

MULTIPLE OBSERVERS FOR DISCRETE-TIME MULTIPLE MODELS

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Abstract: This paper studies the state observer design problem for discrete-time multiple models. New sufficient conditions for the convergence of such multiple observer are derived via Lyapunov approach and BMI (Bilinear Matrix Inequalities) formulation. For facility purpose, a LMI (Linear Matrix Inequality) form is obtained from BMI linearization. The method uses the piecewise quadratic Lyapunov functions and the S-procedure to relax the conservativeness of the quadratic Lyapunov results. *Copyright © 2003 IFAC.*

Key words : Discrete-time nonlinear systems, multiple models approach, Takagi-Sugeno models, multiple observers, Lyapunov method, rank constraints, LMI.

1. INTRODUCTION

The issue of stability, the design of state feedback controller as well as the design of state observer for nonlinear systems described by multiple models (Murray-Smith and Johansen, 1997) have been considered actively during the last decade (Chadli, et al., 2002a; Tanaka, et al., 1998; Narendra and Balakrishnan, 1994). Having the property of universal approximation (Buckley, 1992; Castro, 1995), this representation includes the Takagi-Sugeno model (Takagi and Sugeno, 1985; Sugeno and Kang, 1988) and can be seen also as Polytopic Linear Differential Inclusions (PLDI) (Boyd, et al., 1994). The multiple model consists to construct nonlinear dynamic system by means of interpolating the behavior of several LTI local submodels. Each submodel contributes to the global model in a particular subset of the operating space throughout activation functions.

Many works have been carried out to investigate the stability analysis of multiple models using a quadratic Lyapunov function and sufficient conditions for the stability and stabilizability have been established (Tanaka, et al., 1998; Narendra and Balakrishnan, 1994, Chadli, 2002d). The stability mainly depends on the existence of a common positive definite matrix guarantying the stability of all local subsystems. These stability conditions may be expressed in linear matrix inequalities (LMIs) form (Boyd, et al., 1994). The obtaining of a solution is then facilitated by using numerical toolboxes for solving such problems. To obtain relaxed stability conditions, nonquadratic Lyapunov function formulated as a set of LMIs are used (Chadli, et al., 2002a; Feng, et al., 2000; Johansen, et al., 1999). A certain form of multiple observers has been proposed and sufficient conditions for the asymptotic convergence are obtained which are dual to those for the stability of multiple controllers (Tanaka, et al.,

1998; Chadli, et al. 2002b). LMIs constraints have been also used for pole assignment in LMI regions to achieve desired performances of multiple controllers and multiple observers (Lopez-Toribio and Patton, 1999). In (Chadli, et al. 2002b; Chadli, et al., 2002c) the separation property have been studied for both measurable and estimated decision variables cases.

This paper is organized as follows. Section 2 recalls the structure of discrete-time multiple models. In section 3, under the assumption that the multiple model is locally detectable, sufficient conditions for the global exponential stability are derived in LMIs form for multiple observer. In section 4, the main result is given. Based on the use of the piecewise quadratic Lyapunov function and the S-procedure, new stability conditions are presented. Firstly in the BMI form and secondly in LMI form under rank constraint.

In the rest of the paper, the following useful notation is used: X^T denotes the transpose of the matrix X , $X > 0$ ($X \geq 0$) denotes symmetric positive definite (semidefinite) matrix, $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote respectively the minimum and maximum eigenvalues of the matrix X , X^- denotes the Moore-Penrose inverse of X , $\begin{pmatrix} A & * \\ B & C \end{pmatrix}$ denotes symmetric matrix where $(*) = B^T$, $I_n = \{1, \dots, n\}$ and $\sum_{i < j}^n x_i x_j = \sum_{i=1}^n \sum_{j=1, j > i}^n x_i x_j$.

2. DISCRETE-TIME MULTIPLE MODELS

Consider the following nonlinear dynamic system in the multiple models representation:

$$x(k+1) = \sum_{i=1}^n \mu_i(z(k)) (A_i x(k) + B_i u(k)) \quad (1)$$

where $x(k) \in \mathbb{R}^p$ is the state vector, $u(k) \in \mathbb{R}^m$ is the input vector, n is the number of submodels, $y(k) \in \mathbb{R}^l$ is the output vector, $A_i \in \mathbb{R}^{p \times p}$, $B_i \in \mathbb{R}^{p \times m}$ and $z(k)$ is the *decision variable* vector. The choice of the variable $z(k)$ leads to different class of systems. It can depend on the measurable state variables and possibly on the input; in this case, the system (1) describes a nonlinear system. It can also be an unknown constant value, system (1) then represents a polytopic linear differential inclusion (PLDI) (Boyd, et al., 1994).

The activation function $\mu_i(z(k))$ in relation with the i^{th} submodel is such that

$$\begin{cases} \sum_{i=1}^n \mu_i(z(k)) = 1 \\ \mu_i(z(k)) \geq 0, \quad i \in I_n \end{cases} \quad (2)$$

The final output of discrete-time multiple models is also interpolated as follows:

$$y(k) = \sum_{i=1}^n \mu_i(z(k)) C_i x(k) \quad (3)$$

where $C_i \in \mathbb{R}^{l \times p}$ are the output matrices.

It should be point out that at a specific time, only a number r of local models are activated, depending on the structure of the activation functions $\mu_i(z(k))$.

3. BASIC MULTIPLE OBSERVER DESIGN

In practice, all the states of a system are not fully measurable. Thus, the problem addressed in this section is the construction of a multiple observer to estimate states of the multiple models (1). It is supposed that the decision variables $z(k)$ are measurable and the multiple models (1) is locally detectable, i.e. the pairs (A_i, C_i) , $i \in I_n$ are detectable.

Using the same structure as the one for multiple controller design, the multiple observer for the multiple models (1) is written as follows :

$$\begin{cases} \hat{x}(k+1) = \sum_{i=1}^n \mu_i(z(k)) (A_i \hat{x}(k) + B_i u(k) + L_i (y(k) - \hat{y}(k))) \\ \hat{y}(k) = \sum_{i=1}^n \mu_i(z(k)) C_i \hat{x}(k) \end{cases} \quad (4)$$

where $\hat{x}(k)$ and $\hat{y}(k)$ denote the estimated state vector and output vector respectively. The activation functions $\mu_i(z(k))$ are the same that those used in the multiple models (1). Denoting the state estimation error by

$$\tilde{x}(k) = x(k) - \hat{x}(k) \quad (5)$$

it follows from (1) and (4) that the observer dynamic error is given by the following equation :

$$\tilde{x}(k+1) = \sum_{i=1}^n \sum_{j=1}^n \mu_i(z(k)) \mu_j(z(k)) \Theta_{ij} \tilde{x}(k) \quad (6)$$

where

$$\Theta_{ij} = A_i - L_i C_j \quad (7)$$

The design of the observer consists to determine the local gains L_i to ensure the asymptotic convergence to zero of the estimation error (6). To prove the global exponential stability conditions of the estimation error (6), the following result which is derived from theorem 1 is proposed.

The **quadratic** case uses the quadratic Lyapunov function

$$V(x(k)) = x^T(k) P x(k), P > 0 \quad (8)$$

Which is a radially unbounded Lyapunov function since that $\forall x(k) \in \mathbb{R}^P$

$$\lambda_{\min}(P_1) \|x(k)\|^2 \leq V(x(k)) \leq \lambda_{\max}(P_1) \|x(k)\|^2 \quad (9)$$

Tacking into account the variation of Lyapunov function (8)

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k)) \quad (10)$$

along the trajectory of the model (6), we obtain the following result :

Theorem 1 : Suppose that there exist matrices L_i and symmetric positive definite matrices P and Q such that

$$\Theta_{ii}^T P \Theta_{ii} - P + \left(r - \frac{1}{2}\right) Q < 0 \quad i \in I_n \quad (11a)$$

$$\frac{(\Theta_{ij} + \Theta_{ji})^T}{2} P \frac{(\Theta_{ij} + \Theta_{ji})}{2} - P < \frac{Q}{2} \quad (11b)$$

with $\Theta_{ij} = A_i - L_i C_j, \forall i < j \in I_n$ and $\mu_i(z(k)) \mu_j(z(k)) \neq 0$. Then there exists a discrete-time multiple observer of the form (4) such that the error estimation (6) is globally exponentially stable.

Proof : see (Chadli, et al., 2002b). ■

With the definition (7), the constraints (11) are bilinear in L_i and P . Using the Schur complement, the linearization of (11) gives :

$$P > 0, Q > 0 \quad (12a)$$

$$\begin{pmatrix} P - \left(r - \frac{1}{2}\right) Q & * \\ P A_i - Y_i C_i & P \end{pmatrix} > 0, \quad i \in I_n \quad (12b)$$

$$\begin{pmatrix} P + \frac{Q}{2} & * \\ \frac{1}{2}(P(A_i + A_j) - Y_j C_i - Y_i C_j) & P \end{pmatrix} > 0, \quad i < j \in I_n \quad (12c)$$

which are LMIs in P, Q and $Y_i, i \in I_n$ with $L_i = P^{-1} Y_i$ (13)

4. MAIN RESULT

This section is based on the use of the piecewise quadratic Lyapunov function:

$$V(x(k)) = \max(V_1(x(k)), \dots, V_i(x(k)), \dots, V_n(x(k)))$$

where $V_i(x(k)) = x(k)^T P_i x(k), P_i > 0, i \in I_n$ (14)

The following theorem gives sufficient stability conditions by using the S-procedure lemma and nonquadratic Lyapunov function candidate (14).

Firstly, let us recall the following result (Chadli, et al., 2002a).

Theorem 2: Suppose that there exists symmetric matrices $P_i, i \in I_n$ and scalars $\tau_{ijk} \geq 0$ such that

$$A_i^T P_j A_i - P_j + \sum_{k=1}^n \tau_{ijk} (P_j - P_k), \quad (i, j) \in I_n^2 \quad (15)$$

Then the unforced discrete-time multiple model of (1) is globally asymptotically stable.

Proof : The proof is obtained by using the nonquadratic Lyapunov function (14) and the S-procedure lemma (see appendix). ■

The result obtained in (15) is less conservative than that derived in the quadratic case which is based on the inequalities $A_i^T P A_i - P < 0, P > 0, i \in I_n$. We can prove easily that the quadratic conditions are included in the derived conditions by substituting $P_i, i \in I_n$ by P .

The conditions (15) of theorem 2 will be extended to design multiple observers. We can substitute directly Θ_{ij} by A_i in (15). However, for more relaxed stability conditions, the constraints (11) of theorem 1 will be used.

Theorem 3 : Suppose that there exist matrices L_i and symmetric positive definite matrices P_i $i \in I_n$, Q and scalars $\tau_{ijkl} \geq 0$ such that

$$\begin{pmatrix} P_k - \left(r - \frac{1}{2}\right)Q - \sum_{l=1}^n \tau_{ikl}(P_k - P_l) & * \\ P_k \Theta_{ii} & P_k \end{pmatrix} > 0, (i, k) \in I_n^2 \quad (16a)$$

$$\begin{pmatrix} 4P_k + 2Q - \sum_{l=1}^n \tau_{ijkl}(P_k - P_l) & * \\ P_k(\Theta_{ij} + \Theta_{ji}) & P_k \end{pmatrix} > 0 \quad (16b)$$

$$(i, j, k) \in I_n^3, i < j$$

with $\Theta_{ij} = A_i - L_i C_j, \forall i < j \in I_n$ and $\mu_i(z(k))\mu_j(z(k)) \neq 0$. Then there exists a discrete-time multiple observer of the form (4) such that the error estimation (6) is globally exponentially stable.

Proof : Considering the nonquadratic Lyapunov function candidate (14). It follows that

$$V(x(k)) = V_k(x(k)) \text{ if } V_k(x(k)) \geq V_l(x(k)), k \neq l \in I_n \quad (17)$$

Consequently if $V_k(x(k)) \geq V_l(x(k)), k \neq l \in I_n$ then $\Delta V(x(k)) = \Delta V_k(x(k))$

In the following, for take of simplicity, we denote $\mu_i(z(k))$ by μ_i and $V_i(x(k))$ by V_i . Considering all possible situations, we have when $V_k \geq V_l, \forall x(k)$:

$$\begin{aligned} \Delta V_k &= x(k)^T \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n \mu_i \mu_j \mu_k \mu_l (\Theta_{ij}^T P_k \Theta_{ij} - P_k) x(k) \\ &\leq x(k)^T \sum_{i=1}^n \mu_i^2 (\Theta_{ii}^T P_k \Theta_{ii} - P_k) x(k) + \\ &\quad x(k)^T \sum_{i < j} \mu_i \mu_j \left(\frac{(\Theta_{ij} + \Theta_{ji})^T}{2} P_k \frac{(\Theta_{ij} + \Theta_{ji})}{2} - P_k \right) x(k) \end{aligned}$$

Consequently, when $x(k)^T (P_k - P_l) x(k) \geq 0$ $k \neq l \in I_n$, if

$$\begin{cases} \bullet \Theta_{ii}^T P \Theta_{ii} - P + \left(r - \frac{1}{2}\right)Q < 0 \\ \bullet \frac{(\Theta_{ij} + \Theta_{ji})^T}{2} P \frac{(\Theta_{ij} + \Theta_{ji})}{2} - P < \frac{Q}{2} \end{cases} \quad (18)$$

then $\Delta V_k < 0 \forall x(k) \neq 0$.

Finally, constraints (16) are obtained by applying the S-procedure lemma to (18) and using the Schur complement. ■

Remarks:

1) It should be noted that the quadratic conditions (11) are included in conditions derived in (16). So when $P_i = P, i \in I_n$ we have $P_k - P_l = 0$ and $V(x(k)) = \max_{i \in I_n} (V_i(x(k))) = x(k)^T P x(k)$. Then we obtain conditions (11) derived from the quadratic case.

2) The same result can be obtained by using the nonquadratic Lyapunov function and $V(x(k)) = \min_{i \in I_n} (V_i(x(k)))$.

The use of the S-procedure lemma and the nonquadratic Lyapunov function (14) leads to $n^2(n+1)/2$ BMI conditions. We know that BMI problem is not convex and may have multiple local solutions. However, many control problems that require the solution to BMIs can be formulated as LMIs, which may be solved very efficiently. Unfortunately, obtaining the LMI formulation is very difficult in our case. For solving BMI problem, we can use, for example, the path-following method, developed in (Hassibi, et al., 1999). This method utilizes a first order perturbation approximation to linearize the BMI problem. Hence, the BMIs are converted into a series of LMIs iteratively solved until a desired performance is achieved if any.

For that purpose, let P_{k0} and L_{i0} be initial values such that

$$P_k = P_{k0} + \delta P_k, L_i = L_{i0} + \delta L_i \quad (20)$$

The BMIs term $P_k \Theta_{ij}$ of (16) can be rewritten by neglecting the second order terms $\delta P_k \delta L_i C_i$ as the following LMIs in the variables δP_k and δL_i .

$$P_k \Theta_{ij} \approx (P_{k0} + \delta P_k)(A_i - L_{i0} C_j) - P_{k0} \delta L_i C_j \quad (21)$$

with $P_{k0} + \delta P_k > 0$

Tacking into account the expression (21), the initial BMI problem (16) become LMI in respect to the variables δP_k and δL_i by fixing the scalars τ_{ijkl} .

It is important to note that the following constraints: $\|\delta P_k\| < \zeta \|P_{k0}\|$ and $\|\delta L_i\| < \zeta \|L_{i0}\|, 0 < \zeta < 1$ must be

added in order to ensure that the linear approximation should be valid.

The major weakness of this method is, firstly, the choice of initial values for an acceptable solution and secondly the convergence to a solution which is not guaranteed. Some BMI can be converted into an equivalent LMI problem with a rank constraint. The resolution of this problem appears in many problems of analysis and synthesis. It is a nonconvex NP-hard problem. However, some methods are proposed to find solutions for this kind of problems. Local or global optimization Algorithms can be employed. As an example, in (Henrion, et al., 2000) the authors proposed an heuristic algorithm to solve an LMI problem with a rank-one constraints. In (Apkarian, et al., 2000) a method, based on the Frank & Wolf algorithm, guaranteeing the global optimality of the solution, if any, is given. In the following we propose an LMI formulation under rank constraint of the BMI problem (16).

Theorem 4 : Suppose that there exist symmetric positive matrices definite matrices $P_i, i \in I_n$, $Q \in \mathbb{R}^{p \times p}$, matrices R_{ki} and scalars $\tau_{ijkl} \geq 0$ such that

$$\begin{pmatrix} P_k - \left(r - \frac{1}{2}\right)Q - \sum_{l=1}^n \tau_{ikl}(P_k - P_l) & * \\ P_k A_i + R_{ki} C_i & P_k \end{pmatrix} > 0, (i, k) \in I_n^2 \quad (22a)$$

$$\begin{pmatrix} 4P_k + 2Q - \sum_{l=1}^n \tau_{ijkl}(P_k - P_l) & * \\ P_k(A_i + A_j) - R_{ki}C_j - R_{kj}C_i & P_k \end{pmatrix} > 0 \quad (i, j, k) \in I_n^3, i < j \quad (22b)$$

$$\text{rang}(P \quad R_i) = p, i \in I_n \quad (22c)$$

with

$$P = (P_1 \quad P_1 \quad \dots \quad P_n)^T, R_i = (R_{1i} \quad R_{2i} \quad \dots \quad R_{ni})^T$$

Then there exists a discrete-time multiple observer of the form (4) such that the error estimation (6) is globally exponentially stable where $L_i = P^- R_i$.

Proof : The conditions (22) are obtained directly from (16) (theorem 3) with the change of variables $R_{ki} = P_k L_i$. The rank constraint (22c) is essential in order to guarantee that the change of variables $R_{ki} = P_k L_i$ leads at least to a solution. ■

In the case of using a common output matrix, i. e. $C_i = C, i \in I_n$, we obtain the following result :

Theorem 5 : Suppose that there exist symmetric positive matrices definite matrices $P_i, i \in I_n$, $Q \in \mathbb{R}^{p \times p}$, matrices R_{ki} and scalars $\tau_{ikl} \geq 0$ such that

$$\begin{pmatrix} P_k - \sum_{l=1}^n \tau_{ikl}(P_k - P_l) & * \\ P_k A_i + R_{ki} C_i & P_k \end{pmatrix} > 0, (i, k) \in I_n^2 \quad (23a)$$

$$\text{rang}(P \quad R_i) = p, i \in I_n \quad (23b)$$

with

$$P = (P_1 \quad P_1 \quad \dots \quad P_n)^T, R_i = (R_{1i} \quad R_{2i} \quad \dots \quad R_{ni})^T$$

Then there exists a discrete-time multiple observer of the form (4) such that the error estimation (6) is globally exponentially stable where $L_i = P^- R_i$.

Proof: see theorem 4. ■

Let us notice that it suffices to take $P_k = P_0, k \in I_n$ to obtain the quadratic results. Indeed, since $P_k - P_l = 0$ and the condition of rank constraints becomes commonplace, the conditions of theorems 4 and 5 become LMI in P_0 and R_{0i} . The observer gains are obtained directly by $L_i = P_0^{-1} R_{0i}$.

5. CONCLUSION

This paper presents new stability conditions for discrete-time multiple observer. The result is based on the use of the piecewise quadratic Lyapunov function and the S-procedure. Although these conditions are in the BMI form, the result is less conservative than the quadratic result. An LMI formulation under rank constraint is also proposed.

APPENDIX

Lemma (S-Procedure, (Boyd, et al., 1994)):

Let $F_0(x(t)), \dots, F_q(x(t))$ be quadratic functions of the variable $x(t) \in \mathbb{R}^p$.

If there exists scalars $\tau_1 \geq 0, \dots, \tau_q \geq 0$ such that

$$F_0(x(t)) - \sum_{i=1}^q \tau_i F_i(x(t)) \leq 0$$

then $F_0(x(t)) \leq 0$ for all $x(t)$ such that $F_i(x(t)) \leq 0, \forall i \in I_q$. ■

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