# State estimation of nonlinear discrete-time systems based on the decoupled multiple model approach

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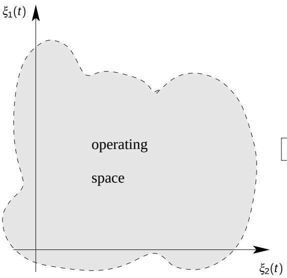
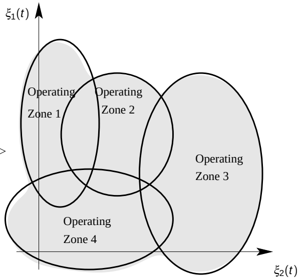
# Introduction




***NONLINEAR SYSTEM***




***MULTIPLE MODEL REPRESENTATION***

## Multiple model approach

- Decomposition of the operating space into operating zones
- Modelling each zone by a single submodel (often linear)
- The contribution of each submodel depends on a weighting function $\mu_i$
- Take judiciously into account the contribution of each submodel

# Plan

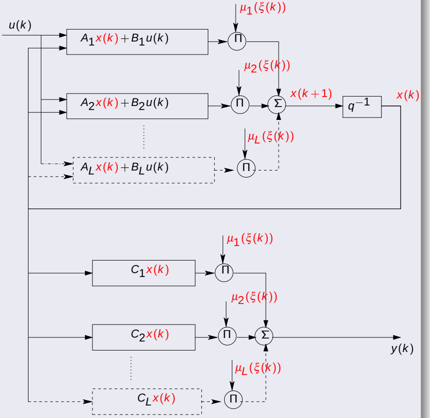
1. Multiple model structures
   - Takagi-Sugeno multiple model
   - Decoupled multiple model
2. Main results
   - Stability of decoupled multiple model
   - State estimation
3. Academic example
4. Conclusion

# Multiple model structures

# Multiple model structures

## Takagi-Sugeno multiple model

$$
\begin{aligned}
x(k+1) &= \tilde{A}(k)x(k) + \tilde{B}(k)u(k), \\
y(k) &= \tilde{C}(k)x(k), \\
\tilde{A}(k) &= \sum_{i=1}^{L} \mu_i(\xi(k))A_i, \\
\tilde{B}(k) &= \sum_{i=1}^{L} \mu_i(\xi(k))B_i, \\
\tilde{C}(k) &= \sum_{i=1}^{L} \mu_i(\xi(k))C_i,
\end{aligned}
$$




$\mu_i(\xi(k))$ : *weighting function*

$$
\begin{cases}
\sum_{i=1}^{L} \mu_i(\xi(k)) = 1, \forall k \\
0 \leq \mu_i(\xi(k)) \leq 1
\end{cases}
$$

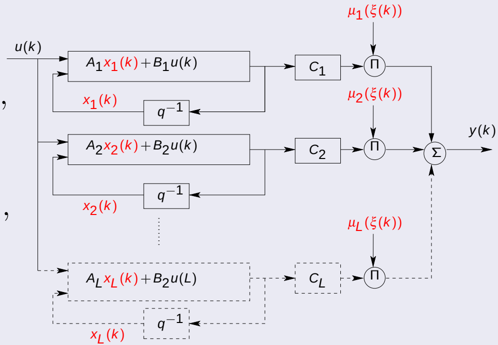
$\xi(k)$ : *is a measurable variable*

# Multiple model structures

## Proposed structure

**Decoupled multiple model**

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + B_i u(k), \\ y_i(k) &= C_i x_i(k), \\ y(k) &= \sum_{i=1}^{L} \mu_i(\xi(k)) y_i(k), \end{aligned}$$




$\mu_i(\xi(k))$ : *weighting function*

$$\begin{cases} \sum_{i=1}^{L} \mu_i(\xi(k)) = 1, \forall k \\ 0 \leq \mu_i(\xi(k)) \leq 1 \end{cases}$$

# Multiple model structures

## Takagi-Sugeno multiple model

- Similar to a system whose parameters vary with time (blending parameters),
- The submodels share the same state vector,
- **Dimension of submodels are identical**.

## Decoupled multiple model

- The output of multiple model is given by a weighted sum of the submodel outputs (blending outputs),
- Each submodel evolves independently (an particular state space),
- **Dimension of submodels can be different**,
- Using for modelling complex nonlinear system whose complexity is not uniform.

# Main results

# Stability of the decoupled multiple model

## Rewrite the equations of the multiple model

$$\begin{aligned} x_i(k+1) &= A_i x_i(k) + B_i u(k), \\ y_i(k) &= C_i x_i(k), \\ y(k) &= \sum_{i=1}^{L} \mu_i(\xi(k)) y_i(k), \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x(k+1) &= \tilde{A} x(k) + \tilde{B} u(k), \\ y(k) &= \tilde{C}(k) x(k), \end{aligned}$$

where:

$$\tilde{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & A_i & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & A_L \end{bmatrix}, \tilde{B} = \begin{bmatrix} B_1 \\ \vdots \\ B_i \\ \vdots \\ B_L \end{bmatrix}, \tilde{C}(k) = \begin{bmatrix} \mu_1(k) C_1^T \\ \vdots \\ \mu_i(k) C_i^T \\ \vdots \\ \mu_L(k) C_L^T \end{bmatrix}^T$$

and $\quad x(k) = \left[ x_1^T(k) \cdots x_i^T(k) \cdots x_L^T(k) \right]^T \in \mathbb{R}^n, n = \sum_{i=1}^{L} n_i.$

# Stability of the decoupled multiple model

## Stability criterion

The decoupled multiple model is stable if and only if all submodels are stable

## Stability analysis

By analysing the eigenvalues of the matrix

$$\tilde{A} = \begin{bmatrix} A_1 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & A_i & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & A_L \end{bmatrix},$$

- $\tilde{A}$ is a block diagonal matrix,
- all eigenvalues of this matrix are inside the unit circle if and only if all eigenvalues of every matrices $A_i$ are inside the unit circle.

## Observer structure

$$\begin{aligned}
\hat{x}_i(k+1) &= A_i\hat{x}_i(k) + B_i u(k) + K_i(y(k) - \hat{y}(k)), \\
\hat{y}_i(k) &= C_i\hat{x}_i(k), \\
\hat{y}(k) &= \sum_{i=1}^{L} \mu_i(k)\hat{y}_i(k).
\end{aligned}$$

Rewrite the equations of the observer:

$$\begin{aligned}
\hat{x}(k+1) &= \tilde{A}\hat{x}(k) + \tilde{B}u(k) + \tilde{K}(y(k) - \hat{y}(k)), \\
\hat{y}(k) &= \tilde{C}(k)\hat{x}(k), \\
\tilde{K} &= \begin{bmatrix} K_1^T & \cdots & K_i^T & \cdots & K_L^T \end{bmatrix}^T, \\
\tilde{C}(k) &= \begin{bmatrix} \mu_1(k)C_1 & \cdots & \mu_i(k)C_i & \cdots & \mu_L(k)C_L \end{bmatrix} \\
\hat{x}(k) &= \begin{bmatrix} \hat{x}_1^T(k) & \cdots & \hat{x}_i^T(k) & \cdots & \hat{x}_L^T(k) \end{bmatrix}^T
\end{aligned}$$

# Observer design

## Lyapunov second method

V(e(k)) is a candidate Lyapunov function.
The convergence of the estimation error is guaranteed if:

1. $V(e(k)) > 0$
2. $\Delta V(e(k)) = V(e(k+1)) - V(e(k)) < 0$

The estimation error is given by:

$$\begin{aligned} e(k) &= x(k) - \hat{x}(k), \\ e(k+1) &= \left(\tilde{A} - \tilde{K}\tilde{C}(k)\right) e(k), \\ e(k+1) &= A_{obs}(k) e(k), \end{aligned}$$

# Observer design

## First case

Let us consider the following quadratic Lyapunov function:

$$V(e(k)) = e^T(k)Pe(k), \quad P = P^T \text{ and } P > 0.$$

After some algebraic manipulations....

## Theorem (Asymptotic convergence)

*The asymptotic convergence towards zero of the estimation error is guaranteed if there exists a symmetric and positive definite matrix P and a matrix G such that:*

$$\begin{bmatrix} P & \tilde{A}^T P - \tilde{C}_i^T G^T \\ P\tilde{A} - G\tilde{C}_i & P \end{bmatrix} > 0, \quad i = 1...L,$$

*where the observer gain is deduced from* $\tilde{K} = P^{-1}G$.

# Less conservative conditions

## Facts

- The asymptotic convergence conditions of the estimation error depend on the existence of the common matrix $P$ which satisfies a set of LMIs.
- When the multiple model has a large number of submodels, the matrix $P$ cannot be found.

## Second case

Let us consider the following Lyapunov function:

$$V(e(k)) = e^T(k) \sum_{i=1}^{L} \mu_i(k) P_i e(k) = e^T(k) P(k) e(k),$$

where $P_i = P_i^T$ and $P_i > 0$.
After some algebraic manipulations....

# Less conservative conditions

## Theorem (Less conservative condition)

*The asymptotic convergence towards zero of the estimation error is guaranteed if there exists symmetric and positive definite matrices $P_i$ and $P_j$ and a some matrix $M$ and $G$ such that:*

$$\begin{bmatrix} P_i & (M\tilde{A}-G\tilde{C}_i)^T \\ M\tilde{A}-G\tilde{C}_i & M+M^T-P_j \end{bmatrix} > 0 \quad \forall i,j = 1...L,$$

*where the observer gain is deduced from $\tilde{K} = M^{-1}G$.*

## Remark

This theorem is less conservative because if one sets $P_i = P_j = M = P$ then this theorem coincides with the previous one.

# Example

# Academic example

## Example

Let us consider the state estimation of the decoupled multiple model with $L = 3$ submodels. The numerical matrices $A_i$, $B_i$ and $C_i$ are:

$$A_1 = \begin{bmatrix} 0.8 & 0 \\ 0.4 & 0.1 \end{bmatrix}, A_2 = \begin{bmatrix} -0.3 & -0.5 & 0.2 \\ 0.7 & -0.8 & 0 \\ -2 & 0.1 & 0.7 \end{bmatrix}, A_3 = \begin{bmatrix} -0.5 & 0.1 \\ -0.6 & -0.5 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.2 & -0.4 \end{bmatrix}^T, B_2 = \begin{bmatrix} 0.7 & -0.5 & 0.3 \end{bmatrix}^T, B_3 = \begin{bmatrix} -0.2 & 0 \end{bmatrix}^T,$$

$$C_1 = \begin{bmatrix} 0.7 & 0 \\ 0.5 & 0.2 \end{bmatrix}, C_2 = \begin{bmatrix} 0.5 & 0 & 0.8 \\ 0.7 & 0.2 & 0.1 \end{bmatrix}, C_3 = \begin{bmatrix} 0.9 & 0.3 \\ -0.6 & 0 \end{bmatrix}.$$

$\xi(k)$ is the input signal $u(k)$.

- The eigenvalues of the matrix $\tilde{A}$ are inside the unit circle, thus the multiple model is stable
- we obtain the following observer gain:

$$\tilde{K} = \begin{bmatrix} 0.041 & 0.020 & 0.160 & 0.190 & 0.221 & -0.090 & -0.181 \\ 0.194 & 0.113 & -0.299 & -0.044 & -0.701 & 0.172 & 0.268 \end{bmatrix}^T.$$

# Academic example

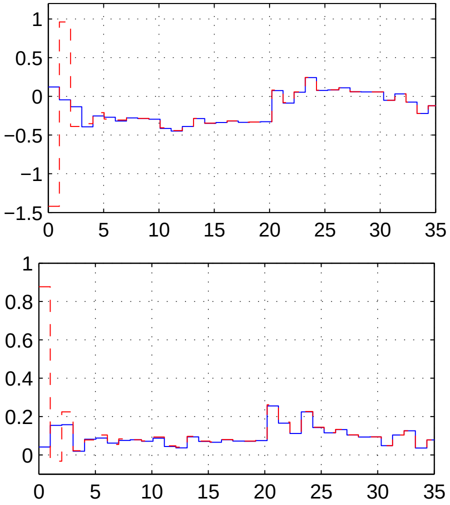



Figure: Output $y_i$ of the multiple model (solid line) and its estimated (dashed line).

# Conclusion and perspectives

## Conclusion

- A decoupled discrete time multiple observer has been presented in order to proceed to the state estimation of a class of nonlinear systems.
- Sufficient conditions that guarantee the asymptotic convergence of the estimation error are given in terms of a set of LMIs.

## Perspectives

- This class of multiple model and observer can be useful for setting up a diagnosis strategy of a nonlinear system.
- The proposed approach will be extended to other observer classes as proportional integral observer or unknown input observer.

**Thank you**