Robust input and state estimation for linear discrete-time systems using PI two stage Kalman filter

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ABSTRACT. In this paper, we have solved the problem of simultaneously estimating the state and input of linear time varying stochastic systems with uncertain noise covariance. The approach suggested rests on the use of the Proportional-Integral Two Stages Kalman Filter (PI-TSKF). This technique is qualified to be robust against the uncertainty. The filter is tested by an illustrative example.

KEYWORDS: Time varying linear system, two stage Kalman filter, Integral action, input estimation.

1. Introduction

The state estimation of the dynamic systems in the presence of unknown input is largely studied in the literature. In the stochastic linear context at discrete time, we can note works which use an approach based on Kalman filtering [1, 2, 4, 6, 8, 9, and 11]. These works represent an extension of Friedland’s work which is the first to introduce the technique of TSKF (Two Stage-Kalman Filter). The principal idea, on which this technique rests, consists in using a double transformation to decouple ASKF (Augmented State Kalman Filter) in two sub-filters: The first sub-filter is dedicated to the estimation of the state and the second one to estimation of the unknown input. The filter proposed by Friedland [2] is optimal only in the presence of a constant unknown input. The fact that encouraged Alouani et al. [1] to extend this filter to a dynamic stochastic unknown input; an algebraic constraint is then introduced to ensure the optimality of the TSKF. Hsieh and Chen [4] developed the OTSKF (Optimal Two Stage Kalman Filter) which is applied in a more general context, i.e. when the unknown input is generated by a stochastic dynamic process and jointly affects the state and the measurement equations of the system. Kim et al. [8] developed an adaptive version of TSKF noted ATSKF (Adaptive Two Stage Kalman
Filter) and presented in [9] an analysis of the stability of this filter. Recently, Khémiri et al [10] have used the proposed filter by Hsieh and Chen [4] to solve the fault estimation and isolation problem. The problem disadvantage of this approach is that the filter lost its optimality in the presence of an uncertainty noise covariance.

In this work, we propose to solve this problem by using the PI TSKF (Proportional Integral Two Stage Kalman Filter) to estimate the state and the fault of a stochastic system. The concept of integral action was used in the observer theory [3, 11]. This technique was introduced first in [7] and further developed in [12] and extended in [13] using a fading integral that was capable of coping transitory effect and improve the stability margin. Our aim is to extend the OTSKF by introducing the integral action technique proposed in [13], and solve the problem of state and fault estimation for systems with uncertainty noise covariance.

This paper is organized as follows. In section 2, we present the statement of the problem. In section 3, we recall the OTSKF. In section 4, the design of the PI TSKF is developed. Finally, an illustrative example of the proposed approach technique is presented in section 5.

2. Statement of the problem

Considering the linear time-varying discrete stochastic systems with unknown inputs, is described by:

\[
\begin{align*}
x_{k+1} &= A_k x_k + B_k u_k + E_k d_k + w_k^x \\
d_{k+1} &= d_k + w_k^d \\
y_k &= H_k x_k + F_k d_k + v_k
\end{align*}
\]

(1)

where \(x_k \in \mathbb{R}^n\) is the state vector, \(u_k \in \mathbb{R}^r\) is the known control input, \(y_k \in \mathbb{R}^m\) is the observation vector and \(d_k \in \mathbb{R}^p\) is the unknown inputs vector. The matrices \(A_k, B_k, H_k, E_k, F_k\) are known and have appropriate dimensions.

Assumptions:

- **A1**: the noises \(w_k^x\) and \(v_k\) are zero-mean white noise sequences with the following covariance’s: \(E \{ w_k^x w_j^x \} = Q_k^x \delta_{kj}, \) \(E \{ v_k v_j^T \} = R_k \delta_{kj}\) and \(E \{ w_k^x v_j^T \} = 0\), where \(^T\) denotes transpose and \(\delta_{kj}\) denotes the Kronecker delta function.

- **A2**: the noises \(w_k^d\) is zero-mean white noise sequence with the following covariance:

\(E \{ w_k^d w_j^d \} = Q_k^d \delta_{kj}\) and \(E \{ w_k^d w_j^d \} = Q_k^d \delta_{kj}\)


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- **A3**: the initial state $x_0$ is uncorrelated with the white noises processes $w_k^d$, $w_k^d$ and $v_k$ and is a gaussian random variable with $E\{x_0\} = \bar{x}_0$ and $E\{(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\} = P_0^d$.

- **A4**: the initial fault $d_0$ is satisfied the following

$$E\{d_0\} = \bar{d}_0, E\{(d_0 - \bar{d}_0)(d_0 - \bar{d}_0)^T\} = P_0^d, E\{(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\} = P_0^{xd}$$

- **A5**: the conditions on matrix ranks:
  - $\text{rank}(H_k) = m \geq p$ : to avoid the redundancy of measurement
  - $\text{rank}(F_k) = p$ : to guarantee the independence of the unknown input directions

We know that, if the model parameters bias from the reality, the estimating precision would become worse. Our problem consists of designing a filter that gives a robust state and unknown inputs estimation for linear time-varying stochastic systems, such that, all covariance matrices are not perfectly known.

### 3. Optimal Two Stage Kalman filter

The OTSKF (Optimal Two Stage Kalman Filter) is derived by Hsieh and Chen [4] where it is equivalent to the ASKF (Augmented State Kalman Filter). This technique is developed to solve the problem of optimal estimation for linear systems with unknown inputs. Recently, Khemiri et al. [10] have used this filter to solve the problem of FDI. This latter permits an optimal estimation of the state and the unknown inputs. The following OTSKF proposed by Hsieh and Chen [4], may be used:

#### Initialization

$$V_0 = P_0^{xd} \left( P_{0/0}^d \right)^{-1}, \bar{x}_{0/0} = \bar{x}_0 - V_0 \bar{d}_0, \bar{d}_{0/0} = \bar{d}_0,$$

$$\bar{P}_{0/0} = P_0^d - V_0 \bar{d}_0 \bar{d}_0^T$$

#### State sub-filter

$$\bar{x}_{k+1/k} = A_k \bar{x}_k + B_k u_k + \bar{u}_k$$

$$\bar{P}_{k+1/k} = A_k \bar{P}_k \bar{x}_k A_k^T + \bar{Q}_k$$

$$\bar{x}_{k+1/k+1} = \bar{x}_{k+1/k} + \bar{K}_{k+1} (y_{k+1} - H_{k+1} \bar{x}_{k+1/k})$$

$$\bar{R}_{k+1} = \bar{F}_{k+1} H_{k+1} (P_{k+1/k} \bar{F}_{k+1} H_{k+1}^T + R_{k+1})^{-1}$$

$$\bar{P}_{k+1/k+1} = \left( I - \bar{R}_{k+1} \bar{K}_{k+1} \right) \bar{P}_{k+1/k}$$
Unknown input sub-filter

\[
\begin{align*}
\bar{d}_{k+1/k} &= \bar{d}_{k/k} \quad \text{(7)} \\
\tilde{P}_{k+1/k}^d &= \tilde{P}_{k/k}^d + Q_{k}^d \quad \text{(8)} \\
\tilde{d}_{k+1/k+1} &= \bar{d}_{k+1/k} + \tilde{R}_{k+1}^d \times \left( y_{k+1} - H_{k+1}\bar{x}_{k+1} - S_{k+1}\bar{d}_{k+1/k} \right) \quad \text{(9)} \\
\tilde{R}_{k+1}^d &= \tilde{P}_{k+1/k}^d S_{k+1}^T + H_{k+1}\tilde{P}_{k+1/k}^d H_{k+1}^T + R_{k+1} \quad \text{(10)} \\
\tilde{P}_{k+1/k+1}^d &= \left(I - \tilde{R}_{k+1}^d S_{k+1}^T\right)\tilde{P}_{k+1/k}^d \quad \text{(11)}
\end{align*}
\]

Correction

\[
\begin{align*}
\hat{x}_{k+1/k+1} &= \bar{x}_{k+1/k+1} + V_{k+1} \bar{d}_{k+1/k+1} \\
\hat{d}_{k+1/k+1} &= \bar{d}_{k+1/k+1}
\end{align*}
\]

where

\[
\begin{align*}
U_{k+1} &= \bar{U}_{k+1} + \left(Q_{k}^d - \bar{Q}_{k}^d \bar{Q}_{k}^d \right)\left(\tilde{P}_{k+1/k}^d \right)^{-1} \\
\bar{U}_{k+1} &= \left(\bar{U}_{k+1} - U_{k+1} \right)\bar{d}_{k/k} \\
\bar{Q}_{k}^x &= Q_{k}^x - Q_{k}^d \bar{Q}_{k}^d \left(\bar{Q}_{k}^d - \bar{Q}_{k}^d \bar{Q}_{k}^d \right)^T \\
V_{k+1} &= U_{k+1} - \tilde{R}_{k+1}^\tau S_{k+1} \\
S_{k+1} &= H_{k+1}U_{k+1} + E_{k+1}
\end{align*}
\]

However, application of OTSKF requires an accurate model of the process and if there are any errors in the process model, the OTSKF may lead to poor performance. So, an uncertainty noise covariance can lead to large error estimation and a loss of optimality of OTSKF [10]. Our contribution consists of extending the last technique by modifying the state and the fault sub-filters with integral actions. The robust PI TSKF (Proportional Integral Two Stage Kalman Filter) will be developed in the next section.

4. Robust PI Two Stage Kalman Filter

This section is devoted to the PI TSKF design. So, we present the modified state and unknown input sub-filters in section 4.1 and 4.2. The filter is summarised in section 4.3.

4.1. The modified state sub-filter

In this section, we will modify the state sub-filter, which is described in section 3 by equations (2) – (6). We will introduce the integral action [13] so the modified sub-filter can have the following form
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\[
\bar{x}_{k+1/k} = A_k \bar{x}_{k/k} + B_k u_k + \bar{\nu}_k \tag{20}
\]

\[
f^x_{k+1} = f^x_k + G^{x^i}_{k+1} \left( y_{k+1} - H_{k+1} \bar{x}_{k+1/k} \right) \tag{21}
\]

\[
\bar{x}_{k+1/k+1} = \bar{x}_{k+1/k} + f^x_{k+1} + G^{x^p}_{k+1} \left( y_{k+1} - H_{k+1} \bar{x}_{k+1/k} \right) \tag{22}
\]

where \( f^x_k \) represents the integral term of error estimation and \( G^{x^p}_{k+1} \) and \( G^{x^i}_{k+1} \) are the proportional and integral gain matrices of the state sub-filter.

In this section we introduce two arbitrary matrices [13] : the fading constant and the integral effect coefficient are both incorporated in the above system. Therefore, the resulting update state sub-filter is given in an augmented system as

\[
\begin{bmatrix}
    \bar{x}_{k+1/k+1} \\
    f^x_{k+1}
\end{bmatrix} = \begin{bmatrix}
    A_k & D^x_k \\
    0 & D^f_k
\end{bmatrix} \begin{bmatrix}
    \bar{x}_{k/k} \\
    f^x_k
\end{bmatrix} + \begin{bmatrix}
    B_k \\
    0
\end{bmatrix} u_k + \begin{bmatrix}
    I \\
    0
\end{bmatrix} \bar{u}_k + \begin{bmatrix}
    G^{x^p}_{k+1} \\
    G^{x^i}_{k+1}
\end{bmatrix} \left( y_{k+1} - H_k \bar{x}_{k+1/k} \right) \tag{23}
\]

where \( D^x_k \), \( D^f_k \) are constant matrices determined according to performance specifications for the system. The optimal design of the gain matrices can be determined using the augmented covariance matrix as follow:

\[
\begin{bmatrix}
    \bar{x}_{k+1/k+1} \\
    f^x_{k+1}
\end{bmatrix} = \begin{bmatrix}
    P^{x11}_{k+1/k} & P^{x12}_{k+1/k} \\
    P^{x21}_{k+1/k} & P^{x22}_{k+1/k}
\end{bmatrix} \begin{bmatrix}
    \bar{x}_{k/k} \\
    f^x_k
\end{bmatrix} + \begin{bmatrix}
    G^{x^p}_{k+1} \\
    G^{x^i}_{k+1}
\end{bmatrix} \left( y_{k+1} - H_k \bar{x}_{k+1/k} \right) \tag{24}
\]

\[
\begin{bmatrix}
    P^{x11}_{k+1/k} & P^{x12}_{k+1/k} \\
    P^{x21}_{k+1/k} & P^{x22}_{k+1/k}
\end{bmatrix} = \begin{bmatrix}
    A^x_{k+1} & \bar{Q}^x_{k+1} \\
    0 & 0
\end{bmatrix} = \begin{bmatrix}
    A^x_{k+1} & \bar{Q}^x_{k+1} \\
    0 & 0
\end{bmatrix} = \begin{bmatrix}
    A^x_{k+1} & \bar{Q}^x_{k+1} \\
    0 & 0
\end{bmatrix} \tag{25}
\]

where

\[
A^x_{k+1} = \begin{bmatrix}
    A_k & D^x_k \\
    0 & D^f_k
\end{bmatrix}, \quad \bar{Q}^x_{k+1} = \begin{bmatrix}
    \bar{Q}^x_k & 0 \\
    0 & 0
\end{bmatrix} \quad \text{and} \quad H^x_k = \begin{bmatrix}
    H_k & 0
\end{bmatrix} \tag{26}
\]

By introducing equation (25) into (24) and using (26) the proportional and the integral gain matrices have the following forms:

\[
G^{x^p}_{k+1} = \left( A_k P^{x11}_{k/k} A_k^T H_k^T + D^x_k P^{x21}_{k/k} A_k^T H_k^T + A_k P^{x12}_{k/k} D^x_k H_k^T + D^f_k P^{x22}_{k/k} D^x_k H_k^T + Q^x_k H_k^T \right) \times \\
\left( H_k A_k P^{x11}_{k/k} A_k^T H_k^T + H_k D^x_k P^{x21}_{k/k} A_k^T H_k^T + H_k A_k P^{x12}_{k/k} D^x_k H_k^T + H_k A_k P^{x11}_{k/k} D^x_k H_k^T + H_k D^x_k P^{x22}_{k/k} D^x_k H_k^T + R_k + H_k Q^x_k H_k^T \right)^{-1} \tag{27}
\]
\[ G^{ii}_{k+1} = \left( D^i_f P^{12}_{k/k} A^T_k H^T_k + D^i_f P^{22}_{k/k} D^i_f H^T_k \right) \times \left( H_k A_k P^{11}_{k/k} A^T_k H^T_k + H_k D^i_f P^{12}_{k/k} A^T_k H^T_k + H_k A_k P^{12}_{k/k} D^i_f H^T_k + H_k D^i_f P^{22}_{k/k} D^i_f H^T_k + R_k + H_k Q^i_k H^T_k \right)^{-1} \] (28)

- \( D^i_f \) as a full rank constant matrix presents the extent of the effect of the integral term on the state estimation.

- \( D^i_f \) is the fading constant matrix for the state estimation, where \( D^i_f = \alpha I \), with \( 0 < \alpha \leq 1 \)

### 4.2. The modified unknown input sub-filter

In this section, we follow the same manner for the last section to introduce the integral action in the unknown input sub-filter. The modified unknown input sub-filter will have the following form:

\[
\begin{align*}
\bar{a}_{k+1/k} &= \bar{a}_{k/k} \\
J^d_{k+1} &= J^d_{k} + C^d_{k+1} \left( y_{k+1} - H_{k+1} \bar{x}_{k+1/k} - S_{k+1} \bar{a}_{k+1/k} \right) \\
\bar{a}_{k+1/k+1} &= \bar{a}_{k+1/k} + J^d_{k+1} + C^{dp}_{k+1} \left( y_{k+1} - H_{k+1} \bar{x}_{k+1/k} - S_{k+1} \bar{a}_{k+1/k} \right)
\end{align*}
\] (29)  (30)  (31)

where, \( J^d_k \) represents the integral term and \( C^{dp}_{k+1} \) and \( C^{di}_{k+1} \) are the proportional and integral gain matrices of the unknown input sub-filter.

In this section we introduce two arbitrary matrices: the fading constant and the integral effect coefficient are both incorporated in the above system. Therefore, the resulting update unknown input sub-filter is given in an augmented system as

\[
\begin{bmatrix}
\bar{a}_{k+1/k+1} \\
\bar{a}_{k+1/k}
\end{bmatrix} =
\begin{bmatrix}
I & D^i_f \\
0 & D^i_f
\end{bmatrix}
\begin{bmatrix}
\bar{a}_{k/k} \\
\bar{a}_{k+1/k}
\end{bmatrix} +
\begin{bmatrix}
C^{dp}_{k+1} \\
C^{di}_{k+1}
\end{bmatrix}
\begin{bmatrix}
y_{k+1} - H_{k+1} \bar{x}_{k+1/k} - S_{k+1} \bar{a}_{k+1/k} \\
y_{k+1} - H_{k+1} \bar{x}_{k+1/k} - S_{k+1} \bar{a}_{k+1/k}
\end{bmatrix}
\] (32)

where \( D^i_f, D^i_f \) are constant matrices determined according to performance specifications for the system. The optimal design of the gain matrices can be determined using the augmented covariance matrix as follow:

\[
\begin{bmatrix}
P^{d11}_{k/k} \\
P^{d12}_{k/k} \\
P^{d21}_{k/k} \\
P^{d22}_{k/k}
\end{bmatrix} =
\begin{bmatrix}
P^{d1}_{k/k} \\
P^{d2}_{k/k}
\end{bmatrix} +
\begin{bmatrix}
\bar{a}^{-1}_{k/k} S_k 
\end{bmatrix}
\] (33)
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\[
\mathbf{P}_{k|k-1}^d = A_{k-1}^d \mathbf{P}_{k-1|k-1}^d A_{k-1}^T + \mathbf{Q}_{k-1}^d \tag{34}
\]

where

\[
A_k^d = \begin{bmatrix} I & D_f^d \\ 0 & D_f^d \end{bmatrix}, \quad \mathbf{Q}_k^d = \begin{bmatrix} Q_k^d & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_k = [S_k \ 0] \tag{35}
\]

By introducing equation (32) into (31) and using (33) the proportional and the integral gain matrices have the following forms:

\[
G_{k+1}^d = \left( A_k^d \mathbf{P}_{k|k}^d A_k^T + D_f^d \mathbf{P}_{k|k}^d A_k^T S_k^T + A_k^d \mathbf{P}_{k|k}^d D_f^T S_k^T + D_f^d \mathbf{P}_{k|k}^d D_f^T S_k^T + Q_k^d S_k^T \right) \times
\]

\[
\left( H_k A_k P_{k|k}^{11} A_k^T H_k^T + H_k D_f P_{k|k}^{21} A_k^T H_k^T + H_k A_k P_{k|k}^{12} D_f^T H_k^T + H_k D_f P_{k|k}^{22} D_f^T H_k^T + S_k A_k^d P_{k|k}^{11} A_k^T S_k^T + S_k D_f^d P_{k|k}^{21} A_k^T S_k^T + S_k A_k^d P_{k|k}^{12} D_f^T S_k^T + S_k D_f^d P_{k|k}^{22} D_f^T S_k^T \right)^{-1} \tag{36}
\]

\[
G_{k+1}^i = \left( D_f^i P_{k|k}^{11} A_k^T S_k^T + D_f^i P_{k|k}^d D_f^i S_k^T \right) \times
\]

\[
\left( H_k A_k P_{k|k}^{11} A_k^T H_k^T + H_k D_f P_{k|k}^{21} A_k^T H_k^T + H_k A_k P_{k|k}^{12} D_f^T H_k^T + H_k D_f P_{k|k}^{22} D_f^T H_k^T + S_k A_k^d P_{k|k}^{11} A_k^T S_k^T + S_k D_f^d P_{k|k}^{21} A_k^T S_k^T + S_k A_k^d P_{k|k}^{12} D_f^T S_k^T + S_k D_f^d P_{k|k}^{22} D_f^T S_k^T \right)^{-1} \tag{37}
\]

- \( D_f^i \) as full rank constant matrix presents the extent of the effect of the integral term on the unknown input estimation.

- \( D_f^i \) is the fading constant matrix for the unknown input estimation, where \( D_f^i = \alpha I \) with \( 0 < \alpha \leq 1 \)

4.3. Summary of filter equations

Finally, the PI TSKF filter is summarized in the Table 1. We suppose to know the following:
- The known input $u_k$
- Matrices $A_k$, $B_k$, $H_k$, $E_k$ and $F_k$
- Matrices $Q_k^e$, $R_k$ and $Q_k^d$
- Initial values $\xi_0$, $\bar{a}_0$, $P_0^x$ and $P_0^d$

**Algorithm**: state and unknown input estimation by PI TSKF

- **Step 0**: Initialization
  
  \[
  V_0 = P_0^d \left( P_0^{d_0} \right)^{-1}, \quad \xi_0 = x_0 - V_0 \bar{\omega}_0, \quad \bar{a}_0 = \bar{\omega}_0, \quad P_0/0 = P_0^x - V_0 P_0^{d_0} V_0^T, \nu \\
  P_0^{d_0} = P_0^d, \quad k = 0 
  \]

- **Step 1**: state sub-filter

  To calculate $\bar{U}_{k+1}$, $\bar{U}_{k+1}$, $\bar{Q}_{k+1}$ and $\bar{P}_{k+1/k}^d$ respectively from (14), (15), (16), (17) and (11)

  To calculate $\bar{F}_{k+1/k}$, $\bar{f}_{k+1}$, $\bar{F}_{k+1/k}$, $\bar{P}_{k+1/k}$ and $\bar{G}_{k+1}^{sp}$ and $G_{k+1}^{sl}$ respectively from (20), (21), (22), (24), (25), (27) and (28)

- **Step 2**: unknown input sub-filter

  To calculate $S_{k+1}$ from (19)

  To calculate $\bar{F}_{k+1/k}$, $\bar{f}_{k+1}$, $\bar{F}_{k+1/k}$, $\bar{P}_{k+1/k}$ and $\bar{G}_{k+1}^{dp}$ and $\bar{G}_{k+1}^{di}$ respectively from (29), (30), (31), (33), (34), (36) and (37)

- **Step 3**: correction

  Update $V_k = U_{k+1} - G_{k+1}^{sp} S_{k+1}$

  To calculate $\hat{x}_{k+1/k+1}$ and $\hat{d}_{k+1/k+1}$ from (12) and (13)

- **Step 4**: $k = k + 1$ go to Step 1

---

**Table 1**: PI TSKF algorithm

The obtained PI TSKF is qualified to be robust against the uncertainty noise covariance of the state, the unknown inputs and the measurement. In the next section, the filter will be tested by an illustrative example.
5. **Illustrative example**

In this section, we present the application of the proposed filters. The parameters of the system (1) are given by:

\[
\begin{align*}
X_k &= \begin{bmatrix} X_{1,k} \\ X_{2,k} \\ X_{3,k} \end{bmatrix}, \quad A_k = \begin{bmatrix} a_k & 0.5 & 0.08 \\ 0.6 & 0.01 & 0.04 \\ 0.1 & 0.7 & 0.05 \end{bmatrix}, \quad a_k = 0.2 + 0.2 \sin(0.1k), \\
B_k &= \begin{bmatrix} 2 \\ -1.5 \\ 0.5 \end{bmatrix}, \\
H_k &= \begin{bmatrix} 0.2 & 0.4 & 0 \\ -0.5 & -0.8 & 0.4 \end{bmatrix}, \\
E_k &= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \\
F_k &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
D^x_l &= I_{3 \times 3}, \quad D^y_f = 0.3 I_{3 \times 3}, \\
D^d_l &= 1.5, \quad D^d_f = 0.3 \\
Q^x_k &= 0.1 I_{3 \times 3}, \quad R_k = 0.01 I_{2 \times 2}, \\
Q_k^{ou} &= \begin{bmatrix} 0.1 & 0.2 & 0.2 \end{bmatrix}^T, \quad Q_k^{id} = 0.2
\end{align*}
\]

Fig. 2 presents the input/output sequence of the system. The sampling time is \( N = 100 \).
In the simulation, we introduce a perturbation on the noise covariance matrices of the state, the measurement and the unknown input as follow:

\[
Q_k' = \begin{cases} 
Q_k^x & \text{if } k \leq 50 \\
(1 + \alpha_1)Q_k^x & \text{if } k > 50 
\end{cases}
\]

\[
R_k' = \begin{cases} 
R_k & \text{if } k \leq 50 \\
(1 + \alpha_2)R_k & \text{if } k > 50 
\end{cases}
\]

\[
Q_k' = \begin{cases} 
Q_k^d & \text{if } k \leq 50 \\
(1 + \alpha_3)Q_k^d & \text{if } k > 50 
\end{cases}
\]

where \( \alpha_1 = \alpha_2 = \alpha_3 = 1 \)

![Figure 3. State estimation](image)

Fig. 3 and Fig. 4 represent the states, the unknown inputs and their estimates. We can show that, with the perturbation in the covariance matrices, the PI TSKF have the better performances than the OTSKF. The last leads to a large error estimation and loss her optimality.
Robust input and state estimation

Figure 4. Unknown input estimation

The simulation results in Table 2 show the average root mean square errors (RMSE) in the estimated states and unknown input. For example, the RMSE of the first component of state vector $x_{1,k}$ is calculated by:

$$RMSE(x_{1,k}) = \frac{1}{N} \sum_{k=1}^{N} (x_{1,k} - \hat{x}_{1,k})^2$$

<table>
<thead>
<tr>
<th>RMSE</th>
<th>$x_{1,k}$</th>
<th>$x_{2,k}$</th>
<th>$x_{3,k}$</th>
<th>$d_{k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>OTSKF</td>
<td>21.15</td>
<td>8.32</td>
<td>54.59</td>
<td>7.08</td>
</tr>
<tr>
<td>PI TSKF</td>
<td>1.47</td>
<td>1.32</td>
<td>2.77</td>
<td>0.31</td>
</tr>
</tbody>
</table>

Table 2. RMSE values

6. Conclusion

In this paper, a robust estimation for linear stochastic discrete-time varying systems with uncertainty noise covariance is proposed by using a PI TSKF. The filter is obtained by a
modification of the state and the unknown inputs sub-filters of the OTSKF with integral actions. This recursive method is tested by an illustrative example. Indeed, the PI TSKF remains powerful in spite of the errors made on the covariance matrices characterizing the noises of the state $Q_k^s$, the measurement $R_k$ and the unknown inputs $Q_k^d$.

7. References