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State, unknown input and uncertainty estimation for nonlinear systems using a Takagi-Sugeno model

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Abstract— **The paper addresses a systematic procedure to deal with the state, unknown input and parameter uncertainty estimation for nonlinear time-varying systems. This is realized by designing a robust observer for dynamic nonlinear systems using a Takagi-Sugeno (T-S) multi-model (MM) approach with nonlinear outputs. The method applies the technique of descriptor systems by considering unknown inputs and parameter uncertainty as auxiliary state variables. This approach allows to apply the tools of the linear automatic to dynamic nonlinear systems by using the Linear Matrix Inequalities (LMI) optimization. The observer estimates the previous mentioned variables and minimizes the effect of external disturbances on the estimation error. The model uncertainties are included in the model in a polynomial way which allows to consider the model uncertainty estimation as a fault detection problem. The residual sensitivity to faults while maintaining robustness according to a noise signal is handled by $\mathscr{H}_\infty/\mathscr{H}_-$ approach.**

I. INTRODUCTION

Most of the fault diagnosis methods for dynamic nonlinear systems (see [1], [2], [3], [4] and the reference therein) treat external faults (sensor/actuator). Only a few works exist on system faults caused by internal process modification [5], [6], [7], [8], due to modeling uncertainties or parameter varying. In the literature, the term uncertainty is related to the model parameters (model parameter uncertainty [5], [7]), to the model inputs (input uncertainty [9]) or to the computer implementation (model technical uncertainty [10]).

In the present work, the authors focus on the firstly mentioned class of uncertainty, model parameter uncertainty, needed to represent accurately the system behavior. This turns into using time-varying parameters in the model and leads to more challenging problems in the estimation process than using time invariant parameters. The main difficulty comes from the lack of knowledge on the parameter evolution. For the joint state and unknown parameter estimation, an idea is to consider the extended system, obtained by appending the unknown parameters into the state vector. Conventional observers, essentially developed for time invariant systems, cannot be used in this case. Adaptive observers developed for joint estimation have been presented [5], but only for linear systems. Extensions to nonlinear systems with unknown constant parameters have been developed in [7], [11]. In [6] robust fault detection for continuous-time switched delay systems is designed, the model uncertainties are norm-bounded and are considered in the model structure in an additive way, which restricts the study to quite a small class of systems. Recent work [8] focus on nonlinear systems with non measurable time-varying parameters that are considered model disturbances acting on the system evolution. The parameters are expressed as functions of their upper and lower bounds, according to the sector nonlinearity transformation [12].

In the field of observer design for fault diagnosis, the extension of linear methods for nonlinear systems is generally a difficult problem. Additionally, the complex behaviors of real systems demand a representation in a large operational domain involving nonlinear relations between the process variables, the model parameters, the control inputs and the external perturbations. Thus, it is a need to build models that can operate over a wide range of operating conditions. The T-S model has proven to be a powerful tool in the analysis/synthesis of nonlinear control and fault detection [13] and also in the representation of nonlinear systems on a compact set of the state space (chapter 14 of [12]). A methodology to rewrite dynamic nonlinear systems into T-S model is addressed in [14] and used in [15] for observer design of descriptor systems; this rewriting method will be called in this article to design dynamic nonlinear systems. In most works concerning the T-S models, the authors assume linear outputs in the T-S model, which is obviously the simplest case. A few works are devoted to the nonlinear output situation [16] and will be considered in the present work.

The main contribution of this paper is to propose a methodology to estimate state variables, model parameter uncertainties and unknown inputs for nonlinear systems, presented under a perturbed T-S multi-model formulation using robust residual generators. The modeling uncertainties occur in the system as polynomials which represents a more general class of uncertainties than those used in the majority of the published publications. The polynomial model uncertainties are then considered multiplicative faults, which transforms the model uncertainty estimation in a fault detection problem. In most of the existing research based on T-S models [17], [18], modeling uncertainties are considered in the model structure additively [19] and are norm bounded [20], [19], which restricts the study to small class of systems. Recent work, based on T-S descriptor systems for discrete-time [21], treats systems affected by sensor faults and bounded distur-

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bances but does not take into account model uncertainties neither nonlinear outputs. Up to the author's knowledge, this is the first contribution where the model uncertainty, unknown input and state estimation problem is treated in such a way for the nonlinear systems.

Robust fault detection observers with respect to external disturbances have been introduced over the years [22], based on $\mathscr{H}_\infty$ norm optimization techniques. A robust observer attenuating the impact of external disturbances while remaining sensitive to model uncertainties is proposed in this work. In a classical framework, the $\mathscr{H}_\infty$ norm maximizes the effect of the fault on the residual, but it can be reformulated as a minimization problem. The residual sensitivity to model uncertainties while maintaining robustness according to external disturbances motivates to introduce mixed $\mathscr{H}_\infty/\mathscr{H}_-$ approach for fault detection observer design. Thus, the robust residual generation can be considered a multi objective optimization problem. The linear matrix inequality (LMI) approach has been widely used for various types of filtering problems [23], [24] thanks to the ease in incorporating several design objectives in his formulation. This motivates us to consider the LMI tool for our methodology development. Thus, the sufficient conditions for $\mathscr{H}_\infty$ and $\mathscr{H}_-$ indexes are derived in the LMI framework.

The paper is organized as follows: section II illustrates the problem formulation by giving some preliminaries on the system structure, the robustness and the sensitivity conditions, section III gives the fault detection observer result. A numerical example is proposed in section IV in order to illustrate the effectiveness of the proposed method.

Notation 1.1: The star symbol $*$ in a symmetric matrix denotes the transposed block in the symmetric position. I and 0 are the identity matrix and the null matrix of appropriate dimensions, respectively. M^T and M^{-1} are the transpose and the inverse of matrix M, respectively. For the sake of simplicity, an abbreviated form will be used for $\mu_i(\xi(t)) \overset{not}{=} \mu_i(t)$, where ξ is the measurable premise variable vector.

II. Uncertain Takagi-Sugeno model formulation

A. Problem statement

Model uncertainty generally refers to a difference between the system model and the reality. It can be caused by changes within the process itself or in the environment around it. Let us consider a nonlinear dynamic model taking into account these changes as follows:

$$\begin{aligned} \dot{x}(t) &= f(x(t),u(t),\theta(t),d(t),w(t)) \\ y(t) &= g(x(t),u(t),\theta(t),d(t)) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^{n_u}$ is the input, $y \in \mathbb{R}^{n_y}$ is the output, $d \in \mathbb{R}^{n_d}$ is the unknown input, $w \in \mathbb{R}^{n_w}$ is the external disturbance and $\theta \in \mathbb{R}^{n_\theta}$ is the modeling uncertainty. f and g are continuous nonlinear functions. Let us consider that the system (1) is equivalently rewritten as the T-S multi-model:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \mu_i(t)\left[A_i(\theta(t))x(t) + B_i(\theta(t))u(t) + E_i d(t) + F_i w(t)\right] \\ y(t) &= C(\theta(t))x(t) + Gd(t) \end{aligned} \tag{2}$$

where r is the number of linear sub-models in the T-S MM form, $A_i(\theta(t))$, $B_i(\theta(t))$ and $C(\theta(t))$ are time varying matrices of appropriate dimensions, E_i, F_i and G are constant matrices of appropriate dimensions and where the weighting functions μ_i have the following property:

$$\sum_{i=1}^{r} \mu_i(t) = 1,\ \ \mu_i(t) \geq 0, \forall t \in \mathbb{R} \tag{3}$$

The T-S multi-model (2) is equivalent to the system (1) and is obtained by using the general methodology described in [14], where no uncertain system is considered. For instance, a slightly different T-S MM form is used here in (2), since the sub-models are linear parameter varying with matrices $A_i(\theta(t))$ and $B_i(\theta(t))$ depending on the uncertain parameter $\theta(t)$. In most studies [17], [20], [19], [25] the modeling uncertainties are norm bounded and are expressed additively in the state matrix of the dynamic nonlinear model [25].

In this paper, a more general class of modeling uncertainties is considered as follows. The uncertainties $\theta(t) = \left[\theta_1(t), \theta_2(t), \cdots \theta_{n_\theta}(t)\right]^T$ occur in a polynomial way in the T-S multi-model (2):

$$A_i(\theta(t)) = A_{i,0} + \sum_{j=1}^{n_A} \theta_{A\,j}(t) A_{i,j} \tag{4a}$$

$$B_i(\theta(t)) = B_{i,0} + \sum_{j=1}^{n_B} \theta_{B\,j}(t) B_{i,j} \tag{4b}$$

$$C(\theta(t)) = C_0 + \sum_{j=1}^{n_C} \theta_{C\,j}(t) C_j \tag{4c}$$

where $n_A, n_B, n_C \leq n_\theta$, $\theta_A(t)$, $\theta_B(t)$ and $\theta_C(t)$ are unknown time functions representing subsets of parameters of $\theta(t)$ to be estimated, matrices $A_{i,j}$ $(i = 1, \cdots, r,\ j = 0, \cdots, n_A)$, $B_{i,j}$ $(i = 1, \cdots, r,\ j = 0, \cdots, n_B)$ and C_j $(j = 0, \cdots, n_C)$ are constants known matrices and of appropriate dimensions. This type of uncertainties is directly related to malfunctions in the process that cause changes in the model parameters. They are also called multiplicative faults and are characterized by their direct influence on the system stability. One of the main objectives of the paper is to be able to estimate the model parameter changes with the final goal to adapt control law strategies for real process. Let us replace the matrices (4) in the system (2)

$$\begin{aligned} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(t) &\left[\left(A_{i,0} + \sum_{j=1}^{n_A} \theta_{A\,j}(t) A_{i,j}\right) x(t)\right. \\ &+ \left.\left(B_{i,0} + \sum_{j=1}^{n_B} \theta_{B\,j}(t) B_{i,j}\right) u(t) + E_i d(t) + F_i w(t)\right] \\ y(t) = &\left(C_0 + \sum_{j=1}^{n_C} \theta_{C\,j}(t) C_j\right) x(t) + Gd(t) \end{aligned} \tag{5}$$

Remark 2.1: Let us introduce the following change of variables

$$\begin{cases} \Psi : \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_f} \\ f = \Psi(\theta) \end{cases} \tag{6}$$

$$\Psi(\theta(t)) = \Phi(\theta(t))\,[Mx(t)+Nu(t)]$$

$$M = [\ \overbrace{I_n \ \cdots \ I_n}^{\times n_A}\ \overbrace{0 \ \cdots \ 0}^{\times n_B}\ \overbrace{I_n \ \cdots \ I_n}^{\times n_C}\ ]^T$$

$$N = [\ \overbrace{0 \ \cdots \ 0}^{\times n_A}\ \overbrace{I_{n_u} \ \cdots \ I_{n_u}}^{\times n_B}\ \overbrace{0 \ \cdots \ 0}^{\times n_C}\ ]^T$$

$$\Phi(\theta(t)) = \mathrm{diag}\,[\,\theta_{A1}(t)I_n\,, \cdots, \theta_{An_A}(t)I_n,\ \theta_{B1}(t)I_{n_u}\,, \cdots$$
$$\cdots, \theta_{Bn_B}(t)I_{n_u},\ \theta_{C1}(t)I_n\,, \cdots, \theta_{Cn_C}(t)I_n\,]$$

In conformity with the change of variables (6), the system (5) can be written as

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \mu_i(t)\left[A_{i,0}x(t)+B_{i,0}u(t)+\bar{E}_i\bar{d}(t)+F_iw(t)\right] \\ y(t) &= C_0x(t)+\bar{G}_0\bar{d}(t) \end{aligned} \tag{7}$$

where

$$\bar{E}_i = \begin{bmatrix} E_i & A_{i,1}\cdots A_{i,n_A} & B_{i,1}\cdots B_{i,n_B} & 0_{n,n_C\cdot n} \end{bmatrix} \tag{8}$$
$$\bar{G}_0 = \begin{bmatrix} G & 0_{n_y,n_A\cdot n} & 0_{n_y,n_B\cdot n_u} & C_1\cdots C_{n_C} \end{bmatrix} \tag{9}$$

with $\bar{d}(t) = [d^T(t) \ \ f^T(t)]^T$. In the following, the vector $f \in \mathbb{R}^{n_f}$ is considered as a fault signal. In this manner, the modeling uncertainties θ are transformed in faults f by using algebraic techniques. In the following we will refer only at faults by keeping in mind the modeling uncertainties.

Remark 2.2: Let us decompose the matrix $\bar{G}_0$ as follows

$$\bar{G}_0 = \bar{G} + G_1 \tag{10a}$$
$$\bar{G} = \begin{bmatrix} G & \Gamma_A & \Gamma_B & C_1 & \cdots & C_{n_C} \end{bmatrix} \tag{10b}$$
$$G_1 = \begin{bmatrix} 0 & -\Gamma_A & -\Gamma_B & 0 & \cdots & 0 \end{bmatrix} \tag{10c}$$

where $\Gamma_A \in \mathbb{R}^{n_y \times n_A \cdot n}$ and $\Gamma_B \in \mathbb{R}^{n_y \times n_B \cdot n_u}$ are given full column rank matrices. This decomposition ensures the full column rank property for the output matrix $\bar{G}$, required in the proposed estimation methodology.

In order to estimate the state x, the unknown input d and the fault f, an observer based on the technique of descriptor systems is synthesized. By using the remark 2.2, the T-S model (7) is rewritten as

$$\begin{aligned} \tilde{E}\dot{\tilde{x}}(t) &= \sum_{i=1}^{r} \mu_i(t)\left[\tilde{A}_i\tilde{x}(t)+\tilde{B}_iu(t)+\tilde{E}_i\bar{d}(t)+\tilde{G}x_s(t)+\tilde{F}_iw(t)\right] \\ y(t) &= \tilde{C}\tilde{x}(t) \\ &= C^*\tilde{x}(t)+x_s(t) \end{aligned} \tag{11}$$

where

$$x_s(t) = \bar{G}\bar{d}(t), \ \ \tilde{x}(t) = \begin{bmatrix} x(t) \\ x_s(t) \end{bmatrix} \tag{12}$$

and with the following matrix definitions

$$\tilde{E} = \begin{bmatrix} I_n & 0 \\ 0 & 0_{n_y} \end{bmatrix}, \tilde{A}_i = \begin{bmatrix} A_{i,0} & 0 \\ 0 & -I_{n_y} \end{bmatrix}, \tag{13a}$$
$$\tilde{B}_i = \begin{bmatrix} B_{i,0} \\ 0_{n_y} \end{bmatrix}, \ \ \tilde{E}_i = \begin{bmatrix} \bar{E}_i \\ 0_{n_y} \end{bmatrix}, \tag{13b}$$
$$\tilde{G} = \begin{bmatrix} 0_n \\ I_{n_y} \end{bmatrix}, \tilde{F}_i = \begin{bmatrix} F_i \\ 0_{n_y} \end{bmatrix}, C^* = \begin{bmatrix} C_0 & G^* \end{bmatrix}, \tag{13c}$$
$$G^* = G_1\left(\bar{G}^T\bar{G}\right)^{-1}\bar{G}^T, \tilde{C} = \begin{bmatrix} C_0 & I_{n_y} \end{bmatrix} \tag{13d}$$

In this paper, an observer that uses the input u, the output y and x_s as unknown input is considered, with the following structure

$$E\dot{z}(t) = \sum_{i=1}^{r} \mu_i(t)\left[K_iz(t)+\tilde{B}_iu(t)\right] \tag{14a}$$
$$\hat{\tilde{x}}(t) = z(t)+Ly(t) \tag{14b}$$
$$\begin{aligned} \hat{y}(t) &= C^*\hat{\tilde{x}}(t) \\ &= C\hat{x}(t) \end{aligned} \tag{14c}$$

where $z \in \mathbb{R}^{n+n_y}$ is an auxiliary state vector of the observer, $\hat{\tilde{x}}(t)$ is the state estimation of the system (11).

The observer gains E, K_i and L are to be determined.

Let us define the residual vector r as follows

$$r(t) = V\,[y(t)-\hat{y}(t)] \tag{15}$$

where V is the residual weighting matrix.

Let us define the error state $e(t)$ as follows

$$e(t) = \tilde{x}(t)-\hat{\tilde{x}}(t) \tag{16}$$

In the following the error dynamic is determined by substituting $z(t) = \hat{\tilde{x}}(t)-Ly(t)$ in (14a) and by subtracting the result from system (11)

$$\begin{aligned} &\left(\tilde{E}+EL\tilde{C}\right)\dot{\tilde{x}}(t) - E\dot{\hat{\tilde{x}}}(t) = \\ &\quad \sum_{i=1}^{r} \mu_i(t)\left[\left(\tilde{A}_i+K_iLC^*\right)\tilde{x}(t) - K_i\hat{\tilde{x}}(t) + \right. \\ &\quad \left. \left(K_iL+\tilde{G}\right)x_s(t)+\tilde{F}_iw(t)+\tilde{E}_i\bar{d}(t)\right] \end{aligned} \tag{17}$$

The dynamic of the error reduces to

$$E\dot{e}(t) = \sum_{i=1}^{r} \mu_i(t)\left[K_ie(t)+\tilde{F}_iw(t)+\tilde{E}_i\bar{d}(t)\right] \tag{18}$$

if the following conditions are fulfilled

$$\begin{aligned} E &= \tilde{E}+EL\tilde{C} \\ K_i &= \tilde{A}_i+K_iLC^*, \quad K_iL = -\tilde{G} \end{aligned} \tag{19}$$

The conditions (19) are respected for

$$K_i = \begin{bmatrix} A_i & 0 \\ -C & -I_{ny} \end{bmatrix}, L = \begin{bmatrix} 0 \\ I_{ny} \end{bmatrix}, E = \begin{bmatrix} I_n+XC & X \\ YC & Y \end{bmatrix} \tag{20}$$

The matrices X and Y with appropriate dimensions are to be determined.

Then, the error dynamic system is given by

$$\dot{e}(t) = S_\mu e(t)+\tilde{E}_\mu^d\bar{d}(t)+E_\mu^w w(t) \tag{21a}$$
$$r(t) = V\left[C^*e(t)+\bar{G}\bar{d}(t)\right] \tag{21b}$$

By using the dynamic of the error (18) and the matrices definitions (13) the compact form definition for S_μ, $\tilde{E}_\mu^d$ and $\tilde{E}_\mu^w$ is obtained

$$S_\mu = \sum_{i=1}^{r}\mu_i(t)S_i,\ \tilde{E}_\mu^d = \sum_{i=1}^{r}\mu_i(t)\tilde{E}_i^d,\ \tilde{E}_\mu^w = \sum_{i=1}^{r}\mu_i(t)\tilde{E}_i^w \tag{22}$$

where

$$S_i = \begin{bmatrix} A_i + XY^{-1}C & XY^{-1} \\ -CA_i - \left(Y^{-1} + CXY^{-1}\right)C & -\left(Y^{-1} + CXY^{-1}\right) \end{bmatrix}$$

$$\tilde{E_i^d} = \begin{bmatrix} \tilde{E}_i \\ C\tilde{E}_i \end{bmatrix}, \quad E_i^w = \begin{bmatrix} K_i \\ CK_i \end{bmatrix} \tag{23}$$

The system (21) can be written under the compact form

$$r(t) = T_{rd}\bar{d}(t) + T_{rw}w(t) \tag{24}$$

where T_{rd} stands for the transfer function from the fault $\bar{d}$ to the residual r and T_{rw} stands for the transfer function from the disturbance w to the residual and are defined by

$$T_{rd} := \left[\begin{array}{c|c} S_\mu & \tilde{E_\mu^d} \\ \hline VC^* & V\bar{G} \end{array}\right], \qquad T_{rw} := \left[\begin{array}{c|c} S_\mu & E_\mu^w \\ \hline VC^* & 0 \end{array}\right] \tag{25}$$

The matrices X and Y are to be determined in order to design the observer (14) and to guarantee the non singularity of the matrix E. Thus, the observer design reduces to find matrices Y and X (i.e. observer gain E) and V such that the matrices S_i are quadratically asymptotically stable and that the generated residual r is sensitive to faults $\bar{d}$ and robust with respect to disturbances w.

Definition 2.1: [4] The observer (14) is asymptotically stable with the $\mathscr{H}_\infty/\mathscr{H}_-$ performances if there exists two positive scalars γ_d and γ_w such that the following conditions hold

$$\|T_{rd}(s)\|_- > \gamma_d \tag{26}$$

$$\|T_{rw}(s)\|_\infty < \gamma_w \tag{27}$$

The goal is to find admissible filter (14) that minimize γ_w and maximize γ_d (i.e. an observer that ensures robustness with respect to disturbances w and sensitivity to faults $\bar{d}$). Various mixed $\mathscr{H}_\infty/\mathscr{H}_-$ optimization criteria have been recommended [26], such as: γ_w/γ_d, $a\gamma_w - b\gamma_d$, $\gamma_w^2 - \gamma_d^2$, $\alpha\gamma_w^2 + (1-\alpha)\gamma_d^2$ etc. Here, the last criterion will be used. The coefficient α is selected in order to balance the compromise between the robustness with respect to disturbance and the sensitivity to fault.

B. $\mathscr{H}_\infty$ robustness conditions

In this section, only robustness with respect to disturbance is considered by using the $\mathscr{H}_\infty$ performance.

Lemma 2.1: If there exist P symmetric positive definite and scalar $\gamma_w > 0$ such that the following conditions are satisfied

$$\begin{bmatrix} S_i^T P + PS_i + C^{*T}V^TVC^* & PE_i^w \\ * & -\gamma_w^2 I \end{bmatrix} < 0, \; i = 1, \cdots r \tag{28}$$

then the system (24) with $\bar{d} = 0$ ($r(t) = T_{rw}w(t)$) is stable with γ_w disturbance attenuation (27).

Proof: The condition (27) can be easily formulated as (28) using the bounded real lemma. See, for example [23]. ■

In the following result, robustness conditions with respect to disturbances w are derived in terms of linear matrix inequalities (LMI).

Theorem 2.1: If there exists symmetric matrices P_1 and P_2, matrices Z_1, Z_2 and V and positive scalar γ_w such that the following conditions are satisfied for $i = 1, \cdots, r$

$$\mathscr{M}_{1,i} = \begin{bmatrix} \Delta_{1,i} & \Delta_{2,i} & P_1F_i & C^TV^T \\ * & -Z_2 - Z_2^T & -P_2CF_i & G^{*T}V^T \\ * & * & -\bar{\gamma}_w I & 0 \\ * & * & * & -I \end{bmatrix} \leq 0 \tag{29}$$

where

$$\Delta_{1,i} = P_1A_i + A_i^TP_1^T - Z_1C - C^TZ_1^T \tag{30a}$$

$$\Delta_{2,i} = Z_1 - A_i^TC^TP_2 - C^TZ_2^T \tag{30b}$$

Then the estimation error (21) with $\bar{d} = 0$ is asymptotically stable with the performance (27) and the observer (14) is completely defined by (20) with

$$Y = \left(P_2^{-1}Z_2 - CP_1^{-1}Z_1\right)^{-1} \tag{31a}$$

$$X = P_1^{-1}Z_1Y \tag{31b}$$

Proof: From (28), by using the Schur complement and based on lemma 2.1, the following nonlinear inequality is obtained

$$\begin{bmatrix} S_i^TP + PS_i & PE_i^w & C^{*T}V^T \\ * & -\gamma_w^2 I & 0 \\ * & * & -I \end{bmatrix} < 0$$

Let us define $P = \text{diag}(P_1, P_2)$. Using (23), the following matrix inequality is obtained

$$\begin{bmatrix} \Psi_{1,i} & \Psi_{2,i} & P_1F_i & C^TV^T \\ * & \Psi_{3,i} & -P_2CF_i & G^{*T}V^T \\ * & * & -\gamma_w^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{32}$$

where

$$\Psi_{1,i} = P_1A_i + A_i^TP_1^T - P_1XY^{-1}C - (P_1XY^{-1}C)^T$$

$$\Psi_{2,i} = P_1XY^{-1} - (CA_i)^TP_2 - C^T(Y^{-1} + CXY^{-1})^TP_2$$

$$\Psi_{3,i} = -P_2(Y^{-1} + CXY^{-1}) - (Y^{-1} + CXY^{-1})^TP_2$$

Due to the coupling between P_1, P_2, X and Y, conditions (32) are nonlinear. In order to obtain the LMIs, the following variable changes are introduced

$$Z_1 = P_1XY^{-1} \tag{33}$$

$$Z_2 = P_2(Y^{-1} + CXY^{-1}) \tag{34}$$

$$\bar{\gamma}_w = \gamma_w^2 \tag{35}$$

By using (33) and (34), the matrices X and Y can be determined as defined in (31). The LMIs (29) are obtained, which completes the proof. ■

Since $\bar{G}$ is supposed of full column rank, in conformity with the remark 2.2, the fault estimation is obtained by using (12)

$$\hat{\bar{d}}(t) = \left(\bar{G}^T\bar{G}\right)^{-1}\bar{G}^T \begin{bmatrix} O_n & I_{n_y} \end{bmatrix} \hat{\bar{x}}(t) \tag{36}$$

C. $\mathscr{H}_-$ sensitivity conditions

In this section, the sensitivity of the residual r to faults $\bar{d}$ is discussed. In faulty case without disturbance w, the residual signal (24) is reduced to

$$r(t) = T_{rd}\bar{d}(t) \tag{37}$$

Theorem 2.2: If there exist symmetric positive definite matrices P_1 and P_2, matrices Z_1 and Z_2 and V and positive scalar γ_d such that the following inequalities are satisfied for $i = 1, \cdots, r$

$$\mathscr{M}_{2,i} = \begin{bmatrix} \Delta_{1,i} & \Delta_{2,i} & -P_1\bar{E}_i & C^TV^T \\ * & -Z_2 - Z_2^T & -P_2C\bar{E}_i & G^{*T}V^T \\ * & * & -\bar{\gamma}_d I & \bar{G}^TV^T \\ * & * & * & -I \end{bmatrix} \leq 0 \tag{38}$$

with $\Delta_{1,i}$ and $\Delta_{2,i}$ defined in (30), then the estimation error (37) is asymptotically stable with the performance (26).

Proof: Let us choose the Lyapunov function as

$$V(t) = e(t)^T Pe(t) \tag{39}$$

where P is symmetric positive definite matrix. By considering (37), we design the performance index

$$\begin{aligned} J_- &= \int_0^\infty r^T(t)r(t)d\tau - \gamma_d^2 \int_0^\infty \bar{d}^T(t)\bar{d}(t)d\tau \\ &= \int_0^\infty \left(r^T(t)r(t) - \gamma_d^2 \bar{d}^T(t)\bar{d}(t) - \frac{dV(t)}{d\tau} \right) d\tau + V(t) \\ &= \int_0^\infty \Big[\left(C^*e(t) + \bar{G}\bar{d}(t)\right)^T V^TV \left(C^*e(t) + \bar{G}\bar{d}(t)\right) \\ &\quad - \gamma_d^2 \bar{d}^T(t)\bar{d}(t) - \sum_{i=1}^r \mu_i(t)(S_ie(t))^T Pe(t) \\ &\quad - \sum_{i=1}^r \mu_i(t)e(t)^T PS_ie(t) \Big] d\tau + V(t) \\ &= -\int_0^\infty \sum_{i=1}^r \mu_i(t) \begin{bmatrix} e(t) \\ \bar{d}(t) \end{bmatrix}^T \Omega_i \begin{bmatrix} e(t) \\ \bar{d}(t) \end{bmatrix} d\tau + V(t) \end{aligned}$$

with

$$\Omega_i = \begin{bmatrix} S_i^TP + PS_i - (VC^*)^TVC^* & -P\tilde{E}_i^d + (VC^*)^T(V\bar{G}) \\ * & -\gamma_d^2 I - (V\bar{G})^T(V\bar{G}) \end{bmatrix}$$

If $\Omega_i \leq 0$ then $J_- \geq 0$. With $P = \text{diag}(P_1, P_2)$, $\bar{\gamma}_d = \gamma_d^2$ and the convex property (3), with (23) and (31) we get

$$\begin{bmatrix} \Delta_{1,i} - C^TV^TVC & \Delta_{2,i} & -P_1\bar{E}_i + C^TV^TV\bar{G} \\ * & -Z_2 - Z_2^T & -P_2C\bar{E}_i + G^{*T}V^TVG^{*T} \\ * & * & -\bar{\gamma}_d I - \bar{G}^TV^TV\bar{G} \end{bmatrix} \leq 0 \tag{40}$$

By using the bounded real lemma [23] and the Schur complement we get the LMIs (38). The observer (14) is completely defined by (20), with (31). ∎

III. $\mathscr{H}_\infty/\mathscr{H}_-$ FAULT DETECTION OBSERVER DESIGN

Theorem 3.1: Consider a positive parameter α. The robust observer (14) is obtained by finding symmetric positive definite matrices P_1, P_2, matrices Z_1, Z_2 and V and positive scalars $\bar{\gamma}_d$ and $\bar{\gamma}_w$ solution of the following optimization problem

$$\min_{P_1, P_2, Z_1, Z_2, V, \bar{\gamma}_d, \bar{\gamma}_w} \alpha\bar{\gamma}_d + (1-\alpha)\bar{\gamma}_w \tag{41}$$

such that the conditions (29) and (38) hold. The fault is estimated by (36). The observer gains are defined by (20) with (31). The attenuation levels are given by

$$\gamma_w = \sqrt{\bar{\gamma}_w} \quad \text{and} \quad \gamma_d = \sqrt{\bar{\gamma}_d} \tag{42}$$

Remark 3.1: As stated in the beginning of the paper, the polynomial model uncertainties θ are considered multiplicative faults f by using the change of variables (6). In addition to that, the unknown inputs d have been inserted in the fault vector $\bar{d}$ using algebraic techniques. In this way, the unknown input and model uncertainty estimation became a fault detection problem. More specifically, a robust residual observer has been proposed, in order to estimate the state variables x and the faults $\bar{d}$ by attenuating the effect of external disturbances w while maintaining the sensitivity to faults $\bar{d}$. In fact, the fault signals $\bar{d}$ give information about the unknown inputs d and the model uncertainties θ, in conformity with the change of variables presented in the remark 2.1 and by taking into account (36). Hence, estimating $\bar{d}$ is equivalent to estimate the unknown inputs d and the modeling uncertainties $\theta_{A_1}, \cdots, \theta_{A_{n_A}}$, $\theta_{B_1}, \cdots, \theta_{B_{n_B}}$, by applying the inverse change of variables: $\hat{\theta} = \Psi^{-1}(\hat{f})$. Using (36) and $\hat{f} = [0 \; I_{n_f}]\hat{\bar{d}}$, it is obtained:

$$\hat{\theta}(t) = \Psi^{-1}\left([0 \; I_{n_f}] \cdot \left(\bar{G}^T\bar{G}\right)^{-1} \bar{G}^T \left[O_n \; I_{ny}\right] \hat{\bar{x}}(t) \right) \tag{43}$$

or, equivalently

$$\Phi(\hat{\theta}(t)) = \hat{f}(t)\hat{q}^T(t)\left(\hat{q}(t)\hat{q}^T(t)\right)^{-1} \tag{44a}$$

$$\hat{q}(t) = M\hat{x}(t) + Nu(t) \tag{44b}$$

The equation (44) is obtained by using the change of variables (6), where the notation (44b) is used, as follows. Let us remind that

$$\Psi(\hat{\theta}(t)) = \Phi(\hat{\theta}(t))\hat{q}(t) \tag{45}$$

and replace $\Phi(\hat{\theta}(t)) = \hat{f}(t)$ in equation (45). By multiplying, at the right side, the obtained equality with $\hat{q}^T$ and then with $\left(\hat{q}\hat{q}^T\right)^{-1}$, we obtain the equality (44a).

IV. NUMERICAL EXAMPLE

Let us illustrate the effectiveness of the proposed robust fault detection method by considering the following academic example. Let us consider the T-S uncertain nonlinear system (2) defined by

$$
\begin{aligned}
A_i(\theta(t)) &= A_{i,0} + \theta_1(t)A_{i,1}, \text{ for } i = 1,2 \\
B_i(\theta(t)) &= B_{i,0} + \theta_2(t)B_{i,2}, \text{ for } i = 1,2 \\
C(\theta(t)) &= C_0 + \theta_3(t)C_3 \\
\theta_j(t) &= \theta_j^{nom} + \delta\theta_j(t), j = 1, \dots, 3 \\
\theta_1^{nom} &= 0.2, \ \theta_2^{nom} = 0.3, \ \theta_3^{nom} = 0.1 \\
\delta\theta_1(t) &= \begin{cases} 0.1\sin(t-2), & 2 \le t < 6 \\ 0, & \text{otherwise} \end{cases} \\
\delta\theta_2(t) &= \begin{cases} 0.15\sin(t-3.8), & 3.8 \le t < 7.8 \\ 0, & \text{otherwise} \end{cases} \\
\delta\theta_3(t) &= \begin{cases} 0.2(t-0.3), & 1 \le t < 3 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
A_{1,0} = \begin{bmatrix} -2 & -3 & 0 \\ 1 & 0 & -5 \\ 0.1 & 0.5 & -1 \end{bmatrix}, B_{1,0} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, A_{1,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.2 & 0 \\ -1 & 0 & 0 \end{bmatrix},
$$

$$
B_{1,2} = \begin{bmatrix} 0 \\ 0 \\ -0.5 \end{bmatrix}, \ A_{2,0} = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ 0 & 0.2 & -2.5 \end{bmatrix}, \ B_{2,0} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},
$$

$$
A_{2,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, B_{2,2} = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}, E_1 = \begin{bmatrix} 0.5 \\ 1 \\ -1 \end{bmatrix},
$$

$$
\begin{aligned}
E_2 &= \begin{bmatrix} 1 \\ 0.3 \\ 0.5 \end{bmatrix}, \ F_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ F_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ C_0 = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \\
C_3 &= \begin{bmatrix} 0 & 0.1 & 0 \\ 0.1 & 0 & -0.2 \end{bmatrix}, \ G = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}
\end{aligned} \tag{46}
$$

The weighting functions are defined as follows

$$
\mu_1(u(t)) = \frac{0.5 - \tanh((u(t)-1)/10)}{2}, \ \mu_2(u(t)) = 1 - \mu_1(u(t))
$$

Three parameter uncertainties are considered: θ_1 and θ_2 affecting the system dynamic by being involved in the matrices $A_i(\theta_1)$ and $B_i(\theta_2)$, and θ_3 affects the output y through the matrix $C(\theta)$. Matrices $A_i(\theta_1)$, $B_i(\theta_2)$ and $C(\theta)$ are defined as in (4), with (46). The resolution of the LMIs deriving from the optimization problem in theorem 3.1 with $\alpha = 0.75$ occurs in $\gamma_d = 0.224$ and $\gamma_w = 0.316$ and the observer gain matrices as follows

$$
V = \begin{bmatrix} 0.14 & -0.23 \end{bmatrix}, \quad K_i = \begin{bmatrix} A_{i,0} & 0_{3,2} \\ -C_0 & I_2 \end{bmatrix}
$$

$$
E = \begin{bmatrix} 1.38 & 0.13 & 0.63 & 0.13 & 0.50 \\ -0.18 & 2.08 & 1.97 & 1.07 & 0.89 \\ 0.14 & -0.19 & 0.75 & -0.19 & -0.05 \\ -0.03 & 0.03 & 0.03 & 0.03 & -0.02 \\ 0.05 & 0.01 & 0.08 & 0.01 & 0.06 \end{bmatrix}
$$

As illustrated in figures 1, 2, 3 the proposed method gives good estimation of the state variables, the unknown inputs but also of the parameter uncertainties.

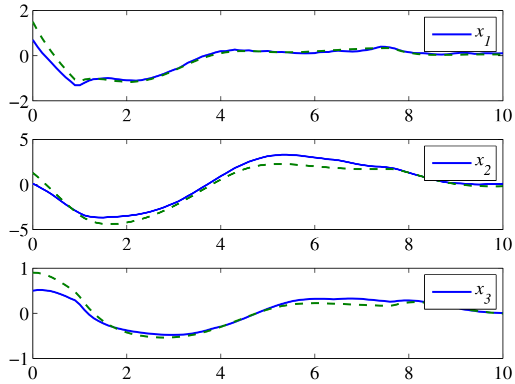

Fig. 1. State estimation: real state x and its estimated $\hat{x}$ (dotted lines)

V. Conclusions

This article designs state, unknown input and parameter uncertainty estimation based on a $\mathscr{H}_\infty/\mathscr{H}_-$ robust fault detection generator. The method applies the technique of descriptor systems by considering unknown inputs and parameter uncertainty as auxiliary state variables. Robustness with regard to external disturbances and sensitivity with respect to parameter uncertainties and unknown inputs are realized through robust fault detection observers. The robustness is based on $\mathscr{H}_\infty$ techniques, although the sensitivity regarding model uncertainties and unknown inputs is expressed using the $\mathscr{H}_-$ index. The nonlinear system is modeled using T-S multi-model approach which represents a very adequate and general way to represent various kinds of systems.

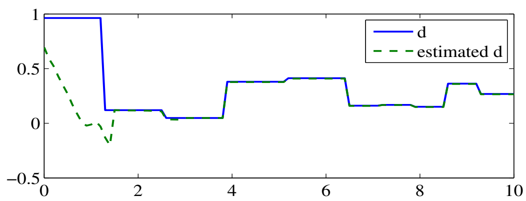

Fig. 2. Unknown input estimation

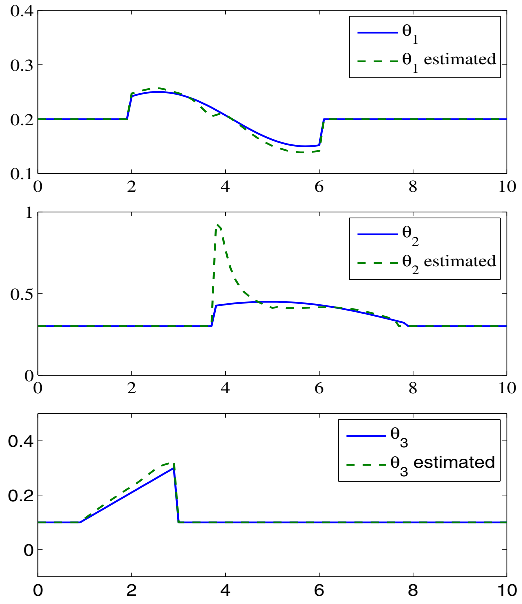

Fig. 3. Parameter estimation: $\theta_1(t)$, $\theta_2(t)$, $\theta_3(t)$

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REFERENCES

[1] A. Xu and Q. Zhang, "Nonlinear system fault diagnosis based on adaptive estimation," *Automatica*, vol. 40, no. 7, pp. 1181–1193, 2004.

[2] J. Ragot and D. Maquin, "Fault measurement detection in an urban water supply network," *Journal of Process Control*, vol. 16, no. 9, pp. 887–902, 2006.

[3] B. Marx, D. Koenig, and J. Ragot, "Design of observers for Takagi-Sugeno descriptor systems with unknown inputs and application to fault diagnosis," *IET Control Theory and Applications*, vol. 1, no. 5, pp. 1487–1495, 2007.

[4] S. X. Ding, *Model-based Fault Diagnosis Techniques*. Springer-Verlag, 2008.

[5] Q. Zhang, "Adaptive observer for multiple-input-multiple-output (MIMO) linear time-varying systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 3, pp. 525–529, 2002.

[6] D. Wang, P. Shi, and W. Wang, "Robust fault detection for continuous-time switched delay systems: an linear matrix inequality approach," *IET Control Theory Applications*, vol. 4, no. 1, pp. 100–108, 2010.

[7] G. Besançon, "Remarks on nonlinear adaptive observer design," *Systems and Control Letters*, vol. 41, no. 4, pp. 271–280, 2000.

[8] S. Bezzaoucha, B. Marx, D. Maquin, and J. Ragot, "Nonlinear joint state and parameter estimation: Application to a wastewater treatment plant," *Control Engineering Practice*, vol. 21, no. 10, pp. 1377 – 1385, 2013.

[9] F. Zouaoui and J. Wilson, "Input uncertainty: accounting for parameter uncertainty in simulation input modeling," in 33^{nd} *Conference on Winter simulation*, Arlington, VA, USA, 2001.

[10] J. Refsgaard, J. Van der Sluijs, A. Hojberg, and P. Vanrolleghem, "Uncertainty in the environmental modelling process - a framework and guidance," *Environmental Modelling & Software*, vol. 22, pp. 1543–1556, 2007.

[11] A. Poznyak and J. Martinez, "Variable structure robust state and parameter estimator," *International Journal of Adaptive Control and Signal Processing*, vol. 15, no. 4, pp. 179–208, 2001.

[12] K. Tanaka and H. O. Wang, *Fuzzy Control System Design and analysis. A Linear Matrix Inequality Approach*. John Wiley & Sons Inc., 2001.

[13] R. Murray-Smith and T. A. Johansen, *Multiple model approaches to modeling and control*. Taylor & Francis, London, 1997.

[14] A. M. Nagy, G. Mourot, B. Marx, G. Schutz, and J. Ragot, "Systematic multi-modeling methodology applied to an activated sludge reactor model," *Industrial & Engineering Chemistry Research*, vol. 49, no. 6, pp. 2790–2799, February, 2010.

[15] A. Nagy Kiss, B. Marx, G. Mourot, G. Schutz, and J. Ragot, "State estimation of two-time scale multiple models. Application to wastewater treatment plant," *Control Engineering Practice*, vol. 19, no. 11, pp. 1354–1362, 2011.

[16] D. Ichalal, B. Marx, J. Ragot, and D. Maquin, "Simultaneous state and unknown inputs estimation with PI and PMI observers for Takagi-Sugeno model with unmeasurable premise variables," in *Mediterranean Conference on Control and Automation*, Thessalonique, Grèce, 2009.

[17] W. Assawinchaichote, S. Nguang, and P. Shi, "$\mathscr{H}_\infty$ output feedback control design for uncertain fuzzy singularly perturbed systems: an LMI approach," *Automatica*, vol. 40, no. 12, pp. 2147 – 2152, 2004.

[18] A. Nagy-Kiss, B. Marx, G. Mourot, G. Schutz, and J. Ragot, "Observers design for uncertain Takagi-Sugeno systems with unmeasurable premise variables and unknown inputs. Application to a wastewater treatment plant," *Journal of Process Control*, vol. 21, no. 7, pp. 1105–1114, 2011.

[19] I. Zamani and M. H. Zarif, "On the continuous-time Takagi-Sugeno fuzzy systems stability analysis," *Applied Soft Computing*, vol. 11, no. 2, pp. 2102–2116, 2011.

[20] J. Yoneyama, "H_∞filtering for fuzzy systems with immeasurable premise variables: An uncertain system approach," *Fuzzy Sets and Systems*, vol. 160, no. 12, pp. 1738–1748, 2009.

[21] M. Chadli, A. Abdo, and S. Ding, "$\mathscr{H}_-/\mathscr{H}_\infty$ fault detection filter design for discrete-time Takagi-Sugeno fuzzy system," *Automatica*, vol. 49, pp. 1996–2005, 2013.

[22] R. Mangoubi, *Robust estimation and failure detection a concise treatment*. Berlin:Springer, 1998.

[23] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*. Philadelphia: Studies in Applied and Numerical Mathematics, 1994.

[24] I. Zamani, N. Sadati, and M. H. Zarif, "On the stability issues for fuzzy large-scale systems," *Fuzzy Sets and Systems*, vol. 174, no. 1, pp. 31–49, July 2011.

[25] A. M. Nagy-Kiss, B. Marx, G. Mourot, G. Schutz, and J. Ragot, "Observer synthesis for uncertain nonlinear systems. application to waste-water treatment plants," in 7^{th} *IFAC Symposium on Robust Control Design*, Aalborg, Denmark, June 2012.

[26] S. Ding, T. Jeinsh, P. Frank, and E. Ding, "A unified approach to the optimization of fault detection systems," *International Journal of Adaptive Control and Signal Processing*, vol. 14, no. 7, pp. 725–745, 2000.