Parameter uncertainties characterisation for linear models

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Purpose and main ideas

- For a system, characterising a parameter set consistent with measurements, the model and the equation error description.
- Finding the set of admissible parameter values corresponding to an admissible error.
- The uncertainties must be treated by a global analysis of the problem : both the equation error and the parameter set are considered unknown.
- The procedure consists in explaining the measurements performed at all time by optimising a precision criterion based on the polytope theory.

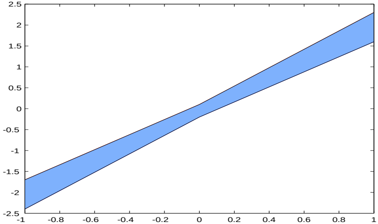
- Model structure

$$y(k) = a(k)x(k) + b(k)$$
$$a \in [1.8\ 2.2]$$
$$b \in [-0.2\ 0.1]$$

- How to use such model ?

$$x(k) = 2$$
$$3.4 \leq y(k) \leq 4.5$$

- Domain



Model with uncertain parameters in the {x, y } plan

Uncertain system : an academic example for identification

- True system with uncertain bounded parameters

$$y(k) = a(k)x(k) + b(k)$$

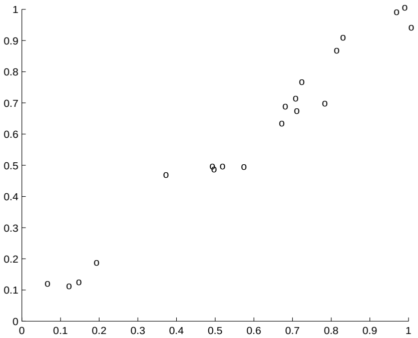
- Model with one uncertain bouded parameter and one certain parameter

$$y(k) = ax(k) + b(k)$$
$$-\delta \leq b(k) \leq \delta$$

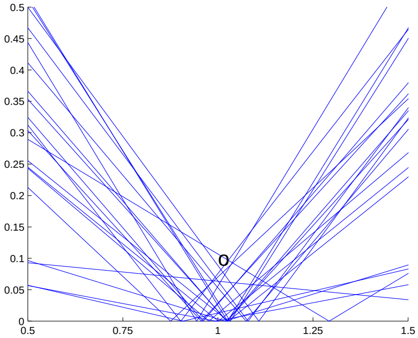
- Estimation : for all k, the parameters a and δ must verify

$$\delta + ax(k) \geq y(k)$$
$$-\delta + ax(k) \leq y(k)$$
$$\text{with } \delta \text{ minimum}$$

Uncertain system : an academic example for identification



Data in the $\{x,\ y\}$ plan



The parameter domain $\{a,\ \delta\}$

Scope of the presentation

- Introductive example
- Historical point of view
- Modelling of an uncertain system
- Principle of parameter estimation
- Example
- Conclusion

Historical point of view

- Feasible Parameter Set determination (FPS). Belforte and Milanese, 1981.
- Approximation of the FPS. Fogel and Huang, 1982.
- Robust identification approach. Reinelt, 2002.
- Set inversion techniques. Jaulin 1993.

- The present paper : parameter estimation in a bounded-error context for MIMO systems. The parameters fluctuate inside a time-invariant bounded domain represented by a zonotope.

Model structure

- MISO model :

$$y_m(k) = x^T(k)\theta(k) \qquad k = 1..N$$
$$y(k) = y_m(k) + e(k)$$

$y_m(k)$ is the model output
$x(k) \in \mathcal{R}^p$ is the regressor vector at the instant k
$y(k)$ is the output measurement
$\theta(k) \in \mathcal{R}^p$ defines the uncertain parameter vector.

- Hypothesis : the error $e(k)$ is bounded, the bounds being supposed invariant along the time :

$$e(k) \in [-\delta, \delta]$$

- MISO model :

$$y_m(k) = x^T(k)\theta(k) \qquad k = 1..N$$
$$y(k) = y_m(k) + e(k)$$

- As $e(k) \in [-\delta, \delta]$, one has :

$$y(k) - \delta \leq x^T(k)\theta(k) \leq y(k) + \delta \qquad k = 1..N$$

  Thus, the known measurement $y(k)$ belongs to an interval.
  The width of this interval depends both on the bound δ and the value set of $\theta(k)$.

- In the following, our objective is to search if there exists, at each instant k, at least one value of $(\theta(k), \delta)$ satisfying the constraints.

Parameter characterization problem

- Hypothesis : the uncertain parameters are described by

$$D_N = \{\theta(k) \in \mathcal{R}^p / \theta(k) = \theta_c + M(\lambda)\nu(k)\}$$
$$M(0) = 0, \qquad \| \nu(k) \|_\infty \leq 1$$

- Estimation : find the central parameter θ_c and the parameter uncertainties λ taking into account the inequality :

$$y(k) - \delta \leq x^T(k)\theta(k) \leq y(k) + \delta \qquad k = 1..N$$

- Model of the system. The MIMO case :

$$Y(k) = X(k)\theta(k) + E(k)$$

- Uncertainties modelling

$$E(k) \in \mathcal{P}_E(\delta) \quad \mathcal{P}_E(\delta) = \{Z(\delta)u, \parallel u \parallel_\infty \leq 1\}$$
$$\theta(k) \in \mathcal{P}_\theta(\lambda, \theta_c) \quad \mathcal{P}_\theta(\lambda, \theta_c) = \{\theta(k) = \theta_c + M(\lambda)\nu(k), \parallel \nu(k) \parallel_\infty \leq 1\}$$

- Example

$$y_1(k) = x_1(k)(\theta_{1c} + \lambda_1\nu(k)) + x_2(k)\theta_{2c} + \delta_1 u(k)$$
$$y_2(k) = x_3(k)\theta_{3c} + x_1(k)(\theta_{1c} + \lambda_1\nu(k)) + \delta_2 u(k)$$
$$\mid \nu(k) \mid \leq 1, \quad \mid u(k) \mid \leq 1$$

- The model has to explain all the available measurements (k)

$$\tilde{Y}(k) \in \mathcal{P}_Y(\lambda, \delta, \theta_c) \quad k = 1..N$$

$$\mathcal{P}_Y(\lambda, \delta, \theta_c) = \{Y(k)/Y(k) = \tilde{X}(k)(\theta_c + M(\lambda)\nu(k)) + Z(\delta)u(k), \\ \| u(k) \|_\infty \leq 1, \| \nu(k) \|_\infty \leq 1\}$$

- Some transformations are needed to isolate the model parameters and the uncertainties. Indeed if $\tilde{Y}(k) \in \mathcal{P}_Y(\lambda, \delta, \theta_c)$ then :

$$\exists w(k) \in \mathcal{H}_{q+n}/\tilde{Y}(k) = \tilde{X}(k)\theta_c + T(k, \lambda, \delta)w(k)$$

$$T(k, \lambda, \delta) = (\tilde{X}(k)M(\lambda) \quad Z(\delta))$$

$$w(k) = \begin{pmatrix} \nu(k) \\ u(k) \end{pmatrix} \qquad \| w(k) \|_\infty \leq 1$$

- Equivalently :

$$\exists w(k) \in \mathcal{H}_{q+n} / \tilde{Y}(k) = \tilde{X}(k)\theta_c \ + T(k,\lambda,\delta)w(k)$$

$$\Leftrightarrow \tilde{X}(k)\theta_c \ - \mid T(k,\lambda,\delta) \mid \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix} \leq \tilde{Y}(k) \leq \tilde{X}(k)\theta_c \ + \mid T(k,\lambda,\delta) \mid \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix}$$

- Additionnal hypothesis : linearity dependancy

$$Z(\delta) = Diag(\delta) \quad \text{and} \quad M(\lambda) = MDiag(\lambda)$$

$$T(k,\lambda,\delta) = (\lambda_1 t_1(k) \dots \lambda_q t_q(k) \ \ \delta_1 e_1 \dots \delta_n e_n)$$

$$\tilde{X}(k)\theta_c - \mid \tilde{T}(k) \mid \begin{pmatrix} \lambda \\ \delta \end{pmatrix} \leq \tilde{Y}(k) \leq \tilde{X}(k)\theta_c + \mid \tilde{T}(k) \mid \begin{pmatrix} \lambda \\ \delta \end{pmatrix}$$

- Influence of the bounded variable $w(k)$ on each component of $\tilde{Y}(k)$

Principle of parameter estimation

- Exact description of the parameter domain : all the dependencies in respect to w have to be taken into account :

$$\exists w(k) \in \mathcal{H}_{q+n} / \tilde{Y}(k) = \tilde{X}(k)\theta_c \; + T(k, \lambda, \delta)w(k)$$

- For that, the matrix $R(k)$ is introduced to combine all components of $\tilde{Y}(k)$ by eliminating some components of $w(k)$:

$$\exists w(k) \in \mathcal{H}_{q+n} / R(k)\tilde{Y}(k) = R(k)\tilde{Y}_c(\theta_c, k) + R(k)\tilde{T}(k) \begin{pmatrix} \lambda \\ \delta \end{pmatrix} w(k)$$

- There exists a systematic procedure to find all the possible elimination (see full paper)

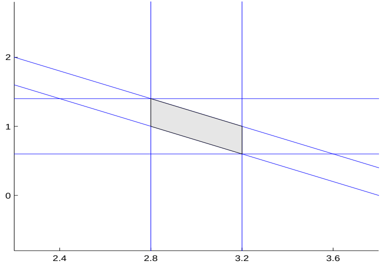
- Example

$$y_1(k) = \theta_c x(k) + w_1(k) - w_2(k)$$
$$y_2(k) = 2x(k) + \theta_c + w_2(k)$$

- Elimination of $w_2(k)$ gives :

$$y_1(k) + y_2(k) = (\theta_c + 2)x(k) + \theta_c + w_1(k)$$

- Domain in the $\{y_1,\ y_2\}$ plan



- Exact description

$$R(k)(Y(k) - Y_c(k)) = R(k)\tilde{T}(k) \begin{pmatrix} \lambda \\ \delta \end{pmatrix} w(k)$$

$$\Downarrow$$

$$R(k)Y_c(k) - \mid R(k)\tilde{T}(k) \mid \begin{pmatrix} \lambda \\ \delta \end{pmatrix} \leq R(k)\tilde{Y}(k) \leq R(k)Y_c(k) + \mid R(k)\tilde{T}(k) \mid \begin{pmatrix} \lambda \\ \delta \end{pmatrix}$$

- Precision criterion : the "best" parameter vector is those explaining all the measurements with the smaller fluctuation of the parameter $\theta(k)$. For that, we compute the distance $\delta_i(k)$ between each vertice $S_i(k)$ of $\mathcal{P}_Y(\lambda, \delta, \theta_c)$ and its center $\tilde{X}(k)\theta_c$:

$$\delta_i(k) = \| S_i(k) - \tilde{X}(k)\theta_c \|^2$$

$$J(\lambda, \delta, \theta_c) = \sum_{k=1}^{N} \sum_{i=1}^{n_k} \delta_i(k)$$

Example

- Structure of the model

$$Y(k) = X\theta(k)$$
$$\theta(k) = \theta_c + M(\lambda)\nu(k)$$

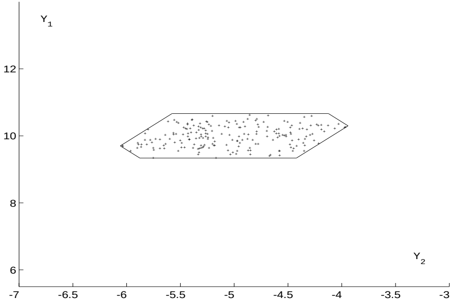
- Numerical values

$$X = \begin{pmatrix} -1.5 & 0.5 \\ -1.0 & 3.0 \end{pmatrix} \quad \theta_c = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \quad M(\lambda) = \frac{1}{10}\begin{pmatrix} \lambda_1 & \lambda_2 & 3\lambda_3 \\ \lambda_1 & -\lambda_2 & \lambda_3 \end{pmatrix}$$
$$\lambda = \begin{pmatrix} 1 & 1.25 & 2 \end{pmatrix}$$

500 observations are used

- Measurements : $Y_1(k),\ Y_2(k)$
- True domain representation

$$Y_1(k) = -5 - 0.1\nu_1(k) - 0.25\nu_2(k) - 0.8\nu_3(k)$$
$$Y_2(k) = 10 + 0.2\nu_1(k) - 0.5\nu_2(k)$$
$$\mid \nu_1(k) \mid \leq 1, \quad \mid \nu_2(k) \mid \leq 1, \quad \mid \nu_3(k) \mid \leq 1$$



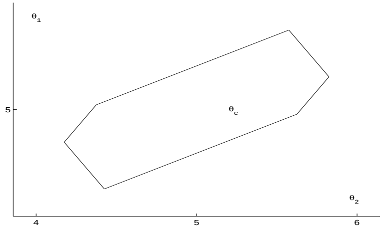
Data and true domain

Example

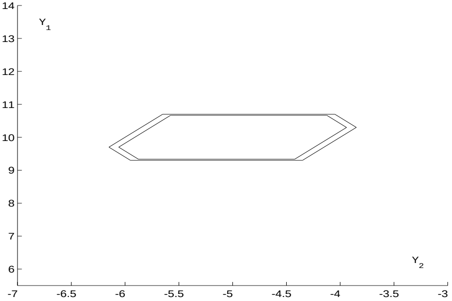
- Estimated parameters

	λ_1	λ_2	λ_3	θ_{c1}	θ_{c2}
true	1	1.5	2.0	5	5
est. (N=500)	0.987	1.246	1.988	.	.
est. (N=1000)	0.989	1.249	1.998	.	.
est. (N=500)	0.966	1.152	1.942	4.998	5.002

- Parameter domain



Example



True and estimated data domains

Conclusion

Summarizing the approach

- Modelling a time-variant parameter system
- All the measurements are needed to explain the model
- Estimation of the most precise model

Further work

- Analysing the abnormal values (outliers) influence
- Using a general polytope instead of a zonotope
- Identifying the structure of the uncertainties
- Optimizing the amount of computation