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State estimation of two-time scale multiple models with unmeasurable premise variables. Application to biological reactors

Anca Maria Nagy-Kiss, Benoît Marx, Gilles Mourot, Georges Schutz, José Ragot

Abstract— **This paper deals with the state estimation of nonlinear systems. The systems under study are characterized by two-time scale models. The state estimation is performed by designing a proportional integral observer (PIO) with unknown inputs. In order to design such an observer, the nonlinear model is transformed into an equivalent multiple model form and the fast dynamics are considered as unknown inputs. An application to an ASM1 model of Wastewater Treatment Plants (WWTP) is considered and the obtained numerical results show the performances of the proposed approach.**

Index Terms— **multiple modeling, singularly perturbed systems, unmeasurable premise variables, LMI, $\mathscr{L}_2$ approach**

I. Introduction

The application of linear methods to nonlinear systems is a difficult problem when talking about the observer/controller synthesis. The multiple model (MM) [12] -also called in the literature fuzzy Takagi-Sugeno model [17], or polytopic linear model [1]- has received a special attention in the last two decades, in order to overcome this difficulty. The MM structure is mainly based on the idea of complexity reduction of nonlinear systems, by constructing linear submodels aggregated by weighting functions [17]. Several techniques were developed in order to obtain such a structure from a general representation of nonlinear system. In this paper, the MM is obtained by applying a method proposed in [13] to represent nonlinear system into an equivalent MM. Only the general steps of this technique are given here.

In many practical situations, systems can have multiple time scale dynamics. In order to deal with such systems, the singularly perturbed theory is often used to highlight the decomposition of the system into various time scales. Nevertheless, it is not obvious to model a process under the standard singularly perturbed form especially if the system is nonlinear.

The first difficult point of this modeling technique is the separation of the slow and fast dynamics. Different methods are proposed in the literature [16], [15], [6]; the most frequently used is based on the evaluation of the jacobian eigenvalues of the linearized system and will be used here.

After the separation of the multiple-time scale dynamics, the standard singularly perturbed form is obtained. In the limit case, when the singularly perturbed parameter tends towards zero, this form has a dynamic part, represented by an ordinary differential equation (ODE), and a static part expressed by an algebraic equation. Thus, a second difficult point is to solve the algebraic system in order to express the fast variables and replace them into the ODE corresponding to the slow dynamics. The method mainly used to deal with this problem is based on a coordinate change [16], [18] requiring a linear transformation in order to eliminate the fast dynamic components. In order to be able to apply this method, the nonlinear system has to respect some structural constraints, which are not always satisfied.

By considering the standard singularly perturbed system, an equivalent MM can be written. The classical MM form is slightly modified in order to separate the slow and the fast dynamics of the system.

The main contribution of this paper is to estimate the state variables of a two-time scale nonlinear system by avoiding the resolution of the algebraic -static- equation corresponding to fast variables. This is possible by constructing an augmented output vector using the static equation and by considering the fast state variables as unknown inputs. Thus, a proportional integral observer (PIO) with unknown inputs can be designed by using the MM singularly perturbed form. Due to the limited number of sensors, this approach turns out to be interesting because of the choice of the fast variables as unknown inputs. This observer enables to reconstruct simultaneously the slow and fast variables and gives better results than a classic unknown input observer concerning the noise reconstruction [8].

In [11] is presented a state estimation method for singular MM affected by unknown inputs and with measurable decision variables. As in [11], most of the existing works, dedicated to MM in general and to observer design based on MM in particular [4], [9], are with measurable decision variables (inputs/outputs). But, in many practical situations, these premise variables depend on the states, thus they are not accessible. Recently, few works [2], [8], [19] are devoted to the case of unmeasurable decision variables. This last case will be treated here. The convergence conditions of the state and unknown input estimation error are expressed through LMIs (Linear Matrix Inequalities) by using the Lyapunov method and the $\mathscr{L}_2$ approach.

In the second part of the paper, the MM structure and the singularly perturbed theory are used in order to reconstruct the states of an ASM1 (Activate Sludge Model 1)[14] describing a biological degradation process characterized by two-time scale dynamics. The proportional integral observer proposed previously is applied to this model for this purpose.

A. M. Nagy-Kiss, B. Marx, G. Mourot, J. Ragot are with Centre de Recherche en Automatique de Nancy, Nancy Université, 2, avenue de la Forêt de Haye, 54516 Vandoeuvre-les-Nancy Cedex France anca-maria.nagy, gilles.mourot, benoit.marx, jose.ragot@ensem.inpl-nancy.fr

G. Schutz is with CRP "Henri Tudor", LTI Unité Business and Systems Analytics, 29, Avenue John F. Kennedy, L-1855 Luxembourg - Kirchberg georges.schutz@tudor.lu

In section II are given the essential tools for modeling nonlinear systems, in section III the observer design is presented. Section IV proposes the application to the ASM1, known to be a realistic model of WWTP. The paper ends with some conclusions and future works.

II. Modeling main tools

A. Multiple model representation

Generally, a dynamic nonlinear system can be described by the following ordinary differential equations:

$$\begin{aligned} \dot{x}(t) &= f(x(t),u(t)) \\ y(t) &= g(x(t),u(t)) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^l$ the output vector.
The multiple model enables to represent a nonlinear dynamic system into a convex combination of r linear submodels:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \mu_i(x,u)\,[A_i x(t) + B_i u(t)] \\ y(t) &= \sum_{i=1}^{r} \mu_i(x,u)\,[C_i x(t) + D_i u(t)] \end{aligned} \tag{2}$$

where A_i, B_i, C_i and D_i are constant matrices of suitable dimensions. The functions $\mu_i(x,u)$ represent the weights of the submodels $\{A_i,B_i,C_i,D_i\}$ in the global model and they have the following properties:

$$\sum_{i=1}^{r} \mu_i(x,u) = 1 \quad \text{and} \quad \mu_i(x,u) \geq 0, \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m \tag{3}$$

In order to obtain the MM form, a method giving an equivalent rewriting of the nonlinear system (1) is used [13]. Firstly, by extracting the state in the input vectors contained in the functions f and g, the system (1) is transformed into a quasi-Linear Parameter Varying (quasi-LPV) form:

$$\begin{aligned} \dot{x}(t) &= A(x(t),u(t))\,x(t) + B(x(t),u(t))\,u(t) \\ y(t) &= C(x(t),u(t))\,x(t) + D(x(t),u(t))\,u(t) \end{aligned} \tag{4}$$

Secondly, some nonlinear entries of the matrices A, B, C and D are considered as "premise variables" and denoted $z_j(x,u)(j = 1,...,q)$. Several choices of these premise variables are possible due to the existence of different equivalent quasi-LPV forms (further details on the selection procedure can be found in [13]).
Thirdly, a convex polytopic transformation is performed for all premise variables $(j = 1,...,q)$, as follows:

$$z_j(x,u) = F_{j,1}(z_j(x,u))\,z_{j,1} + F_{j,2}(z_j(x,u))\,z_{j,2} \tag{5}$$

where

$$z_{j,1} = \max_{x,u} \{z_j(x,u)\} \tag{6a}$$

$$z_{j,2} = \min_{x,u} \{z_j(x,u)\} \tag{6b}$$

and where

$$F_{j,1}(z_j(x,u)) = \frac{z_j(x,u) - z_{j,2}}{z_{j,1} - z_{j,2}} \tag{7a}$$

$$F_{j,2}(z_j(x,u)) = \frac{z_{j,1} - z_j(x,u)}{z_{j,1} - z_{j,2}} \tag{7b}$$

Remark 1: For q decision variables z_j, $r = 2^q$ submodels will be obtained. By multiplying the functions F_{j,σ_i^j}, the weighting functions are obtained:

$$\mu_i(x,u) = \prod_{j=1}^{q} F_{j,\sigma_i^j}(z_j(x,u)), \quad i = 1,\cdots,r \tag{8}$$

where the indexes σ_i^j ($i = 1,...,2^q$ and $j = 1,...,q$), equal to 1 or 2, indicate which partition of the j^{th} decision variable ($F_{j,1}$ or $F_{j,2}$) is involved in the i^{th} submodel.
In definition (2), the constant matrices A_i, B_i, C_i and D_i ($i = 1,...,2^q$) are obtained by replacing the premise variables $z_j(x,u)$ involved in the matrices $A(x,u)$, $B(x,u)$, $C(x,u)$ and $D(x,u)$ with the scalars defined in (6). Here, only the matrices A_i are given, the others being obtained similarly:

$$A_i = A(z_{1,\sigma_i^1},...,z_{q,\sigma_i^q}), \quad i = 1,\cdots,r \tag{9}$$

B. Singularly perturbed systems

Considering equation (1), the standard form of a singularly perturbed system with two-time scales can be expressed by the following system:

$$\varepsilon \dot{x}_f(t) = f_f(x_s(t),x_f(t),u(t),\varepsilon) \tag{10a}$$

$$\dot{x}_s(t) = f_s(x_s(t),x_f(t),u(t),\varepsilon) \tag{10b}$$

where $x_s \in \mathbb{R}^{n_s}$ and $x_f \in \mathbb{R}^{n_f}$ are respectively the slow and fast state variables, $f_f(x,u,\varepsilon) \in \mathbb{R}^{n_f}$, $f_s(x,u,\varepsilon) \in \mathbb{R}^{n_s}$, $n = n_s + n_f$ and ε is a small and positive parameter, known as *singular perturbed parameter*.
In order to obtain the standard singularly perturbed form, the identification and separation of slow and fast dynamics is the keypoint. In this article, this is realized by using the homotopy method for the linearized system [18]. This method enables to link each state variable with an eigenvalue. By comparing all the real parts of the eigenvalues, the biggest (resp. smallest) ones will be associated with the slowest (resp. fastest) dynamics. The comparison has to be performed when linearizing the system around several operating points in order to test if this classification remains the same.
Remark 2. Note that the linearized system is only used to identify the slow and fast dynamics, but not to design the observer in order to estimate the state variables. An equivalent MM representation will be used for this purpose. In the limit case $\varepsilon \rightarrow 0$, the degree of the system (10) degenerates from n to n_s, and the system becomes:

$$0 = f_f(x_s(t),x_f(t),u(t),0) \tag{11a}$$

$$\dot{x}_s(t) = f_s(x_s(t),x_f(t),u(t),0) \tag{11b}$$

By solving the algebraic equations (11a), the solution $x_f(t) = \varphi(x_s(t),u(t))$ is obtained and used in (11b) to derive the reduced system:

$$x_f(t) = \varphi(x_s(t),u(t)) \tag{12a}$$

$$\dot{x}_s(t) = f_s(x_s(t),x_f(t),u(t)) \tag{12b}$$

Remark 3. The fast variables cannot always be explicitly expressed from (11a). The most popular method used to deal with this problem is based on a change of coordinates [16],

[18], requiring a linear transformation in order to eliminate the fast dynamics. So, this method can only be applied to systems for which this linear transformation can be found.

By taking into account the previous remark, no change of coordinates will be considered; the reduced standard singularly perturbed form (11) is taken into account, supposing that the fast variables cannot be obtained by solving the algebraic equation (11a). Thus, this last equation is used to construct an augmented output vector, as it will be described in detail in the following section, where the design of the proportional integral observer is presented.

III. State estimation

A. Multiple model with slow and fast dynamics

Let us consider the reduced system (11) under an equivalent QLPV form

$$0 = A_{ff}(x,u)x_f(t) + A_{fs}(x,u)x_s(t) + B_f(x,u)u(t) \quad (13)$$

$$\dot{x}_s(t) = A_{sf}(x,u)x_f(t) + A_{ss}(x,u)x_s(t) + B_s(x,u)u(t) \quad (14)$$

It is here assumed that in the QLPV form, the matrix function B_f does not depend on $x(t)$, i.e. $B_f(u(t))$. The multiple model form that is obtained using the methodology described in [13] slightly modified in order to separate the slow and fast variables:

$$\begin{aligned} 0 &= \sum_{i=1}^{r} \mu_i(x,u)\left[A^i_{ff}x_f(t) + A^i_{fs}x_s(t)\right] + \sum_{i=1}^{\tilde{r}} \tilde{\mu}_i(u)B^i_f u(t) \\ \dot{x}_s(t) &= \sum_{i=1}^{r} \mu_i(x,u)\left[A^i_{sf}x_f(t) + A^i_{ss}x_s(t) + B^i_s u(t)\right] \\ y(t) &= C_f x_f(t) + C_s x_s(t) \end{aligned} \quad (15)$$

where $\tilde{r} \leq r$, $\mu_i(x,u)$ and $\tilde{\mu}_i(u)$ satisfy (3), the matrices A^i_{ff}, A^i_{fs}, A^i_{sf}, A^i_{ss}, B^i_f, B^i_s correspond to slow and fast dynamics identified in the matrices A_i and B_i:

$$A_i = \begin{bmatrix} A^i_{ff} & A^i_{fs} \\ A^i_{sf} & A^i_{ss} \end{bmatrix} \quad B_i = \begin{bmatrix} B^i_f \\ B^i_s \end{bmatrix} \quad (16)$$

The measurement equation in (15) can be considered as linear and time invariant since, in most practical situations, the sensors do not change according to the operating point. In the first equation of (15) the control term is moved from the right side of the equality to the left side to obtain:

$$\dot{x}_s(t) = \sum_{i=1}^{r} \mu_i(x,u)\left[A^i_{ss}x_s(t) + B^i_s u(t) + A^i_{sf}x_f(t)\right] \quad (17a)$$

$$y_a(t) = \sum_{i=1}^{\tilde{r}} \tilde{\mu}_i(u)\left[C_i x_s(t) + G_i x_f(t)\right] \quad (17b)$$

where $y_a(t)$ is a measurable augmented output vector defined by:

$$y_a(t) = \begin{bmatrix} -\sum_{i=1}^{\tilde{r}} \tilde{\mu}_i(u)B^i_f u(t) \\ y(t) \end{bmatrix} \quad (18)$$

where the matrices C_i and G_i are given by:

$$C_i = \begin{bmatrix} A^i_{fs} \\ C_S \end{bmatrix} \qquad G_i = \begin{bmatrix} A^i_{ff} \\ C_f \end{bmatrix} \quad (19)$$

As it can be seen, the new output vector y_a is no more linear in the state variable, as the initial output vector y, but becomes nonlinear.

The system (17) can be considered as a MM affected by the unknown inputs x_f. Let us note the unknown input $d(t) = x_f(t)$, with the following property:

$$\dot{d}(t) = 0 \quad (20)$$

The assumption of a constant $d(t)$ is classically needed in the framework of PIO design for the theoretical proof of the convergence of the state estimation error [10]. Nevertheless it is well known that the only practical need is to have a low frequency signal. One should not be confused by this assumption made an a signal called *fast*. The vocable fast refers to the dynamics of f_s. Due to this, $x_f(t)$ is -for $\varepsilon \to 0$- a static function of $x_s(t)$ which obeys to a slow dynamic process. As a consequence, when $\varepsilon \to 0$ (i.e. when neglecting the dynamic behavior of the fast part of the system), $x_f(t)$ is also a slow signal.

Let us construct the augmented state vector $x_a^T = \left[x_s^T \; d^T\right]$ and denote:

$$\tilde{A}_i = \begin{bmatrix} A^i_{ss} & A^i_{sf} \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} B^i_s \\ 0 \end{bmatrix}, \quad \tilde{C}_i = \begin{bmatrix} C_i & G_i \end{bmatrix}$$

Using the previous notations and the property of unknown inputs (20), the system (17) is equivalent to a system under an augmented form, as following:

$$\dot{x}_a(t) = \sum_{i=1}^{r} \mu_i(x_a(t),u(t))\left[\tilde{A}_i x_a(t) + \tilde{B}_i u(t)\right] \quad (21a)$$

$$y_a(t) = \sum_{i=1}^{r} \mu_i(x_a(t),u(t))\tilde{C}_i x_a(t) \quad (21b)$$

Since x_a is unknown, the following form with $\mu_i(\hat{x}_a,u)$ is used:

$$\dot{x}_a(t) = \sum_{i=1}^{r} \mu_i(\hat{x}_a,u)\left[\tilde{A}_i x_a(t) + \tilde{B}_i u(t)\right] + \Gamma\omega(t) \quad (22a)$$

$$y_a(t) = \sum_{i=1}^{\tilde{r}} \tilde{\mu}_i(u)\tilde{C}_i x_a(t) \quad (22b)$$

where $\Gamma = [I \; 0]^T$ and the term $\omega(t)$ plays the role of a bounded disturbance of the form:

$$\omega(t) = \sum_{i=1}^{r} \left[\mu_i(x_a,u) - \mu_i(\hat{x}_a,u)\right]\left[\tilde{A}_i x_a(t) + \tilde{B}_i u(t)\right] \quad (23)$$

B. Proportional Integral Observer

The following proportional integral observer is proposed:

$$\dot{\hat{x}}_a(t) = \sum_{i=1}^{r} \mu_i(\hat{x}_a,u)\left[\tilde{A}_i\hat{x}_a(t) + \tilde{B}_i u(t) + K_i(y_a(t) - \hat{y}_a(t))\right] \quad (24a)$$

$$\hat{y}_a(t) = \sum_{i=1}^{\tilde{r}} \tilde{\mu}_i(u)\tilde{C}_i x_a(t) \quad (24b)$$

The state estimation error is given by

$$e_a(t) = x_a(t) - \hat{x}_a(t) \quad (25)$$

Taking into account (22a) and (24) the augmented state estimation error is governed by:

$$\dot{e}_a(t) = \sum_{i=1}^{r}\sum_{j=1}^{\tilde{r}} \mu_i(\hat{x}_a,u)\tilde{\mu}_j(u)(\tilde{A}_i - K_i\tilde{C}_j)e_a(t) + \Gamma\omega(t) \quad (26)$$

One can see that the dynamic of the state estimation error is only disturbed by $\omega(t)$.

Theorem 1: The optimal proportional integral observer (24) for the system (22) is obtained if there exists a symmetric positive definite matrix X, matrices M_i and a positive scalar λ, minimizing λ under the LMI constraints (27) for $i = 1,\dots,r$ and $j = 1,\dots,\tilde{r}$.

$$\begin{bmatrix} \tilde{A}_i^T X + X\tilde{A}_i - \tilde{C}_j^T M_i^T - M_i\tilde{C}_j + I & X\Gamma \\ \Gamma^T X & -\lambda I \end{bmatrix} < 0 \quad (27)$$

The observer gains are given by: $K_i = X^{-1} M_i$.

Proof: The state estimation error $e_a(t)$ converges to zero when $\omega = 0$ and the $\mathscr{L}_2$ gain from $\omega(t)$ to $e_a(t)$ is bounded by γ if there exists $X = X^T > 0$ such that the following inequalities hold for $i = 1,\dots,r$ and $j = 1,\dots,\tilde{r}$ [3]:

$$\begin{bmatrix} \phi_{ij}^T X + X\phi_{ij} + I & X\Gamma \\ \Gamma^T X & -\gamma^2 I \end{bmatrix} < 0 \quad (28)$$

where $\phi_{ij} = \tilde{A}_i - K_i\tilde{C}_j$ is used. With $\lambda = \gamma^2$ and $M_i = XK_i$ the LMI (27) is obtained. ■

Remark 2: In order to improve the estimation quality, the conditions (29) (for all $i = 1,...,r$ and $j = 1,\dots,\tilde{r}$) can be added to (27) to ensure pole clustering [5]:

$$\tilde{A}_i^T X + X\tilde{A}_i - \tilde{C}_j^T M_i^T - M_i\tilde{C}_j + 2\alpha X < 0 \quad (29)$$

These conditions ensure that the eigenvalues of the generating system of the state estimation error (26) lie in the following region of the complex plane:

$$\mathscr{S}(\alpha,\rho) = \{ w \in \mathbb{C} | Re(w) < -\alpha, |w| < \rho\} \quad (30)$$

IV. Application

A. Process description and nonlinear model

The wastewater treatment with activated sludge is widely used in the last two centuries [14]. It consists in putting in contact waste water with a mixture rich in bacteria to degrade and eliminate the polluting constituents contained in the water, in suspension or dissolved. The functioning principle of the process is briefly described after. The simplified diagram, given in Fig. 1, includes a bioreactor and a clarifier. In this figure q_{in} represents the input flow, q_{out} the bioreactor output flow, q_a the air flow, q_R, q_W are respectively the recycled and the rejected flows. The reactor volume is assumed to be constant and thus: $q_{out} = q_{in} + q_R$. In general, q_R and q_W represent fractions of input flow q_{in}:

$$q_R(t) = f_R\, q_{in}(t), \quad 1 \le f_R \le 2 \quad (31)$$
$$q_W(t) = f_W\, q_{in}(t), \quad 0 < f_W < 1 \quad (32)$$

The polluted circulates in the bioreactor in which the bacterial biomass degrades the organic matter. Micro-organisms gather together in colonial structures called flocs and produce sludges. The mixed liqueur is then sent to the clarifier where the separation of the purified water and the flocs is made by gravity. A fraction of settled sludges is recycled towards the bioreactor to maintain its capacity of purification. The purified water is thrown in the natural environment. The

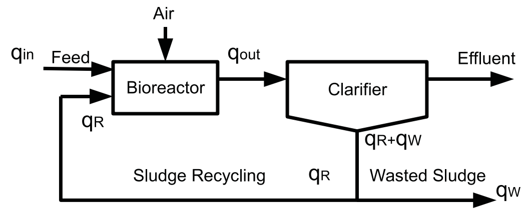

Fig. 1. The diagram of activated sludge wastewater treatment

ASM1 is a commonly used model to describe this process. For simplicity reasons and lack of space, only the carbon pollution of the activated sludge reactor is considered. Thus, the proposed estimation method is illustrated by using a model with three state variables $x = [S_S, S_O, X_{BH}]^T$:

$$\dot{S}_S(t) = -\frac{1}{Y_H}\mu_H\varphi_1(t) + (1 - f_P)b_H\varphi_2(t) + D_1(t)$$
$$\dot{S}_O(t) = \frac{Y_H - 1}{Y_H}\mu_H\varphi_1(t) + D_2(t)$$
$$\dot{X}_{BH}(t) = \mu_H\varphi_1(t) - b_H\varphi_2(t) + D_3(t) \quad (33)$$

where:

$$D_1(t) = \frac{q_{in}(t)}{V}[S_{S,in}(t) - S_S(t)]$$
$$D_2(t) = \frac{q_{in}(t)}{V}[S_{O,in}(t) - S_O(t)] + Kq_a(t)[S_{O,sat} - S_O(t)]$$
$$D_3(t) = \frac{q_{in}(t)}{V}\left[X_{BH,in}(t) - X_{BH}(t) + f_R\frac{1 - f_W}{f_R + f_W}X_{BH}(t)\right] \quad (34)$$

The process kinetics are:

$$\varphi_1(t) = \frac{S_S(t)}{K_S + S_S(t)}\frac{S_O(t)}{K_{OH} + S_O(t)}X_{BH}(t) \quad (35)$$
$$\varphi_2(t) = X_{BH}(t) \quad (36)$$

The variables involved are presented in table I. We suppose that the dissolved oxygen concentration at the reactor input ($S_{O,in}$) is null.

The clarifier is supposed to be perfect, i.e. with no internal dynamic process and no biomass in the effluent. In this case, we can write at each time instant:

$$[q_{in}(t) + q_R(t)]X_{BH}(t) = [q_R(t) + q_W(t)]X_{BH,R}(t) \quad (37a)$$
$$S_{S,R}(t) = S_S(t) \quad (37b)$$

The following heterotrophic growth and decay kinetic parameters are considered [14]: $\mu_H = 3.733[1/24h]$, $K_S = 20[g/m^3]$, $K_{OH} = 0.2[g/m^3]$, $b_H = 0.3[1/24h]$. The stoichiometric parameters are $Y_H = 0.6$[g cell formed], $f_P = 0.1$ and the oxygen saturation concentration is $S_{O,sat} = 10[g/m^3]$. The following numerical values are considered here for the fractions f_R and f_W: $f_R = 1.1$ and $f_W = 0.04$.

	Reactor input	Reactor output	Recycled
X_{BH}	Heterotrophic biomass concentration		
	$X_{BH,in}$	$X_{BH,out}$	$X_{BH,R}$
S_S	Fast biodegradable substrate concentration		
	$S_{S,in}$	$S_{S,out}$	$S_{S,R}$
S_O	Dissolved oxygen concentration		
	$S_{O,in}$	$S_{O,out}$	$S_{O,R}$
q	Flow		
	q_{in}	q_{out}	q_R
q_a	Air flow		
V	Reactor volume		

TABLE I

TABLE OF VARIABLES

B. Slow and fast variables

Let us consider the linearization of the nonlinear system (33) around various equilibrium points (x_0, u_0):

$$\dot{x}(t) = A_0 x(t) + B_0 u(t) \tag{38}$$

where $A_0 = \frac{\partial f(x,u)}{\partial x}\,|_{(x_0,u_0)}$ and $B_0 = \frac{\partial f(x,u)}{\partial u}\,|_{(x_0,u_0)}$.
Considering $Re(\lambda_1) \le Re(\lambda_2) \le ... \le Re(\lambda_n)$ the ordered real part eigenvalues of A_0, the biggest (resp. smallest) real part of eigenvalues correspond to the slowest (resp. fastest) dynamic. This separation will be made by fixing a threshold of separation of both time scales, τ, such as: $Re(\lambda_1) \le ... \le Re(\lambda_{n_f}) << \tau \le Re(\lambda_{n_f+1}) \le ... \le Re(\lambda_n)$.
For the reduced ASM1 (33), the slow and fast separation is confirmed by the eigenvalues of the jacobian A_0, as one can notice on Fig. 2 displaying the real parts of these eigenvalues for forty operating points. The real part of two eigenvalues (λ_2 and λ_3) are included between -50 and -0.4 and the real part of the other (λ_1) between -175 and -250. Setting a threshold at $\tau = 70$, it can be deduced that the system has one fast dynamic ($x_f = S_S$) and two slow dynamics ($x_s = [S_O \;\; X_{BH}]^T$).

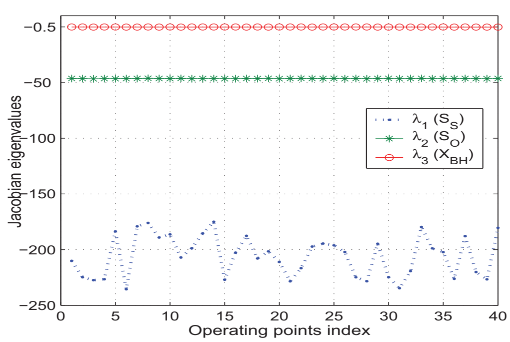

Fig. 2. The real parts of jacobian eigenvalues in various points of the operating space

C. Multiple model

A multiple model is built and used to design an observer allowing slow and fast state estimation.
Considering the process equations (33) and (34), it is natural to define the following decision variables:

$$z_1(u(t)) = \frac{q_{in}(t)}{V} \tag{39a}$$

$$z_2(x(t)) = \frac{1}{K_S + S_S(t)}\frac{S_O(t)}{K_{OH} + S_O(t)} X_{BH}(t) \tag{39b}$$

$$z_3(u(t)) = q_a(t) \tag{39c}$$

The input vector is defined by:

$$u(t) = [\; S_{S,in}(t) \quad q_a(t) \quad X_{BH,in}(t) \;]^T \tag{40}$$

The quasi-LPV form of the model (33) is characterized by matrices $A(t) = A(x(t), u(t))$ and $B(t) = B(u(t))$ decomposed in the following way:

$$A(t) = \begin{bmatrix} A_{ff}(t) & A_{fs}(t) \\ A_{sf}(t) & A_{ss}(t) \end{bmatrix} \quad B(t) = \begin{bmatrix} B_f(t) \\ B_s(t) \end{bmatrix} \tag{41}$$

where

$$A_{ff}(t) = \begin{bmatrix} -z_1(t) - \frac{1}{Y_H}\mu_H z_2(t) \end{bmatrix} \tag{42}$$

$$A_{fs}(t) = \begin{bmatrix} 0 & (1 - f_P) b_H \end{bmatrix} \tag{43}$$

$$B_f(t) = \begin{bmatrix} z_1(t) & 0 & 0 \end{bmatrix} \tag{44}$$

$$A_{sf}(t) = \begin{bmatrix} \frac{Y_H - 1}{Y_H}\mu_H z_2(t) \\ \mu_H z_2(t) \end{bmatrix} \tag{45}$$

$$A_{ss}(t) = \begin{bmatrix} -K z_3(t) - z_1(t) & 0 \\ 0 & \left[\frac{f_R(1-f_W)}{f_W + f_R} - 1\right] z_1(t) - b_H \end{bmatrix} \tag{46}$$

$$B_s(t) = \begin{bmatrix} 0 & K So_{sat} & 0 \\ 0 & 0 & z_1(t) \end{bmatrix} \tag{47}$$

The decomposition of the three premise variables (39) is realized by using the convex polytopic transformation, as in (5), (6) and (7). Multiplying the functions $F_{.,.}$, the $r = 8$ weighting functions $\mu_i(z(x(t)), u(t))$ are obtained:

$$\mu_i(z(x,u)) \quad = \quad F_{1,\sigma_i^1}(x,u) F_{2,\sigma_i^2}(x,u) F_{3,\sigma_i^3}(x,u)$$

The constant matrices A_i and B_i representing the 8 sub-models are defined as in (16) by using the block matrices A and B and the scalars (6), for $i = 1, ..., 8$:

$$\begin{aligned} A_{ff}^i &= A_{ff}(z_{1,\sigma_i^1}, z_{2,\sigma_i^2}) \\ A_{fs}^i &= [0 \quad (1 - f_P) b_H] \\ A_{sf}^i &= A_{sf}(z_{2,\sigma_i^2}) \\ A_{ss}^i &= A_{ss}(z_{1,\sigma_i^1}, z_{3,\sigma_i^3}) \\ B_f^i &= B_f(z_{1,\sigma_i^1}) \\ B_s^i &= B_s(z_{1,\sigma_i^1}) \end{aligned} \tag{48}$$

The model (33) is thus equivalently written under the MM form by using the separation into slow and fast states.
The output vector is defined by $y = C_f x_f + C_s x_s + \eta$, where $\eta(t)$ is a bounded measurement noise and the matrices C_s and C_f are given by:

$$C_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C_s = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{49}$$

By applying *Theorem 1* to the ASM1 model (33), represented into an equivalent MM form, the following state estimation results are obtained and presented in Fig. 3. The $\mathscr{L}_2$ gain from $\omega(t)$ to $e_a(t)$ is bounded by $\gamma = 1.057$.

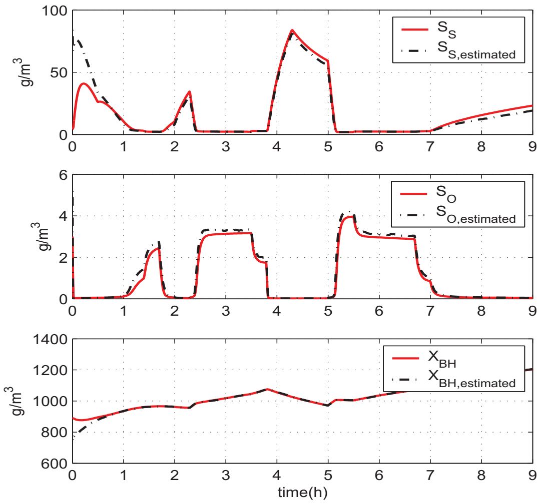

Fig. 3. State estimation using PIO

The estimation of the fast dynamic S_S, considered as unknown input in the global MM, is presented first and is followed by the estimation results of the slow dynamics S_O and X_{BH}. The output estimation results are displayed on Fig. 4, where one can see that the output noise is filtered.

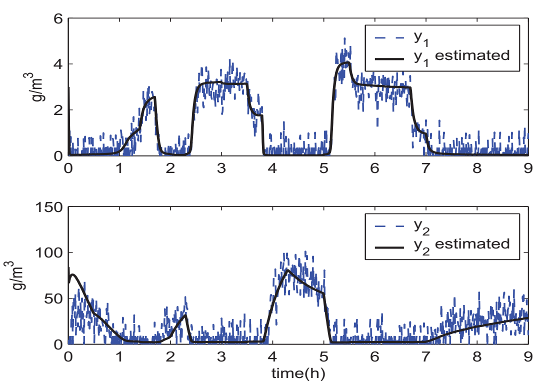

Fig. 4. Outputs

V. Conclusions and Future Works

In this article we propose state estimation of two-time scale systems represented under a MM form with unmeasurable premise variables, by means of a proportional integral observer with unknown inputs. The nonlinear system is put under a MM form that highlights the slow and fast dynamics and then the fast dynamics are considered as unknown inputs and estimated simultaneously with the slow variables. The application to a biological reactor offers good state estimation results. As future works, first the design of a proportional multi-integral observer is envisaged, and second, the extension to the complete ASM1 model.

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