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Fault tolerant control for uncertain descriptor multi-models with application to wastewater treatment plant

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Abstract— **This paper addresses the design of adaptive observer for uncertain descriptor multi-models (MM), affected by unknown inputs (actuator faults) and subject to unknown model parameter variations, in the context of fault tolerant control (FTC) strategy. The FTC strategy contains two parts: the first one aims to estimate the unmeasured states, the actuator faults and the model parameter uncertainties thanks to an observer. In the second part, a control law that takes into account the faults, the uncertainties and the states, provided by the observer, is proposed. This controller is able to compensate the actuator faults even in the presence of parameter uncertainties. The stability of the whole closed-loop system is studied by using the Lyapunov theory. The stability conditions are then expressed in terms of Linear Matrix Inequalities (LMIs). The model parameters are time varying and are involved as polynomials in the dynamic description of the nonlinear system, which corresponds to a more general class of uncertainties in the framework of real process modeling.**

I. INTRODUCTION

Fault Tolerant Control (FTC) is one of the more interesting topics in modern control engineering. It is often studied simultaneously with control and supervision design systems in order to anticipate faults that can occur in the system and avoid some dangerous situations. Some approaches are proposed to deal with the problem of fault tolerance, which can be classified into two categories: Passive Fault Tolerant Control (PFTC) and Active Fault Tolerant Control (AFTC). The PFTC can be viewed as a robust control technique, it takes *a priori* some knowledge on the possible faults that may affect the system. The advantages of such an approach are its design simplicity and only the bounds of faults are required. However, the major disadvantages are the degradation of the performances since the control law is fixed and remains unchanged for all fault situations. In addition, only a particular class of faults can be considered. The AFTC is known as an active control since the structure of the controller can be changed to adapt it to the faulty situation at each time of fault occurrence. The active strategy requires a fault diagnosis unit which allows to detect and isolate even estimate the faults magnitude over time. The information provided by this unit are transmitted to the control unit in order to adapt the control structure and the gain matrices which allow to compensate the faults and preserve some system performances. For interested readers, a good survey on these two approaches can be found in the paper [1] and the references therein.

The idea of AFTC mentioned above is exploited for different classes of linear and nonlinear systems. In the context of nonlinear systems, one can cite the work about actuator fault tolerant control for descriptor nonlinear systems with Lipschitz nonlinearities considered in [2]. New fault tolerant control strategies are proposed in [3] for Takagi-Sugeno (T-S) models by using relaxed stability conditions based on Polya's theorem. Active fault tolerant control for actuator and sensor faults was proposed recently in [4] for regular Takagi-Sugeno models. An observer-based fault tolerant control design for a class of LPV descriptor systems is presented in [5] for time-varying faults, but with no model parameter uncertainties.

For the fault diagnosis unit, [6] proposed a robust sensor Fault Detection and Isolation (FDI) observer for polytopic descriptor Linear Parameter Varying (LPV) systems with unmeasurable premise variables, by using unknown input observer type.

Recent work [7] focuses on nonlinear systems with unmeasurable time-varying parameters that are considered as model disturbances acting on the system evolution. The parameters are expressed as functions of their upper and lower bounds, according to the sector nonlinearity transformation [8]. A proportional integral observer is then proposed to estimate state and model parameter for regular nonlinear systems.

Previous result [9] addresses the state estimation of singular nonlinear systems with unknown-input observer based on multi-models with unmeasurable premise (scheduling) variables. Based on this outcome, a robust adaptive unknown inputs observer is proposed in this article, that estimates states, actuator faults and model parameter uncertainties. The model parameters are time varying and are involved as polynomials in the description of the system dynamics, situation corresponding to a more realistic modeling case than previous results considering only constant and additive parametrization. In addition, the fault tolerant control strategy proposed in this paper use unmeasurable premise variables, comparing to other recent strategies using only measurable premises [3], [4], [5].

The contributions of this paper are twofold: firstly, a theoretical methodology is proposed for active fault tolerant control of nonlinear descriptor systems described by

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a Takagi-Sugeno representation with unmeasurable premise variables, time varying uncertainties and external perturbations. The approach is based on the design of an observer that provides, simultaneously, an estimation of the state vector, the unknown parameters variations and the actuator faults. These information are then used by the control unit in order to deliver the adequate control signals aiming to preserve the stability and the performances of the system. Secondly, the practical contribution is to apply the proposed fault tolerant control strategy to the realistic model of a wastewater treatment process (WWTP) modeled by an Activated Sludge Model no.1 (ASM1). The measures used for the simulation process are those of the European program benchmark Cost 624 for the evaluation of control strategies in WWTP [10]. Previous approach [11] propose estimation and diagnosis for WWTPs using multi-models and gives encouraging results. The choice of different variables involved in the proposed strategy is made by taking into account the real conditions properties of the Bleesbrück treatment station, Luxembourg.

Notation 1.1: For a given square matrix X, $\lambda_{max}(X)$ is the maximum singular value of X, the matrix $\mathbb{S}(X)$ is defined by $\mathbb{S}(X) = X + X^T$ and $*$ inside a matrix denotes the terms induced by the symmetry.

II. PROBLEM STATEMENT

A descriptor nonlinear system affected by unknown inputs, actuator faults and subjected to model parameter uncertainties is written as follows:

$$E\dot{x}(t) = g(x(t), u(t), d(t), f(t), \theta(t)) \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

where E is a singular matrix (i.e. rank$(E) \leq n$), $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^{n_u}$ is the input, $y \in \mathbb{R}^{n_y}$ is the output, $f \in \mathbb{R}^{n_f}$ is the actuator fault, $d \in \mathbb{R}^{n_d}$ is the unknown input, which can model some external perturbations and noises, $\theta \in \mathbb{R}^{n_\theta}$ is the modeling uncertainty, g is continuous nonlinear function and C is a matrix with appropriate dimensions.

Let us consider that the system (1) is equivalently rewritten as the T-S multi-model in a compact set of the state space:

$$\begin{aligned} E\dot{x}(t) = &\sum_{i=1}^{r} \mu_i(\xi(t)) \left[A_i(\theta(t))x(t) \right.\\ &\left. + B_i(\theta(t))(u(t) + f(t)) + E_i d(t)\right] \\ y(t) = &Cx(t) \end{aligned} \tag{2}$$

where r is the number of linear sub-models in the T-S MM form, $A_i(\theta(t))$ and $B_i(\theta(t))$ are time varying matrices of appropriate dimensions, E_i are constant matrices of appropriate dimensions and where the weighting functions μ_i, depending on unmeasurable premise variable $\xi(t)$ (which is function of the unmeasured state variables of the system), have the following property:

$$\sum_{i=1}^{r} \mu_i(\xi(t)) = 1, \ \mu_i(\xi(t)) \geq 0, \forall t \in \mathbb{R} \tag{3}$$

The equivalent rewriting between T-S multi-model (2) and the system (1) can be obtained by using the methodology proposed in [12], where no uncertain system and no actuator fault is considered. For instance, a slightly different T-S MM form is used in (2), since the sub-models are linear parameter varying with matrices $A_i(\theta(t))$ and $B_i(\theta(t))$ depending on the uncertain parameter $\theta(t)$.

In most studies [13] the modeling uncertainties are norm bounded and are expressed additively in the state matrix of the dynamic nonlinear model. In this paper, a more general class of modeling uncertainties is considered, as follows:

$$\theta(t) = \left[\theta_1(t), \theta_2(t), \cdots \theta_{n_\theta}(t)\right]^T \tag{4}$$

are bounded and occur in a polynomial way in (2):

$$A_i(\theta(t)) = A_{i,0} + \sum_{j=1}^{n_\theta} \theta_j(t) A_{i,j}, \tag{5a}$$

$$B_i(\theta(t)) = B_{i,0} + \sum_{j=1}^{n_\theta} \theta_j(t) B_{i,j} \tag{5b}$$

where matrices $A_{i,j}$, $B_{i,j}$ $(i = 1, \cdots, r, \ j = 1, \cdots, n_\theta)$ are constants known matrices and of appropriate dimensions. This type of uncertainties is directly related to malfunctions in the process that cause changes in the parameters and are characterized by their direct influence on the system stability.

By using the sector nonlinearity approach as in [12], each time varying parameter $\theta_j(t)$ can be expressed as a multi-model with two submodels as follows:

$$\theta_j(t) = \bar{\mu}_j^1(\theta_j(t))\theta_j^1 + \bar{\mu}_j^2(\theta_j(t))\theta_j^2 \tag{6}$$

with the weighting functions depending on the upper θ_j^1 and lower θ_j^2 bounds of $\theta_j(t)$:

$$\bar{\mu}_j^1(\theta_j(t)) = \frac{\theta_j(t) - \theta_j^2}{\theta_j^1 - \theta_j^2} \quad \text{and} \quad \bar{\mu}_j^2(\theta_j(t)) = \frac{\theta_j^1 - \theta_j(t)}{\theta_j^1 - \theta_j^2} \tag{7}$$

The functions $\bar{\mu}_j^1(\theta_j(t)) \geq 0$ and $\bar{\mu}_j^2(\theta_j(t)) \geq 0$ satisfy the convex sum property $\bar{\mu}_j^1(\theta_j(t)) + \bar{\mu}_j^2(\theta_j(t)) = 1$. By using this convexity property and by replacing (6) in the expression (5) of the matrices $A_i(\theta(t))$ and $B_i(\theta(t))$, the state equation of system (2) is rewritten as follows

$$E\dot{x}(t) = \sum_{i,j}^{r,2^{n_\theta}} \mu_i \bar{\mu}_j \left[\mathscr{A}_{ij} x(t) + \mathscr{B}_{ij}(u(t) + f(t)) + E_i d(t)\right] \tag{8}$$

where the following notation is used

$$\sum_{i,j}^{r,2^{n_\theta}} \mu_i \bar{\mu}_j \equiv \sum_{i=1}^{r} \sum_{j=1}^{2^{n_\theta}} \mu_i(\xi(t)) \bar{\mu}_j(\theta(t)) \tag{9}$$

and with

$$\bar{\mu}_j(\theta(t)) = \prod_{k=1}^{n_\theta} \bar{\mu}_k^{\sigma_j^k}(\theta_k(t)) \tag{10}$$

$$\mathscr{A}_{ij} = A_{i,0} + \sum_{k=1}^{n_\theta} \theta_k^{\sigma_j^k} A_{i,k} \tag{11}$$

$$\mathscr{B}_{ij} = B_{i,0} + \sum_{k=1}^{n_\theta} \theta_k^{\sigma_j^k} B_{i,k} \tag{12}$$

where the index σ_j^k ($j = 1,...,2^n$, $k = 1,...,n_\theta$) equal 1 or 2 indicates which partition of the k^{th} parameter (μ_k^1 or μ_k^2) is involved in the j^{th} model. For more details, see [12].

The new TS model (8) is characterized by constant matrices $\mathscr{A}_{ij}$, $\mathscr{B}_{ij}$ and E_i ($i = 1,...,r$, $j = 1,...,2^{n_\theta}$), which ease the synthesis of adaptive observer and FTC approach.

Hypothesis 2.1: In this paper is assumed that:

1) The faults f and the uncertain parameters θ and their first time derivatives are bounded

$$\|f(t)\| \leq \beta_1, \ \left\|\dot{f}(t)\right\| \leq \beta_2, \ 0 \leq \beta_1 < \infty, 0 \leq \beta_2 < \infty$$
$$\|\theta(t)\| \leq \chi_1, \left\|\dot{\theta}(t)\right\| \leq \chi_2, \ 0 \leq \chi_1 < \infty, 0 \leq \chi_2 < \infty$$

where $\|.\|$ defines the Euclidean norm.

2) rank$(C\mathscr{B}_{ij}) = n_u$, $i = 1,...,r$ and $j = 1,...,2^{n_\theta}$

The main objective of the paper is to be able to estimate the state, the actuator faults and the model parameter changes with the final goal to adapt control law strategies such that the system remains stable even in actuator fault case and be able to minimize the unknown input effect and the actuator fault influence on the system stability in the presence of parameter uncertainties.

III. FAULT TOLERANT CONTROL APPROACH

A. Observer-based FTC synthesis

In order to simplify the implementation, the observer is chosen to be a nonsingular multi-model:

$$\begin{aligned}
\dot{z}(t) =& \sum_{i=1}^{r}\sum_{j=1}^{2^{n_\theta}} \mu_i(\hat{\xi}(t))\bar{\mu}_j(\hat{\theta}(t))\,[N_{ij}z(t) + G_{ij}u(t) \\
&+ L_{ij}e_y(t) + \mathscr{B}_{ij}\hat{f}(t)] \\
\hat{x}(t) =& z(t) + T_2 y(t) \\
\hat{y}(t) =& C\hat{x}(t) \\
e_y(t) =& y(t) - \hat{y}(t) \\
\dot{\hat{f}}(t) =& \Phi \sum_{i=1}^{r}\sum_{j=1}^{2^{n_\theta}} \mu_i(\hat{\xi}(t))\bar{\mu}_j(\hat{\theta}(t))F_{ij}\,[\dot{e}_y(t) + \sigma e_y(t)] \\
\dot{\hat{\theta}}(t) =& \sum_{i=1}^{r}\sum_{j=1}^{2^{n_\theta}} \mu_i(\hat{\xi}(t))\bar{\mu}_j(\hat{\theta}(t))\left[-\alpha_{ij}\hat{\theta}(t) + P_{ij}e_y(t)\right]
\end{aligned} \tag{13}$$

where $\hat{x}$, $\hat{y}$, $\hat{f}$ and $\hat{\theta}$ are respectively the estimates of the state, the output, the actuator fault and the model parameters. The system stability under control has to be insured despite the presence of actuator faults, model parameter uncertainties and unknown inputs. For this purpose, a robust fault detection and isolation procedure is to be implemented, in order to estimate states, faults and model parameters by minimizing the effect of unknown inputs and faults on the estimation error. In order to stabilize the system to the origin, an active fault tolerant control is chosen as follows:

$$u(t) = -\sum_{i=1}^{r} \mu_i(\hat{\xi}(t))K_i\hat{x}(t) - \hat{f}(t) \tag{14}$$

Thus, a solution to the active FTC problem is given by finding $\Phi \in \mathbb{R}^{n_f \times n_f}$, the scalars $\alpha_{ij} \in \mathbb{R}$, the gains $N_{ij} \in \mathbb{R}^{n \times n}$, $G_{ij} \in \mathbb{R}^{n \times n_u}$, $L_{ij} \in \mathbb{R}^{n \times n_y}$, $F_{ij} \in \mathbb{R}^{n_f \times n_y}$ and $P_{ij} \in \mathbb{R}^{n_\theta \times n_y}$ such that the state estimation error converges towards zero if the fault f is zero (or constant), or converges towards a set around zero if f is time varying.

Let us define the state, fault and model parameter estimation error:

$$e_x(t) = x(t) - \hat{x}(t) \tag{15}$$
$$e_\theta(t) = \theta(t) - \hat{\theta}(t) \tag{16}$$
$$e_f(t) = f(t) - \hat{f}(t) \tag{17}$$

From (13) and (15), the state estimation obey to the following differential equation system:

$$\dot{e}_x(t) = T_1 E\dot{x}(t) - \dot{z}(t) \tag{18}$$

since that for *rank* $\begin{bmatrix} E \\ C \end{bmatrix} = n$, $\exists\ T_1 \in \mathbb{R}^{n \times n}$ and $T_2 \in \mathbb{R}^{n \times n_y}$ nonsingular matrices s.t.

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} \begin{bmatrix} E \\ C \end{bmatrix} = I_n \tag{19}$$

A particular solution of (19) is given by using the generalized inverse matrix:

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} = \begin{bmatrix} E \\ C \end{bmatrix}^+ \tag{20}$$

For more details, please see [9].

In order to facilitate the calculation of the dynamic of the state estimation error e_x, the state equation of system (8) -depending, for instance, on the premise ξ and on the uncertain parameters θ- will be rewritten as a perturbed multimodel as follows:

$$E\dot{x}(t) = \sum_{i,j}^{r,2^{n_\theta}} \hat{\mu}_i\hat{\bar{\mu}}_j\,[\mathscr{A}_{ij}x(t) + \mathscr{B}_{ij}\,(u(t) + f(t)) + E_i d(t) + \omega(t)] \tag{21}$$

where similar notation, as in (9), is used for the weighting functions $\hat{\mu}_i$ and $\hat{\bar{\mu}}_j$ -depending on the estimates $\hat{\xi}$ and $\hat{\theta}$- and where the perturbation-like term is defined by:

$$\omega(t) = \sum_{i,j}^{r,2^{n_\theta}} \left(\mu_i\bar{\mu}_j - \hat{\mu}_i\hat{\bar{\mu}}_j\right)[\mathscr{A}_{ij}x(t) + \mathscr{B}_{ij}\,(u(t) + f(t)) + E_i d(t)]$$

Under the assumption that $x(t)$, $u(t)$, $d(t)$ and $f(t)$ are bounded (this assumption is satisfied in the example WWTP), the perturbation-like term ω is a small bounded signal belonging to a set around the origin if the weighting functions estimation error -that is directly related to the state and parameter estimation- is ensured with a certain precision.

Starting with (18), the dynamic of e_x is obtained after some calculations, by using the expression (21), the state feedback control law (14), the observer (13) and the definitions (15) and (17):

$$\begin{aligned}
\dot{e}_x(t) = \sum_{i,j,k}^{r,2^{n_\theta},r} \hat{\mu}_i\hat{\bar{\mu}}_j\hat{\mu}_k \{&[T_1\mathscr{A}_{ij} - N_{ij}(I_n - T_2C) - (T_1\mathscr{B}_{ij} - G_{ij})\,K_k] \\
&\cdot x(t) + [(T_1\mathscr{B}_{ij} - G_{ij})\,K_k + N_{ij} - L_{ij}C]\,e_x(t) \\
&+ T_1E_i d(t) + T_1\omega(t) + (G_{ij} - \mathscr{B}_{ij})\,f(t) \\
&+ (T_1\mathscr{B}_{ij} - G_{ij} + \mathscr{B}_{ij})\,e_f(t)\}
\end{aligned} \tag{22}$$

If the following conditions hold $\forall i=1,\cdots r,\ j=1,\cdots,2^{n_\theta}$:

$$T_1\mathscr{A}_{ij}-N_{ij}(I_n-T_2C)=0 \tag{23}$$
$$T_1\mathscr{B}_{ij}-G_{ij}=0 \tag{24}$$

the state estimation error dynamic reduces to

$$\dot{e}_x(t)=\sum_{i,j}^{r,2^{n_\theta}}\hat{\mu}_i\hat{\bar{\mu}}_j\left[(N_{ij}-L_{ij}C)\,e_x(t)+T_i\bar{\omega}(t)+M_{ij}f(t)+\mathscr{B}_{ij}e_f(t)\right] \tag{25}$$

with $M_{ij}=(T_1-I_n)\mathscr{B}_{ij}$, $\bar{T}_i=T_1\begin{bmatrix}E_i & I_n\end{bmatrix}$ and $\bar{\omega}(t)=\begin{bmatrix}d^T(t) & \omega^T(t)\end{bmatrix}^T$ an augmented perturbation vector putting together the unknown input d and the perturbation ω.
In order to deal with nonlinear terms in the definition of further stability conditions and avoid new invertible matrix conditions, we define the matrix $R_{ij}=N_{ij}T_2$. Thus, the condition (23) yields to:

$$N_{ij}=T_1\mathscr{A}_{ij}+R_{ij}C \tag{26}$$

B. State space dynamic equation

The dynamic of the closed loop system with the control (14) is given by:

$$E\dot{x}(t)=\sum_{i,j,k}^{r,2^n,r}\hat{\mu}_i\hat{\bar{\mu}}_j\hat{\mu}_k\left[\Psi_{ijk}x(t)+\mathscr{B}_{ij}K_k\,e_x(t)+\mathscr{B}_{ij}\,e_f(t)+\bar{E}_i\,\bar{\omega}(t)\right] \tag{27}$$

where

$$\Psi_{ijk}=\mathscr{A}_{ij}-\mathscr{B}_{ij}K_k \tag{28}$$
$$\bar{E}_i=\begin{bmatrix}E_i & I_n\end{bmatrix} \tag{29}$$

C. Stability analysis

In the context of a fault tolerant control by state-feedback, the stability of the differential equations (25) and (27) has to be ensured. The following well known results [14] will be used for the stability conditions design.

Lemma 3.1: Given a symmetric positive matrix Q, the following inequality holds:

$$2x^Ty\le x^TQx+y^TQ^{-1}y,\quad \text{for}\quad x,y\in\mathbb{R}^n \tag{30}$$

Lemma 3.2: Let us consider P a positive definite matrix and Q a full column rank matrix. It follows that the matrix QPQ^T is a positive definite matrix.
The Lyapunov method is generally used to prove stability of dynamic systems. This method will be also used in this paper to establish stability conditions of the closed loop system -derived under Linear Matrix Inequalities formulation- and ensure convergence of the state, uncertain parameter and fault estimation error.

Let us consider V the following Lyapunov function depending on x, e_x, e_θ and e_f:

$$V(t)=x^T(t)E^TP_1x(t)+e_x^T(t)P_2e_x(t)+e_\theta^T(t)P_3e_\theta(t)+\frac{1}{\sigma}e_f^T(t)\Phi^{-1}e_f(t) \tag{31}$$

where P_1, P_2, P_3 and Φ are symmetric and positive definite matrices with appropriate dimensions. Using the equation (25), the time derivative of $V(t)$ is given by

$$\begin{aligned}\dot{V}(t)=\sum_{i,j}^{r,2^{n_\theta}}\hat{\mu}_i\hat{\bar{\mu}}_j\{&(E\dot{x}(t))^TP_1x(t)+x^T(t)P_1^TE\dot{x}(t)\\&+e_x^T(t)[(N_{ij}-L_{ij}C)^TP_2+P_2(N_{ij}-L_{ij}C)]e_x(t)\\&+2e_x^T(t)P_2\bar{T}_i\bar{\omega}(t)+2e_x^T(t)P_2M_{ij}f(t)\\&+2e_x^T(t)P_2\mathscr{B}_{ij}e_f(t)+\frac{2}{\sigma}e_f^T(t)\Phi^{-1}\dot{e}_f(t)\\&+\dot{e}_\theta^T(t)P_3e_\theta(t)+e_\theta^T(t)P_3^T\dot{e}_\theta(t)\}\end{aligned} \tag{32}$$

With $E^TP_1=P_1^TE\ge 0$, by using the equation of the closed loop system (27), the definitions (16), (17) and the estimates $\hat{f}$ and $\hat{\theta}$ from (13), the time derivative of V becomes

$$\begin{aligned}\dot{V}(t)=\sum_{i,j,k}^{r,2^{n_\theta},r}\hat{\mu}_i\hat{\bar{\mu}}_j\hat{\mu}_k\{&x^T(t)\Pi_{ijk}x(t)+e_x^T(t)\Omega_{ij}e_x(t)\\&+2e_x^T(t)P_2M_{ij}f(t)+2e_x^T(t)P_2\mathscr{B}_{ij}e_f(t)\\&+2e_x^T(t)P_2\bar{T}_i\bar{\omega}(t)+2x^T(t)P_1\mathscr{B}_{ij}K_ke_x(t)\\&+2x^T(t)P_1\mathscr{B}_{ij}e_f(t)+2x^T(t)P_1\bar{E}_i\bar{\omega}(t)\\&+\frac{2}{\sigma}e_f^T(t)\Phi^{-1}\dot{f}(t)-2e_f^T(t)F_{ij}C\sigma e_x(t)\\&-\frac{2}{\sigma}e_f^T(t)F_{ij}C\Theta_{ij}e_x(t)-\frac{2}{\sigma}e_f^T(t)F_{ij}CT_i\bar{\omega}(t)\\&-\frac{2}{\sigma}e_f^T(t)F_{ij}CM_{ij}f(t)-\frac{2}{\sigma}e_f^T(t)F_{ij}C\mathscr{B}_{ij}e_f(t)\\&+2e_\theta^T(t)P_3\dot{\theta}(t)+2e_\theta^T(t)P_3\alpha_{ij}\theta(t)\\&-2e_\theta^T(t)P_3\alpha_{ij}e_\theta(t)-2e_\theta^T(t)P_3P_{ij}Ce_x(t)\}\end{aligned} \tag{33}$$

with

$$\Pi_{ijk}=\mathbb{S}(P_1\Psi_{ijk}) \tag{34a}$$
$$\Theta_{ij}=N_{ij}-L_{ij}C \tag{34b}$$
$$\Omega_{ij}=\mathbb{S}(P_2\Theta_{ij}) \tag{34c}$$

Using lemma 3.1 and hypothesis 2.1 it follows that:

$$\begin{aligned}2e_x^T(t)P_2M_{ij}f(t)&\le e_x^T(t)Q_1e_x(t)+f^T(t)(M_{ij}^TP_2Q_1^{-1}P_2M_{ij})f(t)\\&\le e_x^T(t)Q_1e_x(t)+\eta_{1,ij}\\\eta_{1,ij}&=\beta_1^2\lambda_{max}(M_{ij}^TP_2Q_1^{-1}P_2M_{ij})\end{aligned} \tag{35}$$

and similarly for:

$$\begin{aligned}\frac{2}{\sigma}e_f^T(t)\Phi^{-1}\dot{f}(t)&\le\frac{1}{\sigma}e_f^T(t)Q_2e_f(t)+\eta_2\\\eta_2&=\frac{1}{\sigma}\beta_2^2\lambda_{max}(\Phi^{-T}Q_2^{-1}\Phi^{-1})\end{aligned} \tag{36}$$

$$\begin{aligned}-\frac{2}{\sigma}e_f^T(t)F_{ij}CM_{ij}f(t)&\le\frac{1}{\sigma}e_f^T(t)Q_2e_f(t)+\eta_{2,ij}\\\eta_{2,ij}&=\frac{1}{\sigma}\beta_1^2\lambda_{max}(M_{ij}^TC^TF_{ij}^TQ_2^{-1}F_{ij}CM_{ij})\end{aligned} \tag{37}$$

$$\begin{aligned}2e_\theta^T(t)P_3\dot{\theta}(t)&\le e_\theta^T(t)Q_3e_\theta(t)+\eta_3\\\eta_3&=\chi_2^2\lambda_{max}(P_3^TQ_3^{-1}P_3)\end{aligned} \tag{38}$$

$$\begin{aligned}2e_\theta^T(t)P_3\alpha_{ij}\theta(t)&\le e_\theta^T(t)Q_3e_\theta(t)+\eta_{3,ij}\\\eta_{3,ij}&=\chi_1^2\lambda_{max}(\alpha_{ij}^TP_3^TQ_3^{-1}P_3\alpha_{ij})\end{aligned} \tag{39}$$

$$2e_x^T(t)P_2\bar{T}_i\bar{\omega}(t) \le e_x^T(t)Q_1e_x(t) + \eta_{4,i}$$
$$\eta_{4,i} = \delta^2\lambda_{max}(\bar{T}_i^T P_2 Q_1^{-1} P_2 \bar{T}_i) \quad (40)$$
$$2x^T(t)P_1\bar{E}_i\bar{\omega}(t) \le x^T(t)Q_4x(t) + \eta_{5,i}$$
$$\eta_{5,i} = \delta^2\lambda_{max}(\bar{E}_i^T P_1 Q_4^{-1} P_1 \bar{E}_i) \quad (41)$$
$$-\frac{2}{\sigma}e_f^T(t)F_{ij}CT_i\bar{\omega}(t) \le \frac{1}{\sigma}e_f^T(t)Q_2e_f(t) + \eta_{6,ij}$$
$$\eta_{6,ij} = \frac{1}{\sigma}\delta^2\lambda_{max}(T_i^T C^T F_{ij}^T Q_2^{-1} F_{ij} C T_i) \quad (42)$$

Let us consider the scalar ε the maximum value over i and j of the sum of all constant coefficients obtained in the right hand of the inequalities (35) - (42):

$$\varepsilon = \max_{i,j}(\eta_{1,ij} + \eta_2 + \eta_{2,ij} + \eta_3 + \eta_{3,ij} + \eta_{4,i} + \eta_{5,i} + \eta_{6,ij}) \quad (43)$$

Assumption 3.1: Assume it is possible to find F_{ij} and P_2 such that

$$\mathscr{B}_{ij}^T P_2 = F_{ij}C \quad (44)$$

With the assumption 3.1 and the definition (43) of ε, (33) implies

$$\dot{V}(t) \le x_a^T(t) \sum_{i,j,k}^{r,2^{n_\theta},r} \hat{\mu}_i \hat{\bar{\mu}}_j \hat{\mu}_k \Delta_{ijk} x_a(t) + \varepsilon \quad (45)$$

where $x_a(t) = \begin{bmatrix} x(t)^T & e_x(t)^T & e_f(t)^T & e_\theta(t)^T \end{bmatrix}^T$ and where

$$\Delta_{ijk} = \begin{bmatrix} \Pi_{ijk}+Q_4 & P_1\mathscr{B}_{ij}K_k & P_1\mathscr{B}_{ij} & 0 \\ * & \Omega_{ij}+2Q_1 & -\frac{1}{\sigma}\Theta_{ij}^T P_2\mathscr{B}_{ij} & -(P_{ij}C)^T P_3 \\ * & * & -\frac{1}{\sigma}F_{ij}C\mathscr{B}_{ij} + \frac{3}{\sigma}Q_2 & 0 \\ * & * & * & -P_3\alpha_{ij}+2Q_3 \end{bmatrix} \quad (46)$$

Let us define the positive scalar τ as follows

$$\tau = \min_{t>0} \lambda_{\min}\left(-\sum_{i,j,k}^{r,2^{n_\theta},r} \hat{\mu}_i \hat{\bar{\mu}}_j \hat{\mu}_k \Delta_{ijk}\right) \le \min_{i,j,k} \lambda_{\min}\left(-\Delta_{ijk}\right) \quad (47)$$

Assumption 3.2: Let us consider that the following inequality holds

$$\sum_{i,j,k}^{r,2^{n_\theta},r} \mu_i(\hat{\xi}(t))\bar{\mu}_j(\hat{\theta}(t))\mu_k(\hat{\xi}(t))\Delta_{ijk} < 0 \quad (48)$$

With the assumption 3.2, with the (45), (47) it is obtained

$$\dot{V}(t) < -\tau \|x_a(t)\|^2 + \varepsilon \quad (49)$$

It follows that $\dot{V}(t) < 0$ if $\tau\|x_a(t)\|^2 > \varepsilon$, $\forall t > 0$.

In conformity with the stability Lyapunov theory, we can deduce that the state x, the state estimation error e_x, the fault estimation error e_f and the uncertain parameter estimation error e_θ converge to a small set around the origin. The dimension of this set is depending on the scalar ε (43).

If the two assumptions, 3.1 and 3.2, made during the previous calculations hold, then the stability and convergence conditions are obtained, which completes the proof. Let us derive in the following these conditions under the LMI formulation.

D. The LMI formulation

Lemma 3.3: The following inequality holds for any scalar ψ, $X \succ 0$ and $\Lambda \le 0$:

$$\left(X + \psi\Lambda^{-1}\right)^T \Lambda \left(X + \psi\Lambda^{-1}\right) \le 0 \quad \Leftrightarrow$$
$$X\Lambda X \le -2\psi X - \psi^2\Lambda^{-1} \quad (50)$$

Following the remarks of [14] and [3], it is difficult to obtain a solution satisfying simultaneously to the equality (44) and the inequality (48). The referred citations consider the case where the indexes j and k are constants. The remark stays available for j, k varying indexes. A solution to overcome this difficulty is to reformulate the equality (44) as an optimization problem:

$$min\ \nu \quad \text{subject to} \quad \begin{bmatrix} \nu I & \mathscr{B}_{ij}^T P_2 - F_{ij}C \\ * & \nu I \end{bmatrix} > 0 \quad (51)$$

The following indexed notation is used for the sake of simplicity:

$$Y_{\xi\theta} = \sum_{i,j}^{r,2^{n_\theta}} \mu_i(\hat{\xi})\bar{\mu}_j(\hat{\theta})Y_{ij},\ Y_{\xi\theta\xi} = \sum_{i,j,k}^{r,2^{n_\theta},r} \mu_i(\hat{\xi})\bar{\mu}_j(\hat{\theta})\mu_k(\hat{\xi})Y_{ijk}$$

where Y_{ij} and Y_{ijk} are known matrices. The inequality (48) becomes:

$$\Delta_{\xi\theta\xi} = \begin{bmatrix} \tilde{\Pi}_{\xi\theta\xi} & \Phi_{\xi\theta\xi} \\ * & \tilde{\Lambda}_{\xi\theta} \end{bmatrix} < 0 \quad (52)$$

where Π_{ijk}, Ω_{ij} and Θ_{ij} are defined in (34) and where

$$\begin{aligned} \tilde{\Pi}_{ijk} &= \Pi_{ijk} + Q_4 \\ \Phi_{ijk} &= \begin{bmatrix} P_1\mathscr{B}_{ij}K_k & P_1\mathscr{B}_{ij} & 0 \end{bmatrix} \\ \tilde{\Lambda}_{ij} &= \Lambda_{ij} + \tilde{Q} \\ \tilde{Q} &= \frac{1}{\sigma}\operatorname{diag}(2\sigma Q_1, 3Q_2, 2\sigma Q_3) \\ \Lambda_{ij} &= \begin{bmatrix} \Omega_{ij} & -\frac{1}{\sigma}\Theta_{ij}^T P_2\mathscr{B}_{ij} & -(P_{ij}C)^T P_3 \\ * & -\frac{1}{\sigma}F_{ij}C\mathscr{B}_{ij} & 0 \\ * & * & -P_3\alpha_{ij} \end{bmatrix} \end{aligned} \quad (53)$$

In order to deal with the nonlinear term $P_1\mathscr{B}_{ij}K_k$ appearing in Φ_{ijk}, let us consider a matrix $X \succ 0$ defined as

$$X = \begin{bmatrix} P_1^{-1} & 0 \\ 0 & X_1 \end{bmatrix}, \quad X_1 = \begin{bmatrix} P_1^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad (54)$$

By calculating $X\Delta_{\xi\theta\xi}X$ and using lemma 3.2, the inequality (52) becomes:

$$\begin{bmatrix} P_1^{-1}\tilde{\Pi}_{\xi\theta\xi}P_1^{-1} & P_1^{-1}\Phi_{\xi\theta\xi}X_1 \\ * & X_1\tilde{\Lambda}_{\xi\theta}X_1 \end{bmatrix} < 0 \quad (55)$$

With lemma 3.3 (50) and Schur complement, the inequality (55) holds if

$$\begin{bmatrix} P_1^{-1}\tilde{\Pi}_{\xi\theta\xi}P_1^{-1} & P_1^{-1}\Phi_{\xi\theta\xi}X_1 & 0 \\ * & -2\psi_1 X_1 & \psi_1 I \\ * & * & \tilde{\Lambda}_{\xi\theta} \end{bmatrix} < 0 \quad (56)$$

By applying the same principle with the nonlinear terms in $\tilde{\Lambda}_{\xi\theta}$, let us define a matrix $Z \succ 0$ as follows

$$Z = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Z_1 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & P_3^{-1} \end{bmatrix} \quad (57)$$

By pre- and post-multiplying the inequality (56) by Z, it follows that (56) is equivalent with

$$\begin{bmatrix} P_1^{-1}\tilde{\Pi}_{\xi\theta\xi}P_1^{-1} & P_1^{-1}\Phi_{\xi\theta\xi}X_1 & 0 \\ * & -2\psi_1 X_1 & \psi_1 I \\ * & * & Z_1\tilde{\Lambda}_{\xi\theta}Z_1 \end{bmatrix} < 0 \quad (58)$$

Using again the lemma 3.3 for the last term of the matrix in inequality (58), with the scalar ψ_3 and with Schur complement, it follows that the inequality (58) holds if

$$\begin{bmatrix} P_1^{-1}\tilde{\Pi}_{\xi\theta\xi}P_1^{-1} & P_1^{-1}\Phi_{\xi\theta\xi}X_1 & 0 & 0 \\ * & -2\psi_1 X_1 & \psi_1 I & 0 \\ * & * & -2\psi_3 Z_1 & \psi_3 I \\ * & * & * & \tilde{\Lambda}_{\xi\theta} \end{bmatrix} < 0 \quad (59)$$

With the definitions of $\tilde{\Pi}_{\xi\theta\xi}$, $\Phi_{\xi\theta\xi}$, $\tilde{\Lambda}_{\xi\theta}$ and $\Omega_{\xi\theta}$ given by (53), (34), replacing N_{ij} by (26) and by introducing the change of variables:

$$\begin{aligned} &\mathscr{X}_1 = P_1^{-1}, && \mathscr{X}_3 = P_3^{-1}, && M_k = K_k\mathscr{X}_1, \\ &\tilde{R}_{ij} = P_2 R_{ij}, && \tilde{L}_{ij} = P_2 L_{ij}, && \\ &\tilde{\alpha}_{ij} = \mathscr{X}_3^{-1}\alpha_{ij}, && \tilde{P}_{ij} = \mathscr{X}_3^{-1}P_{ij}C && \end{aligned} \quad (60)$$

the inequalities (62) are easily obtained, where

$$\begin{aligned} \mathscr{W}_{ijk} &= \mathbb{S}(\mathscr{A}_{ij}\mathscr{X}_1 - \mathscr{B}_{ij}M_k) - 2\psi_2\mathscr{X}_1 \\ \mathscr{S}_{ij} &= \mathbb{S}(P_2 T_1\mathscr{A}_{ij} + (\tilde{R}_{ij} - \tilde{L}_{ij})C) + \frac{1}{\sigma}\tilde{Q} \\ \mathscr{T}_{ij} &= -\frac{1}{\sigma}\left(F_{ij}CT_1\mathscr{A}_{ij} + \mathscr{B}_{ij}^T(\tilde{R}_{ij} - \tilde{L}_{ij})C\right) \end{aligned} \quad (63)$$

Theorem 3.1: Given the assumption 3.2 and given positive scalars σ, ψ_1, ψ_3. If there exist symmetric and positive definite matrices $\mathscr{X}_1 \in \mathbb{R}^{n\times n}$, $P_2 \in \mathbb{R}^{n\times n}$, $\mathscr{X}_3 \in \mathbb{R}^{n_\theta\times n_\theta}$, $\tilde{Q} \in \mathbb{R}^{n\times n}$, $Q_4 \in \mathbb{R}^{n\times n}$, matrices $M_k \in \mathbb{R}^{n_u\times n}$, $F_{ij} \in \mathbb{R}^{n_u\times n_y}$, $\tilde{P}_{ij} \in \mathbb{R}^{n\times n_\theta}$, $\tilde{\alpha}_{ij} \in \mathbb{R}^{n_\theta\times n_\theta}$, $\tilde{R}_{ij} \in \mathbb{R}^{n\times n_y}$, $\tilde{L}_{ij} \in \mathbb{R}^{n\times n_y}$ and $\nu > 0$ solution to the optimization problem:

$$\min \nu \quad \text{subject to} \quad \begin{bmatrix} \nu I & \mathscr{B}_{ij}^T P_2 - F_{ij}C \\ * & \nu I \end{bmatrix} > 0 \quad (64a)$$

$$\mathscr{Q}_{ijk} < 0 \quad (64b)$$

$$\mathscr{X}_1 E^T = E\mathscr{X}_1 \geq 0 \quad (64c)$$

where $\mathscr{Q}_{ijk}$ is defined in (62) with (63), then, the state estimation error e_x, the fault estimation error e_f and the parameter uncertainty estimation error e_θ are bounded. In addition to that, if the $\dot{f}(t) = 0$ and $\dot{\theta}(t) = 0$ (i.e. $\beta = \chi = 0$) then these variables converge towards zero. The gains of the observer (13) and the fault tolerant control are given by:

$$\begin{aligned} N_{ij} &= T_1\mathscr{A}_{ij} + P_2^{-1}\tilde{R}_{ij}C, \quad G_{ij} = T_1\mathscr{B}_{ij}, \quad L_{ij} = P_2^{-1}\tilde{L}_{ij}, \\ \alpha_{ij} &= \mathscr{X}_3\tilde{\alpha}_{ij}, \quad P_{ij} = \mathscr{X}_3\tilde{P}_{ij}C^+, \quad K_i = M_i\mathscr{X}_1 \end{aligned} \quad (65)$$

where T_1 and T_2 are given in (20).

IV. WASTEWATER TREATMENT PLANT (WWTP)

A. Process description

The wastewater treatment with activated sludge is widely used in the last two decades [15] and consists in mixing used waters with a rich mixture of bacteria in order to degrade the organic matter. In this work, a part of the COST Benchmark is considered, based on the most common WWTP: a continuous flow activated sludge plant, performing nitrification and de-nitrification. A configuration with a single tank with a settler/clarifier was developed. The objective of this study is to use the data generated by this benchmark.

For observer/controller design, models of reduced complexity are generally used. A simplification with respect to components [15] is considered here. Thus, the biological removal of carbon and nitrogen from wastewater involving the six components are considered: soluble carbon S_S, particulate X_S, dissolved oxygen S_O, heterotrophic biomass X_{BH}, ammonia S_{NH}, nitrate S_{NO} and autotrophic biomass X_{BA}. The following components are not considered: inert components (S_I, X_I, X_P) and the alkalinity (S_{alk}). As in practical situation, a single organic compound (denoted X_{DCO}) will be considered by adding the soluble part S_S and the particulate part X_S. The state vector is:

$$x = [X_{DCO}, S_O, S_{NH}, S_{NO}, X_{BH}, X_{BA}]^T \quad (66)$$

It is supposed that the dissolved oxygen concentration at the reactor input ($S_{O,in}$) is null. In the same time, we can also suppose that $S_{NO,in} \cong 0$ and $X_{BA,in} \cong 0$, which is in conformity with the benchmark of European program Cost 624 [10].

In practice, the concentrations $X_{DCO,in}$, $S_{NH,in}$ and $X_{BH,in}$ are not measured on line. The system becomes unobservable if these concentrations are all taken as unknown inputs. Thus, an approximation often used in practice will be considered: the daily mean value of the $X_{DCO,in}$. The others can be taken as unknown inputs, since four measurements are available on line (X_{DCO}, S_O, S_{NH}, S_{NO}). Thus, the control vector and the unknown input vector are taken under the form:

$$u = [X_{DCO,in}, \quad q_a]^T, \quad d = [S_{NH,in}, \quad X_{BH,in}]^T \quad (67)$$

Let us consider the ASM1 model with the state vector (66):

$$\begin{aligned} \dot{X}_{DCO}(t) &= -\frac{1}{Y_H}[\varphi_1(t) + \varphi_2(t)] + (1 - f_P)[\varphi_4(t) + \varphi_5(t)] + D_1(t) \\ \dot{S}_O(t) &= \frac{Y_H - 1}{Y_H}\varphi_1(t) + \frac{Y_A - 4.57}{Y_A}\varphi_3(t) + D_2(t) \\ \dot{S}_{NH}(t) &= -i_{XB}[\varphi_1(t) + \varphi_2(t)] - \left(i_{XB} + \frac{1}{Y_A}\right)\varphi_3(t) \\ &\quad + (i_{XB} - f_P\, i_{XP})[\varphi_4(t) + \varphi_5(t)] + D_3(t) \\ \dot{S}_{NO}(t) &= \frac{Y_H - 1}{2.86Y_H}\varphi_2(t) + \frac{1}{Y_A}\varphi_3(t) + D_4(t) \\ \dot{X}_{BH}(t) &= \varphi_1(t) + \varphi_2(t) - \varphi_4(t) + D_5(t) \\ \dot{X}_{BA}(t) &= \varphi_3(t) - \varphi_5(t) + D_6(t) \end{aligned} \quad (68)$$

where $\varphi_1(t), \cdots, \varphi_5(t) \in \mathbb{R}$ is given by:

$$\begin{aligned} \varphi_1(t) &= \mu_H\frac{X_{DCO}(t)}{K_{DCO} + X_{DCO}(t)}\frac{S_O(t)}{K_{OH} + S_O(t)}X_{BH}(t) \\ \varphi_2(t) &= \frac{\mu_H\eta_{NOg}X_{DCO}(t)}{K_{DCO} + X_{DCO}(t)}\frac{S_{NO}(t)}{K_{NO} + S_{NO}(t)}\frac{K_{OH}}{K_{OH} + S_O(t)}X_{BH}(t) \\ \varphi_3(t) &= \mu_A\frac{S_{NH}(t)}{K_{NH,A} + S_{NH}(t)}\frac{S_O(t)}{K_{O,A} + S_O(t)}X_{BA}(t) \\ \varphi_4(t) &= b_H X_{BH}(t) \\ \varphi_5(t) &= b_A X_{BA}(t) \end{aligned} \quad (69)$$

$$
\mathscr{Q}_{ijk}=\begin{bmatrix}
\mathscr{W}_{ijk} & \psi_2 I & \mathscr{B}_{ij}M_k & \mathscr{B}_{ij} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & -Q_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & -2\psi_1\mathscr{X}_1 & 0 & 0 & \psi_1 I_{n_x} & 0 & 0 & 0 & 0 & 0\\
* & * & * & -2\psi_1 I_{n_u} & 0 & 0 & \psi_1 I_{n_u} & 0 & 0 & 0 & 0\\
* & * & * & * & -2\psi_1 I_{n_\theta} & 0 & 0 & \psi_1 I_{n_\theta} & 0 & 0 & 0\\
* & * & * & * & * & -2\psi_3 I_{n_x} & 0 & 0 & \psi_3 I_{n_x} & 0 & 0\\
* & * & * & * & * & * & -2\psi_3 I_{n_u} & 0 & 0 & \psi_3 I_{n_u} & 0\\
* & * & * & * & * & * & * & -2\psi_3\mathscr{X}_3 & 0 & 0 & \psi_3 I_{n_\theta}\\
* & * & * & * & * & * & * & * & \mathscr{S}_{ij} & \mathscr{T}_{ij} & -\tilde{P}_{ij}\\
* & * & * & * & * & * & * & * & * & -\frac{1}{\sigma}F_{ij}C\mathscr{B}_{ij} & 0\\
* & * & * & * & * & * & * & * & * & * & -\tilde{\alpha}_{ij}
\end{bmatrix}<0 \quad (62)
$$

where $K_{DCO}=K_S\frac{X_{DCO}}{S_S}=\frac{K_S}{f_{SS}}$. With $\tilde{f}=f_R\frac{1-f_W}{f_R+f_W}$, the variables $D(t)$ expressing the input/output balance is defined by:

$$
\begin{aligned}
D_1(t)&=\frac{q_{in}(t)}{V}\left[X_{DCO,in}(t)-X_{DCO}(t)\right]\\
D_2(t)&=\frac{q_{in}(t)}{V}\left(-S_O(t)\right)+Kq_a(t)\left[S_{O,sat}-S_O(t)\right]\\
D_3(t)&=\frac{q_{in}(t)}{V}\left[S_{NH,in}(t)-S_{NH}(t)\right], \qquad (70)\\
D_4(t)&=\frac{q_{in}(t)}{V}\left[-S_{NO}(t)\right]\\
D_5(t)&=\frac{q_{in}(t)}{V}\left[X_{BH,in}(t)-X_{BH}(t)+\tilde{f}X_{BH}(t)\right]\\
D_6(t)&=\frac{q_{in}(t)}{V}\left[-X_{BA}(t)+\tilde{f}X_{BA}(t)\right] \qquad (71)
\end{aligned}
$$

For numerical applications, the following kinetic parameters are used [15]: $\mu_H=3.733[1/24h]$, $\mu_A=0.3[1/24h]$, $K_S=20[g/m^3]$, $f_{SS}=0.79$, $K_{OH}=0.2[g/m^3]$, $K_{OA}=0.4[g/m^3]$, $K_{NO}=0.5[g/m^3]$, $K_{NH,A}=1[g/m^3]$, $b_H=0.3[1/24h]$, $b_A=0.05[1/24h]$, $\eta_{NOg}=0.8$. The stoichiometric parameters are $Y_H=0.6$[g cell formed], $Y_A=0.24$[g cell], $i_{XB}=0.086$[g N], $i_{XP}=0.06$[g N], $f_P=0.1$ and the oxygen saturation is $S_{O,sat}=10[g/m^3]$ and the tank volume is $V=1333[m^3]$.

B. Uncertain descriptor multi-model design

In order to obtain the descriptor form, the identification of the slow and fast dynamics for the reduced ASM1 model (68) is realized first, by using the homotopy method (please see [9]). The method is essentially based on the eigenvalue analysis of the linearized system. Here, only the result is given, for the sake of brevity and lack of space. The ASM1 model (68) has one fast dynamic, X_{DCO}, and five slow dynamics, represented by the rest of the state variables. Thus, the matrix E from (27) is given by: $E=\mathrm{diag}(0\ 1\ 1\ 1\ 1)$. Taking into account the process equations (68), it is natural to define the premise variables:

$$
\begin{aligned}
\xi_1(t)&=\frac{q_{in}(t)}{V},\quad \xi_2(t)=\frac{X_{DCO}(t)}{K_S+X_{DCO}(t)}\frac{S_O(t)}{K_{OH}+S_O(t)},\\
\xi_3(t)&=\frac{X_{DCO}(t)}{K_S+X_{DCO}(t)}\frac{S_{NO}(t)}{K_{NO}+S_{NO}(t)}\frac{K_{OH}}{K_{OH}+S_O(t)},\\
\xi_4(t)&=\frac{1}{K_{OA}+S_O(t)}\frac{S_{NH}(t)}{K_{NH,A}+S_{NH}(t)}X_{BA}(t) \qquad (72)
\end{aligned}
$$

depending on unmeasurable variables, the states x, and on inputs u. The decomposition of the premises (72) is realized by using the convex polytopic transformation, as in (6), the scalars ξ_j^1 and ξ_j^2 are respectively the upper and lower bounds of $\xi_j(t)$ and the functions $\bar{\mu}_j^1(\xi_j(t))$ and $\bar{\mu}_j^2(\xi_j(t))$ are defined similarly as in (7). The $r=2^4$ weighting functions $\mu_i(\xi(t))$ are obtained:

$$
\mu_i(\xi(t))=\bar{\mu}_1^{\sigma_i^1}(\xi_1(t))\bar{\mu}_2^{\sigma_i^2}(\xi_2(t))\bar{\mu}_3^{\sigma_i^3}(\xi_3(t))\bar{\mu}_4^{\sigma_i^4}(\xi_4(t)) \qquad (73)
$$

Let us note that the indexes σ_i^j $(i=1,...,16,\ j=1,...,4)$ previously used take values 1 or 2. The constant matrices A_i, B_i and E_i defining the $r=16$ submodels, are determined by using the scalars $\xi_j^{\sigma_i^j}$ and the matrices A, B and E:

$$
A_i=A(\xi_1^{\sigma_i^1},\xi_2^{\sigma_i^2},\xi_3^{\sigma_i^3},\xi_4^{\sigma_i^4}),\ B_i=B(\xi_1^{\sigma_i^1}),\ E_i=\bar{E}(\xi_1^{\sigma_i^1}) \qquad (74)
$$

where the matrices $A[a_{i,j}]\in\mathbb{R}^{6\times6}$, $B[b_{i,j}]\in\mathbb{R}^{6\times4}$ and $\bar{E}[e_{i,j}]\in\mathbb{R}^{6\times2}$ are defined by the coefficients $a_{11}(t)=a_{33}(t)=a_{44}(t)=-\xi_1(t)$, $b_{2,2}=KS_{O,sat}$, $b_{1,1}(t)=e_{3,1}(t)=e_{5,2}(t)=\xi_1(t)$,

$$
\begin{aligned}
&a_{15}(t)=-\frac{\mu_H}{Y_H}\xi_2(t)+(1-f_P)b_H-\frac{\mu_H\eta_{NOg}}{Y_H}\xi_3(t),\\
&a_{16}(t)=(1-f_P)b_A,\ a_{22}(t)=-\xi_1(t)-Kq_a-\frac{4.57-Y_A}{Y_A}\mu_A\xi_4(t),\\
&a_{25}(t)=\frac{(Y_H-1)\mu_H}{Y_H}\xi_2(t),\ a_{32}(t)=-\left(i_{XB}+\frac{1}{Y_A}\right)\mu_A\xi_4(t),\\
&a_{35}(t)=(i_{XB}-f_P i_{XP})b_H-i_{XB}\mu_H\xi_2(t)-i_{XB}\mu_H\eta_{NOg}\xi_3(t),\\
&a_{36}(t)=(i_{XB}-f_P i_{XP})b_A,\ a_{42}(t)=\frac{1}{Y_A}\mu_A\xi_4(t),\\
&a_{45}(t)=\frac{Y_H-1}{2.86Y_H}\mu_H\eta_{NOg}\xi_3(t),\\
&a_{55}(t)=\mu_H\xi_2(t)-b_H-\xi_1(t)\tilde{f}+\mu_H\eta_{NOg}\xi_3(t),\\
&a_{62}(t)=\mu_A\xi_4(t),\ a_{66}(t)=-\xi_1(t)\tilde{f}-b_A \qquad (75)
\end{aligned}
$$

The other unmentioned coefficients are null.

Let us now consider that the parameters $K=K(t)$, $b_H=b_H(t)$ and $\mu_A=\mu_A(t)$ are time-varying. Thus, $\theta(t)$ from (4) is $\theta(t)=[K(t),\ b_H(t),\ \mu_A(t)]^T$. In order to obtain the polynomial form (5) for the matrices A and B, a separation of the terms containing these varying parameters has to be done in matrices (74); these time-varying terms will be included in a new tilde matrix and put under the form $\sum_{j=1}^3\theta_j(t)A_{i,j}$ and $\sum_{j=1}^3\theta_j(t)B_{i,j}$, as in (5). The non-time-varying parameter terms will be included in the matrices $A_{i,0}$ and $B_{i,0}$.

By analyzing (75), we deduce that the parameter $K(t)$ is involved in a_{22} and b_{22}, the parameter $b_H(t)$ appears in a_{15}, a_{35} and a_{55}, and $\mu_A(t)$ in a_{22}, a_{32}, a_{42} and a_{62}. Thus, the matrices $A_{i,j}$ and $B_{i,j}$ are obtained similarly as A_i and B_i from (74), by using some intermediary matrices $\tilde{A}_1[\tilde{a}_{1,kl}]$ and $\tilde{B}_1[\tilde{b}_{1,kl}]$ (multiplying K), $\tilde{A}_2[\tilde{a}_{2,kl}]$ (multiplying b_H) and $\tilde{A}_3[\tilde{a}_{3,kl}]$ (multiplying μ_A) defined by: $\tilde{a}_{1,22} = q_a$, $\tilde{b}_{1,22} = S_{O,sat}$, $\tilde{a}_{2,15} = 1 - f_P$, $\tilde{a}_{2,35} = i_{XB} - f_P i_{XP}$, $\tilde{a}_{2,55} = -1$, $\tilde{a}_{3,22} = \frac{4.57 - Y_A}{Y_A}\xi_4(t)$, $\tilde{a}_{3,32} = -\left(i_{XB} + \frac{1}{Y_A}\right)\xi_4(t)$, $\tilde{a}_{3,42} = \frac{1}{Y_A}\xi_4(t)$ and $\tilde{a}_{3,62} = \xi_4(t)$. The other unmentioned coefficients are null. Noticing that only $\tilde{A}_3$ depends on $\xi_4(t)$, the matrices $A_{i,j}$ and $B_{i,j}$ $(i = 1, ..., r,\ j = 1, ..., 3)$ are defined by:

$$A_{i,1} = \tilde{A}_1, \quad A_{i,2} = \tilde{A}_2, \quad A_{i,3} = \tilde{A}_3(\xi_{4,\sigma_i^4}),$$
$$B_{i,1} = \tilde{B}_1, \quad B_{i,2} = 0, \quad B_{i,3} = 0$$

With these definitions and applying the sector nonlinearity approach (6) - (7) for θ, the ASM1 model (68) can be put under the form (8).

C. Active FTC based on adaptive observer

The proposed FTC is designed by solving the optimization problem from theorem 3.1. The parameter values are chosen to $\sigma = 0.7$, $\psi_1 = 25$, $\psi_3 = 18$ and $\Phi = 160$. The gains of the observer are not given here for space reasons. Figure 2 illustrates the results of the proposed control law obtained after applying the optimization problem. The observer rapidly and accurately estimates the faults and the parameter uncertainties, as shown in figures 1. In this approach, the adaptive observer may be considered as an FDI block for diagnosis.

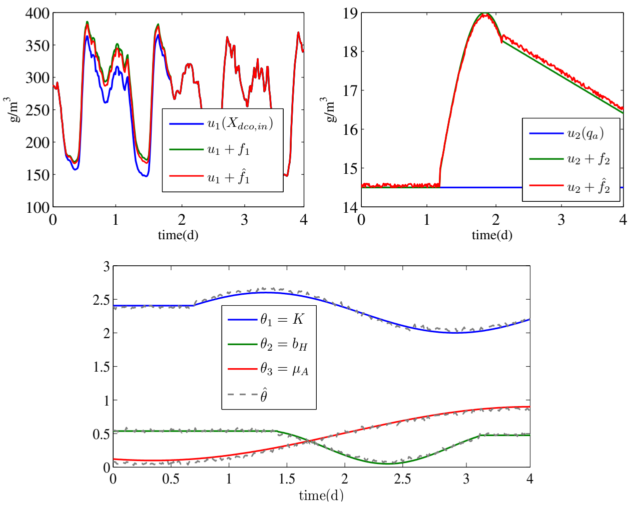

Fig. 1. Actuator faults and uncertainty estimation

V. CONCLUSIONS

This paper presented a new active fault tolerant control strategy for uncertain descriptor nonlinear systems represented by T-S models with unmeasurable premise variables and subjected to actuator faults. The originality of this work is the consideration of the estimation of the time varying parameters which can be viewed as parametric faults. A new adaptive observer is then proposed which provides, simultaneously, the estimation of states, faults and the time varying parameters (uncertainties). These estimated informations are used by the controller in order to compensate efficiently and robustly the faults even in the presence of time varying uncertainties. The proposed strategy is then applied for a real process of wastewater treatment. The obtained results illustrate clearly the performances of the proposed approach.

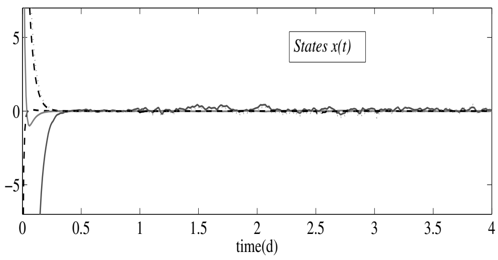

Fig. 2. Fault tolerant control: system states

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