Switching Systems Mode Estimation Using A Model-Based Diagnosis Method.

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Abstract. Switching systems are a particular class of hybrid systems. They are represented by several operating regimes, called modes, each of them being active under certain particular conditions. The modes can be associated with normal or faulty operating conditions. Therefore, the determination of the active mode at any moment constitutes a fault diagnosis approach for such systems. This paper addresses the issue of the determination of the active mode at any moment, using only the system's input/output data. To perform this task, we propose an adaptation of the well-known parity space method to this class of system. Conditions that guarantee the uniqueness of the determined active mode are also given.

Keywords: switching system, estimation, mode, discernability

1. Introduction

The modeling of complex systems often leads to complex non-linear models. To get rid of the obtained model's complexity, a widely used modeling strategy is to represent the system's behavior using a set of models with simple structure, each model describing the behavior of the system in a particular operating zone. Within this modeling framework, hybrid models are very successful in representing such processes.

Hybrid models (Heemels *et al*., 2001) characterize physical processes governed by continuous differential equations and discrete variables. The process is described by several operating regimes, called modes, and the transition from one mode to another is governed by a mechanism which depends of the system's variables (input, output, state) or external variables (human operator for instance). When the transition from one mode to another is abrupt, one obtains a particular, but significant, class of systems namely switching models. This class of models is widely used because tools for analysis and control of linear systems are well mastered and, moreover, many real processes can be represented by models belonging to this class.

Switching system (Juloski *et al*., 2005) are described by several operating regimes that are often linked to linear models. The transition from one mode to another is governed by a mechanism which depends on the system's variables (input, output, state) or external variables (human operator for instance). When the nature of the switching mechanism is unknown, the determination of the mode describing the behavior of the system at any mo-

ment, this mode being called active mode, is a crucial information. We assume here that the modes describe the behavior of the real system either in normal operating conditions or in faulty operating conditions. The aim of this paper is to present a method for the determination of the active mode at any moment using the measurement data (input and output) of the system. This task is equivalent to a fault diagnostic scheme as it leads to the detection of the modes associated with faulty operating conditions. The determination of the active mode is performed thanks to an extension of the well-known parity space method (Chow and Willsky, 1984) to switching system.
The paper starts in section 2 with the problem statement. The determination of the active mode is tackled in section 3. Conditions guaranteeing the discernability of the various modes are formulated in section 5 and an illustrative example is proposed in section 6.

2. Problem statement

Let us consider the switching system represented by equation (1):

$$\begin{cases} x(k+1) = A_{\mu_k} x(k) + Bu(k) \\ y(k) = Cx(k) \end{cases} \\ A_{\mu_k} \in A = \{A_1, A_2, \ldots, A_s\}, \mu_k \in \{1, 2, \ldots, s\}, s \in \mathbb{N}^* \backslash \{1\} \tag{1}$$

In equation (1), the variables $u(\cdot)$, $y(\cdot)$ and $x(\cdot)$ respectively stand for the input, the output and the state of the system. The regime switchings are introduced by means of the state matrix which takes its value in a finite set A which is *a priori* known. The results presented in this paper can be extended to the case where the matrices B and C also take different values. The variable $\mu_{(\cdot)}$ denotes the active mode at any moment. For example, if one has $\mu_k = i$, $i \in \{1, 2, \ldots, s\}$, the system is said to be in the mode i at the instant k. We assume that the switchings are triggered by unknown external variables and then, the mode sequence is arbitrary and independent of the system's variable (input, output and state). Coming from (1), we wish to recover the active mode (or the value taken by $\mu_{(\cdot)}$) at any moment, using only the system's input/output data on a finite observation window. We introduce the following definitions:

Definition 1 (Path) *A path μ is a finite sequence of modes: $\mu = (\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_h)$. The length of a path μ is denoted $|\mu|$ and Θ_h denotes the set of all paths of length $|\mu|$.*
$\mu_{[i,j]}$ is the infix of the path μ between i and j: $\mu_{[i,j]} = (\mu_i \cdot \mu_{i+1} \cdot \ldots \cdot \mu_j)$.

Definition 2 (Observability matrix) *The observability matrix $\mathcal{O}_{\mu,h}$ of a path $\mu \in \Theta_h$ is defined as:*

$$\mathcal{O}_{\mu,h} = \begin{pmatrix} C \\ CA_{\mu_1} \\ \vdots \\ C\underbrace{A_{\mu_{h-1}} A_{\mu_{h-2}} \cdots A_{\mu_1}}_{h-1} \end{pmatrix} \tag{2}$$

Definition 3 (Active path) *On a finite observation window $[k-h,\ k]$, the active path μ^* is the one describing the true mode sequence on the observation window.*

From definitions 1 and 3, the estimation of the active mode at any moment is equivalent to the determination of the path describing the true mode sequence on a finite observation window. For that, throughout the remainder of this paper, we will focus on the recovery of the active path on an observation window.

3. Determination of the active path

The determination of the active path task can be formulated as a recursive problem applied to a sliding window. On a time window $[k-h,\ k]$, equation (1) can be written in a compact way as:

$$Y_{k-h,k} - \mathcal{T}_{\mu,h}U_{k-h,k} = \mathcal{O}_{\mu,h}x(k-h) \tag{3}$$

where $Y_{k-h,k} = \begin{pmatrix} y(k) & \dots & y(k-h) \end{pmatrix}^T$, $U_{k-h,k} = \begin{pmatrix} u(k) & \dots & u(k-h) \end{pmatrix}^T$ and $\mathcal{T}_{\mu,h}$ is a Toeplitz matrix defined by:

$$\mathcal{T}_{\mu,h} = \begin{pmatrix} 0 & \dots & 0 & 0 \\ CB & & 0 & 0 \\ \vdots & & \vdots & \vdots \\ C\underbrace{A_{\mu_{k-1}}\dots A_{\mu_{k-h+1}}}_{h-1}B & \dots & CB & 0 \end{pmatrix} \tag{4}$$

The relation (3) links on the time window the system's input and output to the initial state $x(k-h)$ oon the observation window. We introduce the following proposition:

Proposition 1 *The observability matrices $\mathcal{O}_{\mu,h}$ of the paths μ generated on the observation window $[k-h,\ k]$ are all of full rank: $rank\left(\mathcal{O}_{\mu,h}\right) = \dim(x) = n, \forall h \geq n$.*

The existence of an integer h, such that proposition 1 holds, has been analyzed in (Babaali *et al*., 2003) and is linked to pathwise observability that have been furthermore shown to be decidable.
Using proposition 1, a projection matrix[1] $\Omega_{\mu,h}$ is defined in such a way that $\Omega_{\mu,h}\mathcal{O}_{\mu,h} = 0$, i.e. $\Omega_{\mu,h}$ is selected as a basis for the left null space of $\mathcal{O}_{\mu,h}$.
Next, residuals $r_{\mu,h}(\cdot)$, independent of the initial state $x(k-h)$, can be defined as:

$$r_{\mu,h}(k) = \Omega_{\mu,h}(Y_{k-h,k} - \mathcal{T}_{\mu,h}U_{k-h,k}) \tag{5}$$

The residuals $r_{\mu,h}(\cdot)$ depends only on the system's input and output and their calculation requires the preliminary determination of all matrices $\Omega_{\mu,h}$.

Theorem 1 (Active path determination) *The active path μ^* describing the true mode sequence on a time window $[k-h,\ k]$ satisfies :*

$$r_{\mu^*,h}(k) = \Omega_{\mu^*,h}\left(Y_{k-h,k} - \mathcal{T}_{\mu^*,h}U_{k-h,k}\right) = 0 \tag{6}$$

To recover the true mode sequence μ^* from the system's measurements, one can proceed in the following way:

- first, all the possible paths of length h are built on the time window $[k-h,\ k]$. This is equivalent to finding all the matrices $\mathcal{O}_{\mu,h}$.
- knowing the matrices $\mathcal{O}_{\mu,h}$, the projection matrices $\Omega_{\mu,h}$ are easily calculated.
- from the matrices $O_{\mu,h}$ and $\Omega_{\mu,h}$, one can form the residuals $r_{\mu,h}(\cdot)$ using the system's measurements.

[1] In fact, the existence of the projection matrix is directly linked to the observability of the system and to the length of the observation window (Gertler, 1997)

- the active path is recovered from the system's measurements by testing the residuals $r_{\mu,h}(\cdot)$ and it corresponds to the one which residual is equal to zero.

Let us noticed that the same methodology can be applied for the determination of the active in the case of noisy measurement by allowing the test (6) to take into account the presence of noise on the system measurement (Domlan *et al.*, 2006). Finally, once the active mode is identified, the state $x(\cdot)$ of the system can be estimated using a finite memory observer (Ragot *et al.*, 2003).

4. On the number of path

It is easy to see that the enumeration of all paths on a time window $[k-h,\ k]$ introduces a problem of combinative explosion related on the number of modes and the length of the observation window. Indeed, the number of residuals $r_{\mu,h}(\cdot)$, $\mu \in \Theta_h$, to be calculated is equal to s^h and quickly grows with the length $h+1$ of the observation window and the number s of modes. Then, the use of all paths on a time window is awkward and computationally demanding. In practice, all paths $\mu \in \Theta_h$ do not have to be considered at every moment. When at a time k_0, the active path on an observation window $[k_0-h, k_0]$ is identified, it is not necessary to test the s^h residuals at the next instant k_0+1. Only the paths $\mu \in \Theta_h$ with infixes $\mu_{[k_0+1-h,k_0-1]}$ identical to the infix $\mu^*_{[k_0+1-h,k_0-1]}$ of the path μ^* recovered previously at k_0 are considered at the next instant k_0+1. Moreover, assuming that the minimum sojourn time in a mode is less than the length of the observation window, one can limit the number of generated paths by only considering paths that describe the mode sequence when the system remains in the same mode all over the duration of the observation window, i.e. $\mu = (i \cdot i \cdot \ldots \cdot i)$, $i \in \{1, 2, \ldots, s\}$. Nevertheless, the reduction of the number of residuals comes at the expense of a delay in the estimation of the switching time from one mode to another. The recognition of the active path cannot take place as long as the switching instant is in the observation window. Thus, a maximum delay equals to the length of the observation window exists.
Prior knowledge on the process such as "prohibited" switching sequences or minimal time between two consecutive switchings, can also help to limit the number of generated residuals or paths to be considered.

5. Path discernability

In what follows, we are interested in the conditions guaranteeing the discernability of the various paths enumerated on an observation window. These conditions ensure the uniqueness of the recovered active path μ^* during the path recognition process.

Definition 4 (Path discernability) *Two paths $\mu^1 \in \Theta_h$ and $\mu^2 \in \Theta_h$ are discernible on an observation window $[k-h,\ k]$ if their respective corresponding residuals $r_{\mu^1,h}(\cdot)$ and $r_{\mu^2,h}(\cdot)$ are not simultaneously null when one of the two paths is active on the considered observation window.*

In order to establish the discernability conditions of two different paths, let us consider two paths $\mu^1 \in \Theta_h$ and $\mu^2 \in \Theta_h$ on an observation window $[k-h,\ k]$. We denote $Y^{\mu^1}_{k-h,k}$ (respectively $Y^{\mu^2}_{k-h,k}$) the output vector related to the system when it undergoes the path μ^1 (respectively μ^2). We suppose that at an instant k, the active path on the observation window is the path μ^1. This information being unknown, we have to analyze the possibilities that the path μ^1 or the path μ^2 are in adequacy with the system's data. Using equation (5), the expressions of the residuals $r_{\mu^1,h}(\cdot)$ and $r_{\mu^2,h}(\cdot)$, when μ^1 is the active path, are given by:

$$\begin{cases} r_{\mu^1,h}(k) = 0 \\ r_{\mu^2,h}(k) = \Omega_{\mu^2,h}\left(Y^{\mu^1}_{k-h,k} - \mathcal{T}_{\mu^2,h}U_{k-h,k}\right) \end{cases} \tag{7}$$

Adding and taking away $Y^{\mu^2}_{k-h,k}$ from the expression of $r_{\mu^2,h}(\cdot)$, one obtains:

$$\begin{cases} r_{\mu^1,h}(k) = 0 \\ r_{\mu^2,h}(k) = \Omega_{\mu^2,h}\left(Y^{\mu^1}_{k-h,k} - Y^{\mu^2}_{k-h,k} + Y^{\mu^2}_{k-h,k} - \mathcal{T}_{\mu^2,h}U_{k-h,k}\right) \end{cases} \tag{8}$$

As by definition $\Omega_{\mu^2,h}\left(Y^{\mu^2}_{k-h,k} - \mathcal{T}_{\mu^2,h}U_{k-h,k}\right) = 0$, one has:

$$\begin{cases} r_{\mu^1,h}(k) = 0 \\ r_{\mu^2,h}(k) = \Omega_{\mu^2,h}\left(Y^{\mu^1}_{k-h,k} - Y^{\mu^2}_{k-h,k}\right) \end{cases} \tag{9}$$

Equation (9) clearly points out that the residual calculated for the path μ^2 (nonactive path) directly depends on the difference between the system's outputs when the mode sequence evolves according to the two paths μ^1 and μ^2, the system being excited by the same inputs in both cases. Therefore, a necessary and sufficient condition for the residuals $r_{\mu^1,h}(k)$ and $r_{\mu^2,h}(k)$ not to be simultaneously equal to zero is:

$$Y^{\mu^1}_{k-h,k} - Y^{\mu^2}_{k-h,k} \notin \mathcal{N}_r(\Omega_{\mu^2,h}) \tag{10}$$

where $\mathcal{N}_r$ stands for the operator "right null space".
Condition (10) has to be analyzed in order to deduce the discernability conditions. According to (3), one has:

$$Y^{\mu^1}_{k-h,k} - Y^{\mu^2}_{k-h,k} = \left(\mathcal{O}_{\mu^1,h} - \mathcal{O}_{\mu^2,h}\right)x(k-h) + \left(\mathcal{T}_{\mu^1,h} - \mathcal{T}_{\mu^2,h}\right)U_{k-h,k} \tag{11}$$

where $x(k-h)$ is the value of the system's state at the initial instant of the observation window.
From (11), we obtain:

$$\Omega_{\mu^2,h}(Y^{\mu^1}_{k-h,k} - Y^{\mu^2}_{k-h,k}) = \Omega_{\mu^2,h}\mathcal{O}_{\mu^1,h}x(k-h) + \Omega_{\mu^2,h}\left(\mathcal{T}_{\mu^1,h} - \mathcal{T}_{\mu^2,h}\right)U_{k-h,k} \tag{12}$$

If $Y^{\mu^1}_{k-h,k} - Y^{\mu^2}_{k-h,k}$ belongs to the right null space of $\Omega_{\mu^2,h}$, one has:

$$\Omega_{\mu^2,h}\mathcal{O}_{\mu^1,h}x(k-h) + \Omega_{\mu^2,h}\left(\mathcal{T}_{\mu^1,h} - \mathcal{T}_{\mu^2,h}\right)U_{k-h,k} = 0 \tag{13}$$

The relation is satisfied "for almost every initial state"[2] $x(k-h)$ if the following necessary and sufficient condition is satisfied:

$$\begin{cases} \Omega_{\mu^2,h}\mathcal{O}_{\mu^1,h} = 0 \\ \Omega_{\mu^2,h}\left(\mathcal{T}_{\mu^1,h} - \mathcal{T}_{\mu^2,h}\right)U_{k-h,k} = 0 \end{cases} \tag{14}$$

Therefore, the paths μ^1 and μ^2 are not discernible on a time window $[k-h,\ k]$ if the relations (14) are satisfied.

[2]see remark 1 for the explanation of the expression "for almost every initial state"

Theorem 2 (Path discernability) *Two paths μ^1 and μ^2 of a switching system are discernible on an observation window $[k-h,\ k]$, "for almost every initial state" $x(k-h)$, if:*

$$\Omega_{\mu^i,h}\mathcal{O}_{\mu^j,h} \neq 0, \quad i,j \in \{1,2\}, i \neq j \tag{15}$$

or

$$\Omega_{\mu^i,h}\left(\mathcal{T}_{\mu^j,h} - \mathcal{T}_{\mu^i,h}\right)U_{k-h,k} \neq 0 \quad i,j \in \{1,2\}, i \neq j \tag{16}$$

where $U_{k-h,k}$ is the vector containing the system's input stacked on the observation window.

The proof of this theorem directly comes from the preceding remarks.

Remark 1 (Dependency to the initial state) *In theorem 2, the expression "for almost every initial state" holds owing to the fact that the discernability of the paths cannot be ensured for any initial state $x(k-h)$. In fact, for certain particular values of $x(k-h)$, the relation (13) is always satisfied independently of the input sequence $U_{k-h,k}$. For example, in the situation where $\mathcal{O}_{\mu^1,h}$ has full rank, for $x(k-h) = \left(\mathcal{O}_{\mu^1,h}\right)^\dagger\left(\Phi - \left(\mathcal{T}_{\mu^1,h} - T_{\mu^2,h}\right)U_{k-h,k}\right)$, equation (13) is satisfied for every input sequence $U_{k-h,k}$, where Φ belongs to the right null space of $\Omega_{\mu^2,h}$ and $\left(\mathcal{O}_{\mu^1,h}\right)^\dagger$ is the pseudo-inverse of $\mathcal{O}_{\mu^1,h}$.*

6. Academic example

We present here an academic example of a switching system characterized by three modes and the matrices of the models describing the different modes are:

$$\begin{array}{l} A_1 = \begin{pmatrix} -0.211 & 0 \\ 0 & 0.521 \end{pmatrix} A_2 = \begin{pmatrix} 0.691 & 0 \\ 0 & -0.310 \end{pmatrix} \\ A_3 = \begin{pmatrix} 0.153 & 0 \\ 0 & 0.410 \end{pmatrix} B = \begin{pmatrix} 2 & -1 \end{pmatrix}^T C = \begin{pmatrix} 1 & 2 \end{pmatrix} \end{array} \tag{17}$$

Figure 1 shows the input $u(\cdot)$, the output $y(\cdot)$, the state $x(\cdot)$ and the mode sequence $\mu_{(\cdot)}$. The vertical dashed lines on the third graphic of figure 1 mark the time instants where switchings occur. The fourth graphic plots the mode sequence described by the mode's selection variable $\mu_{(\cdot)}$.

As $\Omega_{l^i}^{k,h} O_{l^j}^{k,h} \neq 0$, $\mu^i, \mu^j \in \Theta_2$, $\mu^i \neq \mu^j$, Θ_2 being the set of all paths of length 2, the condition (15) of theorem 2 is respected. Condition (16) is tested at every moment. If it is not satisfied, no decision is taken concerning the recognition of the active path.
In order to perform the determination of the active path at every moment from the system's input and output signals, we consider an observation window of length 3, leading to the consideration of the set Θ_2 of all paths of length 2 on the observation window.
Figure 2 presents the evolution of the calculated residuals. The different graphics on the figure show the residuals $r_{(i\,\cdot\,j),h}(\cdot)$, $i,j \in \{1,2,3\}$ corresponding to the paths of length 2. Only one residual equals zero at each instant, the index $(i \cdot j)$ of this residual corresponding to the active path on the considered time window.

As explained in section 4, to reduce the number of residuals to be analyzed during the active mode determination process, it is possible to only consider the paths describing the mode sequence when the system remains in the same mode all over the duration of the observation

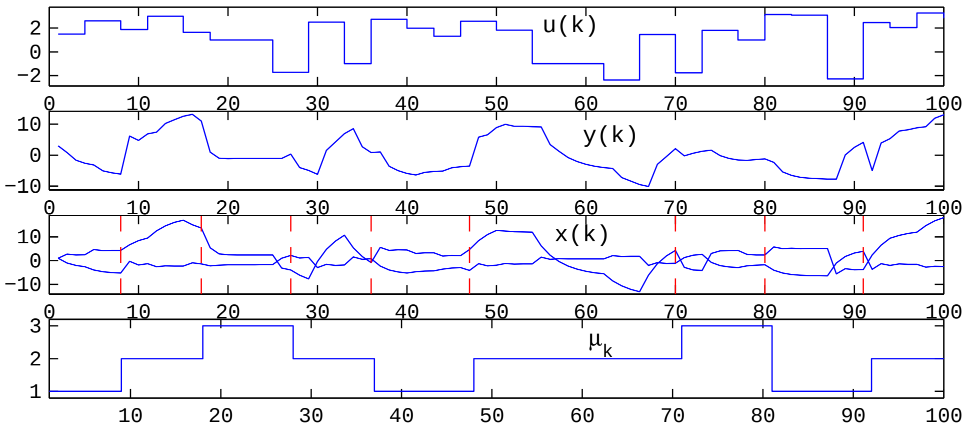

Fig. 1. Input $u(\cdot)$, output $y(\cdot)$, state $x(\cdot)$, mode sequence $\mu_{(\cdot)}$

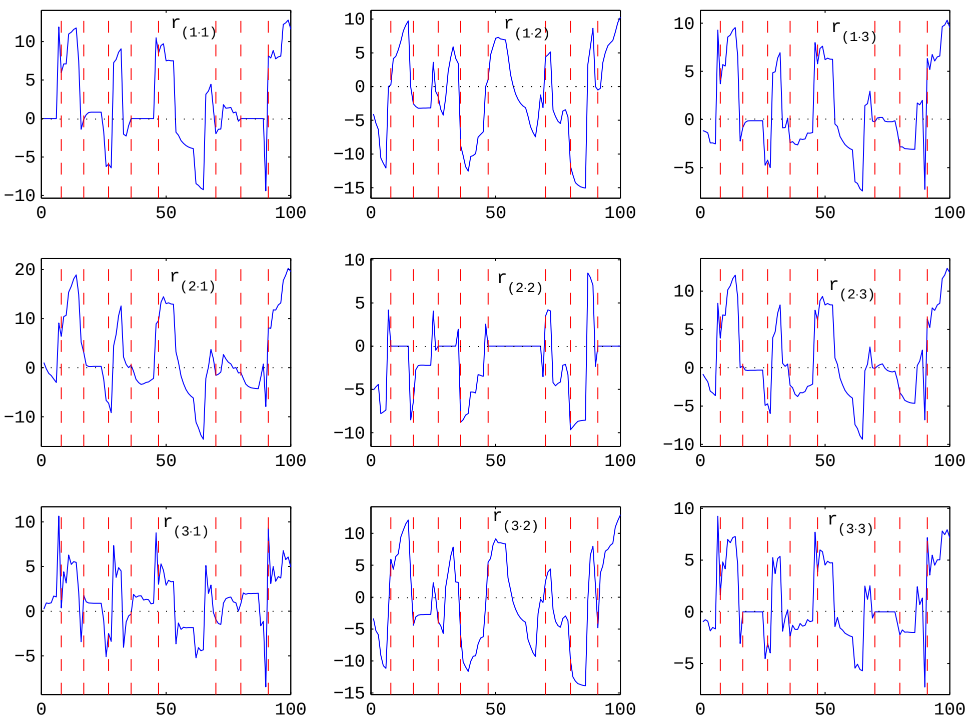

Fig. 2. Residuals $r_{\mu,h}(\cdot)$, $\mu \in \Theta_h$

window. In this case, only the paths $(1 \cdot 1)$, $(2 \cdot 2)$, and $(3 \cdot 3)$ have to be considered. While having a close look at the residuals $r_{(1\cdot1),h}(\cdot)$, $r_{(2\cdot2),h}(\cdot)$ and $r_{(3\cdot3),h}(\cdot)$ on the graphics of figure 2, one can see that , except in a vicinity of the switching instants, only one of the three residuals is null at every moment, the index $(i \cdot i)$, $i \in \{1, 2, \ldots, s\}$ of this residual being the active path on the considered time window. When a switching occurs, none of the paths $(1 \cdot 1)$, $(2 \cdot 2)$ and $(3 \cdot 3)$ matches the active path, this situation highlighting the occurrence of a switching.

The mode sequence (first graphic of figure 3) and its estimation (second graphic of figure

3) while analyzing the residuals are depicted on figure 3. The figure shows that the mode sequence is well reconstructed.

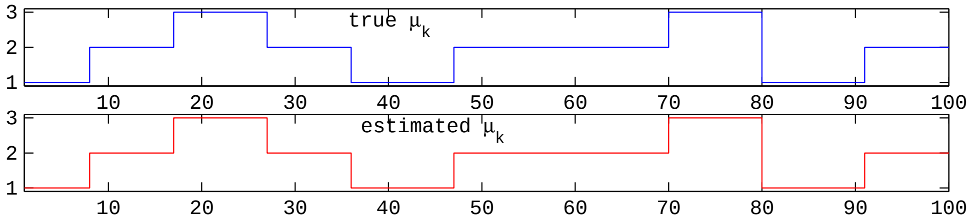

Fig. 3. Mode's recognition

7. Conclusion

This paper proposed a method for the determination of the active mode and the switching instants of a switching system, using only the system's input and output data. Discernability conditions have also been examined.

A point to be deepened is the situation where all the modes of the system are not previously indexed. In this case, one does not have a complete knowledge of all the operating regimes of the system. Therefore, it is necessary to simultaneously proceed to the detection of not indexed modes and to the estimation of their parameters.

Bibliography

Babaali, M., Egerstedt, M. and Kamen, E. W. (2003). Pathwise observability and controllability are decidable, *Proceedings of the 42nd Conference on Decision and Control*, Hawaii, USA, pp. 5771–5776.

Chow, A. Y. and Willsky, A. S. (1984). Analytical redundancy and the design of robust failure detection systems, *IEEE Transactions on Automatic Control* **29(7)**: 603–614.

Domlan, E. A., Ragot, J. and Maquin, D. (2006). Systèmes à commutation : diagnostic de fonctionnement et identification de la loi de commutation, *e-STA, Sciences et Technologies de l'Automatique* **3(3)**.

Gertler, J. (1997). Fault detection and isolation using parity relations, *Control Engineering Practice* **5(5)**: 653–661.

Heemels, W. P. M. H., De Schutter, B. and Bemporad, A. (2001). Equivalence of hybrid dynamical models, *Automatica* **37(7)**: 1085–1091.

Juloski, A., Heemels, M., Ferrari-Trecate, G., Vidal, R., Paoletti, S. and Niessen, H. (2005). *Hybrid Systems: Computation and Control*, Vol. 3141 of *Lecture Notes on Computer Science*, Springer-Verlag, Zurich, Switzerland, chapter Comparison of Four Procedures for Identification of Hybrid Systems.

Ragot, J., Maquin, D. and Domlan, E. A. (2003). Switching time estimation of piecewise linear system. Application to diagnosis, *Proceedings of the 5th IFAC Symposium on Fault Detection, Supervision and Safety of Technical Processes*, Washington DC, USA, pp. 669–704.