k)$.

State estimation

Multiple Model

$$\begin{cases} x(k+1) = \sum_{i=1}^{r} \; {}_i(x(k))(A_i x(k) + B_i u(k)) \\ y(k) = Cx(k) \end{cases}$$

Multiple Observer

$$\begin{cases} \hat{x}(k+1) = \sum_{i=1}^{r} \; {}_i(\hat{x}(k))(A_i \hat{x}(k) + B_i u(k) + G(y(k) - \hat{y}(k))) \\ \hat{y}(k) = C\hat{x}(k) \end{cases}$$

Equivalent form of the multiple model

Variations around a nominal linear model:

$$A_0 = \frac{1}{r}\sum_{i=1}^{r} A_i, \quad A_i = \overline{A}_i + A_0$$

The system is written then:

$$x(k+1) = A_0 x(k) + \sum_{i=1}^{r} \,_i(x(k))(\overline{A}_i x(k) + B_i u(k))$$
$$y(k) = Cx(k) \qquad .$$

Re-writting the model

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Multiple Model

$$\begin{cases} x(k+1) = A_0 x(k) + \sum_{i=1}^{r} \;\; {}_i(x(k))(\overline{A}_i x(k) + B_i u(k)) \\ y(k) = Cx(k) \end{cases}$$

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$$\begin{cases} \hat{x}(k+1) = A_0 \hat{x}(k) + \sum_{i=1}^{r} \;\; {}_i(\hat{x}(k))(\overline{A}_i \hat{x}(k) + B_i u(k) + G(y(k) - \hat{y}(k))) \\ \hat{y}(k) = C\hat{x}(k) \end{cases}$$

State estimation

The state estimation error is given by:

$$e(k) = x(k) - \hat{x}(k)$$

Its dynamic is:

$$e(k+1) = (A_0 - GC)e(k) + \sum_{i=1}^{r} (\overline{A}_i \delta_i(k) + B_i \Delta_i(k))$$

$$\delta_i(k) = \ _i(x(k))x(k) - \ _i(\hat{x}(k))\hat{x}(k)$$

$$\Delta_i(k) = (\ _i(x(k)) - \ _i(\hat{x}(k)))u(k)$$

Assumptions

$_i(x)$ is lipschitz $\Rightarrow | \ _i(x(k)) - \ _i(\hat{x}(k))| \leq N_i \, |x(k) - \hat{x}(k)|$.
$_i(x)x$ is lipschitz $\Rightarrow | \ _i(x(k))x(k) - \ _i(\hat{x}(k))\hat{x}(k)| \leq \gamma_{1i} \, |x(k) - \hat{x}(k)|$.
bounded input $\Rightarrow \|u(k)\| \leq \beta_2 = \frac{\gamma_{2i}}{N_i}$.

Second method of Lyapunov

Consider the quadratic candidate Lyapunov function $V(e(k))$:

$$V(e(k)) = e(k)^T P e(k).$$

The convergence of state estimation error to zero is assured if:

1. $V(e(k)) > 0, \forall k$
2. $\Delta V(e(k)) < 0, \forall k$

Theorem 1 : Asymptotic Convergence

The state estimation error converges globally asymptotically toward zero, if there exists a matrix $P = P^T > 0$, gain matrix K and positive scalars τ, ε_1,ε_2 and ε_3 such that the following conditions hold for $i \in \{1, ..., r\}$:

$$\begin{bmatrix} \Theta_i & \Xi^T & \Xi^T & \Xi^T & \bar{A}_i^T P \\ * & -rP & 0 & 0 & 0 \\ * & 0 & -r\varepsilon_1 I & 0 & 0 \\ * & 0 & 0 & -r\varepsilon_2 I & 0 \\ P\bar{A}_i & 0 & 0 & 0 & -\frac{\varepsilon_3}{r\gamma_{2i}^2} I \end{bmatrix} < 0, \quad \Xi = PA_0 - KC$$

$$(r\varepsilon_2 + r\varepsilon_3)B_i^T B_i + rB_i^T PB_i - \tau I < 0$$

$$\Theta_i = -r^{-1}P + \tau\gamma_{2i}^2 I + \gamma_{1i}^2 (r\varepsilon_1 + 1)\bar{A}_i^T \bar{A}_i + \gamma_{1i}^2 r\bar{A}_i^T P\bar{A}_i$$

The observer gain is given by $G = P^{-1}K$.

Quadratic Lyapunov Function

$$V(e(k)) = e(k)^T P e(k)$$

The variation of V is given by:

$$\begin{aligned}
\Delta V &= e^T\Phi^T P\Phi e + \sum_{i=1}^{r} e^T\Phi^T P\overline{A}_i\delta_i + \sum_{i=1}^{r} e^T\Phi^T PB_i\Delta_i + \sum_{i=1}^{r}\delta_i^T\overline{A}_i^T P\sum_{j=1}^{r}\overline{A}_j\delta_j \\
&+ \sum_{i=1}^{r}\delta_i^T\overline{A}_i^T P\Phi e + \sum_{i=1}^{r}\delta_i^T\overline{A}_i^T P\sum_{j=1}^{r}B_j\Delta_j + \sum_{i=1}^{r}\Delta_i^T B_i^T P\Phi e \\
&+ \sum_{i=1}^{r}\Delta_i^T B_i^T P\sum_{j=1}^{r}\overline{A}_j\delta_j + \sum_{i=1}^{r}\Delta_i^T B_i^T P\sum_{j=1}^{r}B_j\Delta_j - e^T P e
\end{aligned}$$

Lemma

$$X^T Y + Y^T X \leq \varepsilon X^T X + \varepsilon^{-1} Y^T Y, \quad \text{with} \quad \varepsilon > 0$$

Quadratic Lyapunov Function

$$V(e(k)) = e(k)^T P e(k)$$

The variation of V is given by:

$$\begin{aligned}
\Delta V &= e^T \Phi^T P \Phi e + \sum_{i=1}^{r} e^T \Phi^T P \overline{A}_i \delta_i + \sum_{i=1}^{r} e^T \Phi^T P B_i \Delta_i + \sum_{i=1}^{r} \delta_i^T \overline{A}_i^T P \sum_{j=1}^{r} \overline{A}_j \delta_j \\
&+ \sum_{i=1}^{r} \delta_i^T \overline{A}_i^T P \Phi e + \sum_{i=1}^{r} \delta_i^T \overline{A}_i^T P \sum_{j=1}^{r} B_j \Delta_j + \sum_{i=1}^{r} \Delta_i^T B_i^T P \Phi e \\
&+ \sum_{i=1}^{r} \Delta_i^T B_i^T P \sum_{j=1}^{r} \overline{A}_j \delta_j + \sum_{i=1}^{r} \Delta_i^T B_i^T P \sum_{j=1}^{r} B_j \Delta_j - e^T P e
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Lemma

$$X^T Y + Y^T X \leq \varepsilon X^T X + \varepsilon^{-1} Y^T Y, \quad \text{with} \quad \varepsilon > 0$$

Quadratic Lyapunov Function

$$V(e(k)) = e(k)^T P e(k)$$

The variation of V is given by:

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\Delta V &= e^T \Phi^T P \Phi e + \sum_{i=1}^{r} e^T \Phi^T P \overline{A}_i \delta_i + \sum_{i=1}^{r} e^T \Phi^T P B_i \Delta_i + \sum_{i=1}^{r} \delta_i^T \overline{A}_i^T P \sum_{j=1}^{r} \overline{A}_j \delta_j \\
&+ \sum_{i=1}^{r} \delta_i^T \overline{A}_i^T P \Phi e + \sum_{i=1}^{r} \delta_i^T \overline{A}_i^T P \sum_{j=1}^{r} B_j \Delta_j + \sum_{i=1}^{r} \Delta_i^T B_i^T P \Phi e \\
&+ \sum_{i=1}^{r} \Delta_i^T B_i^T P \sum_{j=1}^{r} \overline{A}_j \delta_j + \sum_{i=1}^{r} \Delta_i^T B_i^T P \sum_{j=1}^{r} B_j \Delta_j - e^T P e
\end{aligned}$$

Lemma

$$X^T Y + Y^T X \leq \varepsilon X^T X + \varepsilon^{-1} Y^T Y, \quad \text{with} \quad \varepsilon > 0$$

Proof 2/5

Reducing the double sum:

$$\sum_{i=1}^{r} X_i^T \sum_{j=1}^{r} X_j \leq r \sum_{i=1}^{r} X_i^T X_i$$

The variation of V is bounded as follows:

$$\Delta V \leq e^T \Psi_1 e + \sum_{i=1}^{r} \delta_i^T \Psi_{2i} \delta_i + \sum_{i=1}^{r} \Delta_i^T \Psi_{3i} \Delta_i$$

where:

$$\Psi_1 = \Phi^T P \Phi + \varepsilon_1^{-1} \Phi^T P P \Phi + \varepsilon_2^{-1} \Phi^T P P \Phi - P$$

$$\Psi_{2i} = r \varepsilon_1 \overline{A}_i^T \overline{A}_i + r \overline{A}_i^T P \overline{A}_i + \varepsilon_3^{-1} r \overline{A}_i^T P P \overline{A}_i$$

$$\Psi_{3i} = r(\varepsilon_1 + \varepsilon_2) B_i^T B_i + r B_i^T P B_i$$

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$$\Psi_{3i} = r(\varepsilon_1 + \varepsilon_2) B_i^T B_i + r B_i^T P B_i$$

Lipschitz property of δ_i in assumption 1 gives:

$$\sum_{i=1}^{r} \delta_i^T \delta_i \leq \sum_{i=1}^{r} \gamma_{1i}^2 e^T e$$

we obtain:

$$\sum_{i=1}^{r} \delta_i^T \Psi_{2i} \delta_i < \sum_{i=1}^{r} e^T (\gamma_{1i}^2 r \varepsilon_1 \overline{A}_i^T \overline{A}_i + \gamma_{1i}^2 r \overline{A}_i^T P \overline{A}_i + \gamma_{1i}^2 \varepsilon_3^{-1} r \overline{A}_i^T P P \overline{A}_i) e$$

Lipschitz property of Δ_i in assumption 1 gives:

$$\sum_{i=1}^{r} \Delta_i^T \Delta_i \leq \sum_{i=1}^{r} \gamma_{2i}^2 e^T e$$

that can be written in the form:

$$-\sum_{i=1}^{r} \gamma_{2i}^2 e^T e + \sum_{i=1}^{r} \Delta_i^T \Delta_i \leq 0$$

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that can be written in the form:

$$-\sum_{i=1}^{r} \gamma_{2i}^2 e^T e + \sum_{i=1}^{r} \Delta_i^T \Delta_i \leq 0$$

Applying the S-procedure:

$$\Delta V < \Delta V - \tau\Gamma$$

with:

$$\Gamma = -\sum_{i=1}^{r} \gamma_{2i}^2 e^T e + \sum_{i=1}^{r} \Delta_i^T \Delta_i \leq 0$$

we obtain :

$$\Delta V < \sum_{i=1}^{r} e^T \Omega_{1i} e + \sum_{i=1}^{r} \Delta_i^T \Omega_{2i} \Delta_i$$

where:

$$\begin{aligned} \Omega_{1i} &= r^{-1}\Phi^T P \Phi + r^{-1}\varepsilon_1^{-1}\Phi^T P P \Phi + r^{-1}\varepsilon_2^{-1}\Phi^T P P \Phi - r^{-1} P + \tau \gamma_{2i}^2 I \\ &+ \gamma_i^2 r \varepsilon_1 \overline{A}_i^T \overline{A}_i + \gamma_i^2 r \overline{A}_i^T P \overline{A}_i + \gamma_i^2 \varepsilon_3^{-1} r \overline{A}_i^T P P \overline{A}_i \end{aligned}$$

and:

$$\Omega_{2i} = (r\varepsilon_2 + r\varepsilon_3) B_i^T B_i + r B_i^T P B_i - \tau I$$

Proof 4/5

Applying the S-procedure:

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with:

$$\Gamma = -\sum_{i=1}^{r} \gamma_{2i}^2 e^T e + \sum_{i=1}^{r} \Delta_i^T \Delta_i \leq 0$$

we obtain :

$$\Delta V < \sum_{i=1}^{r} e^T \Omega_{1i} e + \sum_{i=1}^{r} \Delta_i^T \Omega_{2i} \Delta_i$$

where:

$$\begin{aligned} \Omega_{1i} &= r^{-1}\Phi^T P \Phi + r^{-1}\varepsilon_1^{-1}\Phi^T P P \Phi + r^{-1}\varepsilon_2^{-1}\Phi^T P P \Phi - r^{-1}P + \tau\gamma_{2i}^2 I \\ &+ \gamma_i^2 r \varepsilon_1 \overline{A}_i^T \overline{A}_i + \gamma_i^2 r \overline{A}_i^T P \overline{A}_i + \gamma_i^2 \varepsilon_3^{-1} r \overline{A}_i^T P P \overline{A}_i \end{aligned}$$

and:

$$\Omega_{2i} = (r\varepsilon_2 + r\varepsilon_3) B_i^T B_i + r B_i^T P B_i - \tau I$$

Applying the S-procedure:

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with:

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Proof 5/5

The negativity of ΔV is guaranteed if:

$$\Omega_{1i} < 0$$

$$\Omega_{2i} < 0$$

$$i \in \{1, \ldots, r\}$$

We consider the change of variable:

$$K = PG$$

and by using the Schur complement we obtain the inequalities expressed in theorem 1.

Proof 5/5

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Extension of the method: $\mathscr{L}_2$ approach

The state estimation error can be expressed like a perturbed system:

$$e(k+1) = \Phi e(k) + H\omega(k)$$

where:

$$\begin{aligned}
\Phi &= A_0 - GC \\
H &= \begin{bmatrix} H_1 & \cdots & H_r \end{bmatrix} \\
\omega &= \begin{bmatrix} v_1^T & \cdots & v_r^T \end{bmatrix}^T \\
H_i &= \begin{bmatrix} \bar{A}_i & B_i \end{bmatrix} \\
v_i &= \begin{bmatrix} \delta_i^T & \Delta_i^T \end{bmatrix}^T
\end{aligned}$$

Theorem 2

The state estimation error converges globally asymptotically toward zero, if there exists matrices $P = P^T > 0$, K and positive scalar $\tilde{\gamma}$ such that the following condition holds:

$$\begin{bmatrix} -P+I & \Psi_1 & \Psi_2 \\ \Psi_1^T & H^T P H - \tilde{\gamma} I & 0 \\ \Psi_2^T & 0 & -P \end{bmatrix} < 0$$

where:

$$\begin{aligned} \Psi_1 &= (A_0^T P - C^T K^T) H \\ \Psi_2 &= A_0^T P - C^T K^T \end{aligned}$$

The observer gain is given by $G = P^{-1} K$.

Quadratic Lyapunov function:

$$V(e(k)) = e(k)^T Pe(k), P = P^T > 0$$

The observer converges and the $\mathcal{L}_2$-gain from $\omega(k)$ to $e(k)$ is bounded by γ if the following holds :

$$\Delta V(e) + e(k)^T e(k) - \gamma^2 \omega(k)^T \omega(k) < 0$$

Variation of Lyapunov function:

$$\Delta V(e) = e^T \Phi^T P \Phi e - e^T Pe + e^T \Phi^T PH\omega + \omega^T H^T P \Phi e + \omega^T H^T PH\omega$$

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$$\Delta V(e) = e^T \Phi^T P \Phi e - e^T P e + e^T \Phi^T P H \omega + \omega^T H^T P \Phi e + \omega^T H^T P H \omega$$

We obtain:

$$e^T\Phi^T P\Phi e - e^T Pe + e^T\Phi^T PH\omega + \omega^T H^T P\Phi e + \omega^T H^T PH\omega + e^T e - \gamma^2\omega^T\omega < 0$$

That can be expressed under the following form:

$$\begin{bmatrix} e \\ \omega \end{bmatrix}^T \begin{bmatrix} \Phi^T P\Phi - P + I & \Phi^T PH \\ H^T P\Phi & H^T PH - \gamma^2 I \end{bmatrix} \begin{bmatrix} e \\ \omega \end{bmatrix} < 0$$

We consider the changes of variables:

$$K = PG, \quad \tilde{\gamma} = \gamma^2$$

Then, by using Schur Complement, we obtain the LMIs expressed in the theorem 2.

We obtain:

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We consider the changes of variables:

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Then, by using Schur Complement, we obtain the LMIs expressed in the theorem 2.

Simulation results

We consider the following example with variable parameters a and b:

$$A_1 = \begin{bmatrix} a & -0.3 \\ 0 & -0.5 \end{bmatrix}, A_2 = \begin{bmatrix} 0.4 & 0.1 \\ -0.2 & b \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The weighting functions are

$$\begin{cases} {}_1(x) = \frac{1-\tanh(x_1)}{2} \\ {}_2(x) = 1 - {}_1(x) = \frac{1+\tanh(x_1)}{2} \end{cases}$$

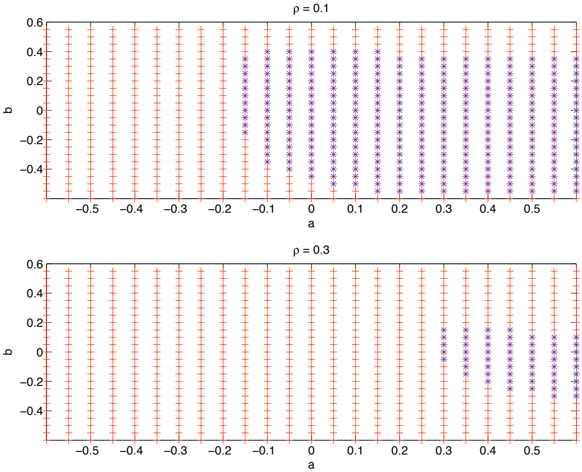

Figure: (*) first method,(+) $\mathscr{L}_2$ method

Example

We consider the previous example with $a = -0.6$ and $b = 0.1$. A stable observer with $\mathscr{L}_2$ attenuation of the considered perturbation terms for the above system can be designed using Theorem 2. Conditions in Theorem 2 are satisfied with :

$$\mathrm{P} = \begin{bmatrix} 2.55 & -1.76 \\ -1.76 & 2.99 \end{bmatrix}, G = \begin{bmatrix} -0.18 \\ -0.27 \end{bmatrix}$$

Given the initial conditions $x(0) = [0.7 \quad -0.5]^T$, $\hat{x}(0) = [0 \quad 0]^T$

Example

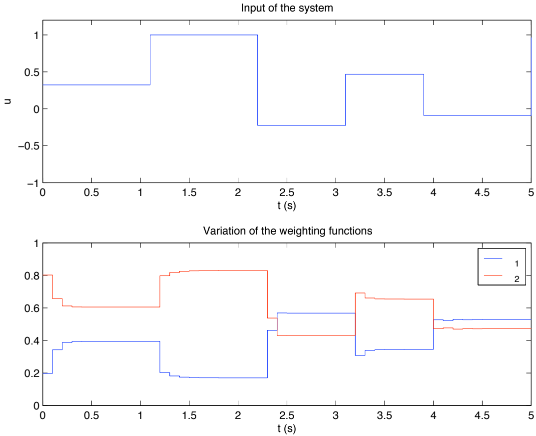

Figure: Input of the system and variation of weighting functions

Example

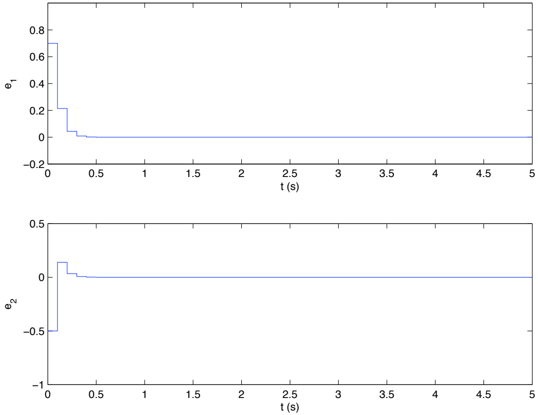

Figure: State estimation error

Conclusions

- Representation of nonlinear systems by the multiple model structure with unmeasurable premise variables.
- Sufficient conditions for state estimation are derived using the second method of Lyapunov and $\mathscr{L}_2$ approach, and given in the form of LMIs.

Future works

- Diagnosis of nonlinear systems using this observer.
- Reducing the conservatism of conditions expressed in theorem 1.
- Extension of the method to unknown input and state estimation.

Thank you