State and multiplicative sensor fault estimation for nonlinear systems

Souad Bezzaoucha, Benoît Marx, Didier Maquin, José Ragot

***Abstract*— This paper addresses the state and sensor fault estimation for nonlinear systems represented by Takagi-Sugeno (T-S) models. The considered faults are time-varying and with multiplicative effect on the sensor output signals. The proposed estimation procedure is based firstly on the sector nonlinearity approach using the convex polytopic transformation where the original system is equivalently rewritten as a Takagi-Sugeno system with unmeasurable premise variables and, secondly on the design of an observer allowing the fault estimation by solving an LMI optimization problem. An application of the proposed approach to a simplified model of an activated sludge reactor model is proposed.**

***Index Terms*— Sensor faults estimation, Takagi-Sugeno models, sector nonlinearity approach, convex polytopic transformation, state and fault observer, linear matrix inequality.**

I. INTRODUCTION

Due to an increasing demand for higher performances, industrial processes tend to be, more and more safety critical, robust and reliable. For this reason, state and fault estimation, and more generaly diagnosis, especially the model-based ones, receive considerable interest from the control community.

In the literature, the available works either address the problems of both sensor fault tolerance and estimation accuracy (see [6] for example), where the Extended Kalman Filter is used as state estimator, or sensor redundancy management logic to maintain estimation functionality in nonlinear systems. In this kind of approach, only the state estimation is considered without any detection or estimation of the sensor fault. However, in the Fault Detection and Diagnosis (FDD) design, both fault detection and isolation are considered, usually by characterizing the normal behaviour of sensor readings, and identify significant deviation from this normal situation as faults [10].

In [2] for example, a strategy for detecting and isolating sensor faults using a bank of residuals within a generalized observer scheme (GOS) is presented. This scheme provides an estimator dedicated to a particular sensor, this estimator being driven by all outputs except that of the considered sensor. In [7], the authors proposed methods for state estimation and fault diagnosis for nonlinear systems represented by Takagi-Sugeno (T-S) models. In [3] and [1], both sensor fault and unknown input estimation were considered.

The main aim of the bank of observers is to generate an incidence matrix or a coupling graph between the fault and the residuals. In fact, a signal that is obtained from the generated residuals defines the effects associated with the fault. This strategy proved its efficiency for the detection and isolation part, but, it requires the same number of observers as outputs. Moreover, it should be highlighted that most of the considered sensor faults are additive ones, i.e. the measured outputs are linear in respect to the faults. The few available works on multiplicative faults use a sliding mode observer and are based on their re-modelling into the framework of additive faults and then apply previous works on reconstructing additive faults (see [12]).

In the present work, is tackled a simultaneous state and multiplicative sensor fault observer design for nonlinear systems represented by T-S models with unmeasurable premise variables. Based on the sector nonlinearity transformation (SNT) [13], [8], both state observer with sensor fault tolerance and estimation accuracy and time-varying fault observer (for detection, isolation and estimation at once) are designed. The proposed method has the advantage to be analytical and systematic without any loss of information since it consists in rewriting the time-varying sensor fault and the system nonlinearities in a polytopic form allowing to transform the original nonlinear system into a T-S model based on the sector nonlinearity approach and the convex polytopic transformation. The T-S model obtained with this transformation has the major interest to exactly represent the system without any loss of informations or any approximation.

The proposed method is an extention to the sensor fault estimation of the author's previous work presented in [5] and [4]. This procedure allows to design a state and fault observer by minimizing the $\mathscr{L}_2$ gain from the fault to the state and fault estimation errors. Using the Lyapunov theory, the $\mathscr{L}_2$ gain minimization is expressed as a LMI constraints. The paper is organized as follows. Section II introduces the T-S representation of the nonlinear system with multiplicative sensor fault. In section III, joint state and unknowm time-varying fault estimation is proposed for T-S systems with unmeasurable premise variables. An application of the proposed approach to a simplified model of an activated sludge reactor model with some simulation results are given in section IV. Conclusions are summarized in section V.

II. PROBLEM STATEMENT: T-S MODELLING OF MULTIPLICATIVE TIME-VARYING SENSOR FAULT

A first contribution of this work is to model nonlinear time-varying systems using the T-S representation. An efficient way consists in rewriting the original nonlinear system in a simplier form, like the T-S one. Originaly introduced by [11], the T-S representation is based on time-varying interpolation between time invariant linear submodels and

The authors are with niversité de Lorraine, Centre de Recherche en Automatique de Nancy CRAN, UMR 7039, 2 avenue de la Forêt de Haye, Vandoeuvre-lès-Nancy Cedex, 54516, France; CNRS, CRAN, UMR 7039, France. email: `firstname.name@univ-lorraine.fr`

allows to describe the exact nonlinear behaviour of a system, under the condition that its nonlinearities are bounded. This is reasonable as variables of physical systems are real and always bounded. See for example [11], [13] and the references therein.
The T-S representation is very interesting in the sense that it simplifies the mathematical developments for observer and controller design compared to the original nonlinear models Moreover, it allows to extend the use of some tools developed in the linear framework to the nonlinear systems, for the stability study, the control design and the observer synthesis.
In the present paper, the T-S models with multiplicative time-varying sensor fault are considered. In order to design a joint state and fault observer, each time-varying sensor fault is rewritten under a particular form.
Let us consider the nonlinear T-S system with multiplicative time-varying sensor fault represented by equation (1)

$$\begin{cases} \dot{x}(t) &= \sum_{i=1}^{r} \mu_i(x(t))(A_i x(t) + B_i u(t)) \\ y(t) &= C(t)x(t) \end{cases} \tag{1}$$

The weighting functions $\mu_i(\xi(t))$ of the r submodels satisfy the convex sum property

$$\begin{cases} \sum_{i=1}^{r} \mu_i(\xi(t)) = 1 \\ 0 \leq \mu_i(\xi(t)) \leq 1, \quad i = 1, \dots, r \end{cases} \tag{2}$$

It is importnt to note that in the remaining of the paper, the weighting functions depend on the state variable $x(t)$. This case is more difficult to deal with and then less studied than the one with measurable premise variables, but naturally appears when the T-S system is obtained with the SNT.
The matrice $C(t)$ is defined as follows

$$C(t) = (I_m + F(t))C \tag{3}$$

s.t. $F(t) \in \mathbb{R}^{m \times m}$ defined by:

$$F(t) = \text{diag}(f(t)) \tag{4}$$

where $\text{diag}(f(t))$ refers to a diagonal matrix with the terms $f_j(t)$ (sensor faults) on its diagonal. $F(t)$ is also expressed as

$$F(t) = \sum_{j=1}^{m} f_j(t) F_j \tag{5}$$

with F_j matrices of dimension $\mathbb{R}^{m \times m}$ and where the element of coordinate (i,i) is equal to 1 and 0 elsewhere. The terms $f_j(t)$ are time-varying unknown parameters and represent multiplicative sensor faults. Each fault component is unknown but bounded where the lower bound f_j^2 and the upper one f_j^1 are known. Thus $f_j(t)$ can be written as:

$$f_j(t) = \widetilde{\mu}_j^1(f_j(t)) f_j^1 + \widetilde{\mu}_j^2(f_j(t)) f_j^2, \; f_j(t) \in [f_j^2, f_j^1] \tag{6}$$

with

$$\widetilde{\mu}_j^1(f_j(t)) = \frac{f_j(t) - f_j^2}{f_j^1 - f_j^2}, \quad \widetilde{\mu}_j^2(f_j(t)) = \frac{f_j^1 - f_j(t)}{f_j^1 - f_j^2} \tag{7}$$

$$\widetilde{\mu}_j^1(f_j(t)) + \widetilde{\mu}_j^2(f_j(t)) = 1, \; \forall t$$

Replacing (6) in (5), it becomes:

$$F(t) = \sum_{j=1}^{m} \sum_{k=1}^{2} \widetilde{\mu}_j^k(f_j(t)) f_j^k F_j \tag{8}$$

In order to write $F(t)$ as a simple polytopic matrix, the convex sum property of the functions $\widetilde{\mu}_j^k(f_j(t))$ can be exploited (see [5] for calculation details):

$$F(t) = \sum_{j=1}^{2^m} \widetilde{\mu}_j(f(t)) \overline{F}_j \tag{9}$$

with

$$\begin{cases} \widetilde{\mu}_j(f(t)) = \prod_{k=1}^{m} \widetilde{\mu}_k^{\sigma_j^k}(f_k(t)) \\ \overline{F}_j = \sum_{k=1}^{m} f_k^{\sigma_j^k} F_j \end{cases} \tag{10}$$

where the $\widetilde{\mu}_j(f(t))$ satisfy the convex sum property.
In the following, $f(t)$ is the fault vector of components $f_j(t), j = 1, \dots, m$. The indices σ_j^k equal to 1 or 2, indicate which partition of the k^{th} parameter ($\widetilde{\mu}_k{}^1$ or $\widetilde{\mu}_k{}^2$) is involved in the j^{th} submodel. The relation between the j^{th} submodel and the σ_j^k indices are given by the following equation

$$j = 2^{m-1}\sigma_j^1 + 2^{m-2}\sigma_j^2 + \dots + 2^0 \sigma_j^m - (2^1 + 2^2 + \dots + 2^{m-1}) \tag{11}$$

Finally, using equations (10) and (3), (1) becomes:

$$\begin{cases} \dot{x}(t) &= \sum_{i=1}^{r} \mu_i(x(t))(A_i x(t) + B_i u(t)) \\ y(t) &= \sum_{j=1}^{2^m} \widetilde{\mu}_j(f(t)) \widetilde{C}_j x(t) \end{cases} \tag{12}$$

$$\widetilde{C}_j = C + \overline{F}_j C \tag{13}$$

III. State and sensor fault observer

Based on the obtained T-S model (12), a simultaneous state and sensor fault observer may be designed and implemented. An $\mathscr{L}_2$ attenuation approach will be proposed to minimize the effect of the time-varying fault on the state and fault error estimation.
The state and sensor fault observer is defined by

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t)) \left(A_i \hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t))\right) \\ \dot{\hat{f}}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t))(K_i(y(t) - \hat{y}(t)) - \alpha_i \hat{f}(t)) \\ \hat{y}(t) = \sum_{j=1}^{2^m} \widetilde{\mu}_j(\hat{f}(t)) \widetilde{C}_j \hat{x}(t) \end{cases} \tag{14}$$

where $L_i \in \mathbb{R}^{\,n_x \times m}$, $K_i \in \mathbb{R}^{\,m \times m}$ and $\alpha_i \in \mathbb{R}^{\,m \times m}$ are the gains to be determined such that the estimated state and fault converge to the system state and fault.
Let us define the state and fault estimation error $e_x(t)$ and $e_f(t)$ as

$$\begin{aligned} e_x(t) &= x(t) - \hat{x}(t) \\ e_f(t) &= f(t) - \hat{f}(t) \end{aligned} \tag{15}$$

Their dynamics cannot be easily computed directly from (15) since in (12) the weighting functions depend on the unmeasurable variables ($f(t)$ and $x(t)$). Because of that, based on the convex sum property of the weighting functions, (12) is rewritten as follow

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} [\mu_i(\hat{x}(t))(A_i x(t) + B_i u(t)) + \\ \quad (\mu_i(x(t)) - \mu_i(\hat{x}(t)))(A_i x(t) + B_i u(t))] \\ y(t) = \sum_{j=1}^{2^m} \left[\widetilde{\mu}_j(\hat{f}(t))\widetilde{C}_j x(t) + (\widetilde{\mu}_j(f(t)) - \widetilde{\mu}_j(\hat{f}(t)))\widetilde{C}_j x(t) \right] \end{cases} \tag{16}$$

This form allows a better comparison of $x(t)$ with $\hat{x}(t)$, since $\mu_i(\hat{x}(t))$ and $\widetilde{\mu}_j(\hat{f}(t))$ not only appears in (14), but also in (16). Let us define:

$$\Delta A(t) = \sum_{i=1}^{r} [\mu_i(x(t)) - \mu_i(\hat{x}(t))] A_i = \mathscr{A}\Sigma_A(t) E_A \tag{17}$$

$$\Delta B(t) = \sum_{i=1}^{r} [\mu_i(x(t)) - \mu_i(\hat{x}(t))] B_i = \mathscr{B}\Sigma_B(t) E_B \tag{18}$$

$$\Delta C(t) = \sum_{j=1}^{2^m} (\widetilde{\mu}_j(f(t)) - \widetilde{\mu}_j(\hat{f}(t)))\widetilde{C}_j = \mathscr{C}\Sigma_C(t) E_C \tag{19}$$

with

$$\begin{array}{l} \mathscr{A} = \left[\begin{array}{ccc} A_1 & \dots & A_r \end{array} \right], \Sigma_A(t) = \mathrm{diag}(\delta_1(t), \dots, \delta_r(t)), \\ \mathscr{B} = \left[\begin{array}{ccc} B_1 & \dots & B_r \end{array} \right], \Sigma_B(t) = \mathrm{diag}(\delta_1(t), \dots, \delta_r(t)), \\ \mathscr{C} = \left[\begin{array}{ccc} C_1 & \dots & C_{2^m} \end{array} \right], \Sigma_C(t) = \mathrm{diag}(\widetilde{\delta}_1(t), \dots, \widetilde{\delta}_{2^m}(t)), \\ E_A = \left[\begin{array}{ccc} I_{n_x} & \dots & I_{n_x} \end{array} \right]^T, \ E_B = \left[\begin{array}{ccc} I_{n_u} & \dots & I_{n_u} \end{array} \right]^T \\ E_C = \left[\begin{array}{ccc} I_{2^m} & \dots & I_{2^m} \end{array} \right]^T \\ \delta_i(t) = \mu_i(x(t)) - \mu_i(\hat{x}(t)), \widetilde{\delta}_j(t) = \widetilde{\mu}_j(f(t)) - \widetilde{\mu}_j(\hat{f}(t)) \end{array} \tag{20}$$

Thanks to property (2), it follows

$$-1 \leq \delta_i(t) \leq 1, -1 \leq \widetilde{\delta}_j(t) \leq 1 \tag{21}$$

which implies from definition (20)

$$\Sigma_A^T(t)\Sigma_A(t) \leq I, \quad \Sigma_B^T(t)\Sigma_B(t) \leq I, \quad \Sigma_C^T(t)\Sigma_C(t) \leq I \tag{22}$$

Using (17), (18) and (19), the system (16) is then written as an uncertain system given by:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t))((A_i + \Delta A(t))x(t) + (B_i + \Delta B(t))u(t)) \\ y(t) = \sum_{j=1}^{2^m} \widetilde{\mu}_j(\hat{f}(t))(\widetilde{C}_j + \Delta C(t))x(t) \end{cases} \tag{23}$$

From equations (23), (14) and (15), the dynamics of the state estimation error is given by

$$\begin{array}{l} \dot{e}_x(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t))(A_i e_x(t) + \Delta A(t)x(t) \\ \quad - L_i(y(t) - \hat{y}(t)) + \Delta B(t)u(t)) \end{array} \tag{24}$$

The output error $y(t) - \hat{y}(t)$ is then calculated as follows

$$y(t) - \hat{y}(t) = \sum_{j=1}^{2^m} \widetilde{\mu}_j(\hat{f}(t))(\widetilde{C}_j e_x(t) + \Delta C(t)x(t)) \tag{25}$$

Replacing (25) in (24), the dynamics of the state estimation error is given by

$$\begin{array}{l} \dot{e}_x(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^m} \mu_i(\hat{x}(t))\widetilde{\mu}_j(\hat{f}(t))((A_i - L_i\widetilde{C}_j)e_x(t) + \\ (\Delta A(t) - L_i\Delta C(t))x(t) + \Delta B(t)u(t)) \end{array} \tag{26}$$

From equations (25) and (14), the dynamics of the fault estimation is given by

$$\begin{array}{l} \dot{e}_f(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^m} \mu_i(\hat{x}(t))\widetilde{\mu}_j(\hat{f}(t))(-K_i\widetilde{C}_j e_x(t) - \alpha_i e_f(t) \\ \quad \dot{f}(t) - K_i\Delta C(t)x(t) + \alpha_i f(t)) \end{array} \tag{27}$$

Due to the coupling between the errors $e_f(t)$ and $e_x(t)$, it is convenient to consider the augmented vectors $e_a(t)$ and $\omega(t)$

$$e_a(t) = \begin{pmatrix} e_x(t) \\ e_f(t) \end{pmatrix}, \ \omega(t) = \begin{pmatrix} x(t) \\ f(t) \\ \dot{f}(t) \\ u(t) \end{pmatrix} \tag{28}$$

From (26), (27) and (28), it follows

$$\dot{e}_a(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^m} \mu_i(\hat{x}(t))\widetilde{\mu}_j(\hat{f}(t))(\Phi_{ij} e_a(t) + \Psi_i(t)\omega(t)) \tag{29}$$

with

$$\begin{array}{rcl} \Phi_{ij} & = & \begin{pmatrix} A_i - L_i\widetilde{C}_j & 0 \\ -K_i\widetilde{C}_j & -\alpha_i \end{pmatrix} \\ \Psi_i(t) & = & \begin{pmatrix} \Delta A(t) - L_i\Delta C(t) & 0 & 0 & \Delta B(t) \\ -K_i\Delta C(t) & \alpha_i & I & 0 \end{pmatrix} \end{array} \tag{30}$$

Considering (29), the objective is to design a joint state and fault observer with a minimal $\mathscr{L}_2$ gain of the transfer from $\omega(t)$ to $e_a(t)$. The computation of the observer gains is detailed in the next theorem.

Theorem 1: There exists a joint robust state and multiplicative sensor fault observer (14) for a nonlinear system (1) with an $\mathscr{L}_2$ gain from $\omega(t)$ to $e_a(t)$ bounded by β ($\beta > 0$) if there exists matrices $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, Γ_1, Γ_2, Γ_3, $\Gamma_4 > 0$, $\overline{\alpha}_i$, $\overline{K}_i$, R_i and scalars β, λ_1, $\lambda_{1C} > 0$, $\lambda_{2C} > 0$ and $\lambda_B > 0$ solutions of the optimization problem (31) under LMI constraints (32) and (33) (see next page)

$$\min_{P_1, P_2, R_i, \overline{K}_i, \overline{\alpha}_i, \lambda_1, \lambda_{1C}, \lambda_{2C}, \lambda_B} \beta \tag{31}$$

for $i = 1, \dots, r$ and $j = 1, 2^m$

$$\Gamma_k < \beta I \text{ for } k = 1, 2, 3, 4 \tag{32}$$

with

$$\begin{array}{l} Q_{ij}^{11} = P_1 A_i + A_i^T P_1 - R_i\widetilde{C}_j - \widetilde{C}_j^T R_i^T + I_{n_x} \\ Q^{33} = -\Gamma_1 + \lambda_1 E_A^T E_A + \lambda_{1C} E_C^T E_C + \lambda_{2C} E_C^T E_C \end{array} \tag{34}$$

The observer gains are given by

$$\begin{cases} L_i = P_1^{-1} R_i \\ K_i = P_2^{-1} \overline{K}_i \\ \alpha_i = P_2^{-1} \overline{\alpha}_i \end{cases} \tag{35}$$

$$
\begin{pmatrix}
Q_{ij}^{11} & -\widetilde{C}_j^T\overline{K}_i^T & 0 & 0 & 0 & 0 & P_1\mathscr{A} & P_1\mathscr{B} & R_i\mathscr{C} & 0 \\
* & -\overline{\alpha}_i-\overline{\alpha}_i^T+I_m & 0 & \overline{\alpha}_i & P_2 & 0 & 0 & 0 & 0 & \overline{K}_i\mathscr{C} \\
* & * & Q^{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\Gamma_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\Gamma_3 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\Gamma_4+\lambda_B E_B^T E_B & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\lambda_1 I & 0 & 0 & 0 \\
* & * & * & * & * & * & 0 & -\lambda_B I & 0 & 0 \\
* & * & * & * & * & * & 0 & 0 & -\lambda_{1C} I & 0 \\
* & * & * & * & * & * & 0 & 0 & 0 & -\lambda_{2C} I
\end{pmatrix} < 0 \tag{33}
$$

Proof: In the remaining of the paper, the following lemma is used:

Lemma 1: [14] Consider two matrices X and Y with appropriate dimensions, a time-varying matrice $\Delta(t)$ and a positive scalar ε. The following property is verified

$$X^T\Delta^T(t)Y+Y^T\Delta(t)X \leq \varepsilon X^TX+\varepsilon^{-1}Y^TY \tag{36}$$

for $\Delta^T(t)\Delta(t)\leq I$.

Let us consider the following quadratic Lyapunov function

$$V(e_a(t)) = e_a^T(t)Pe_a(t),\ P=P^T>0 \tag{37}$$

Using (29), its time derivative is given by

$$\dot{V}(e_a(t)) = \sum_{i=1}^{r}\sum_{j=1}^{2^m}\mu_i(\hat{x}(t))\widetilde{\mu}_j(\hat{f}(t))\left[e_a^T(t)((\Phi_{ij})^TP + P\Phi_{ij})e_a(t)+e_a^T(t)P\Psi_i(t)\omega(t)+\omega^T(t)\Psi_i^T(t)Pe_a(t)\right] \tag{38}$$

It is known that $e_a(t)$ asymptotically converges toward zero when $\omega(t)=0$ and that the $\mathscr{L}_2$ gain from $\omega(t)$ to $e_a(t)$ is bounded by β if the following inequality holds

$$\dot{V}(e_a(t))+e_a^T(t)e_a(t)-\omega^T(t)\Gamma\omega(t)<0 \tag{39}$$

with

$$\Gamma=\mathrm{diag}(\Gamma_k),\ \Gamma_k<\beta\ I,\ \text{for } k=1,2,3,4 \tag{40}$$

An adequate choice of Γ allows to attenuate the transfer from some components of $\omega(t)$ to $e_a(t)$.

From (38), (39) becomes:

$$\sum_{i=1}^{r}\sum_{j=1}^{2^m}\mu_i(\hat{x}(t))\mu_j(\hat{f}(t))\begin{pmatrix} e_a(t)\\ \omega(t)\end{pmatrix}^T \left(\left(\begin{array}{c|c}\Phi_{ij}^TP+P\Phi_{ij}+I_{2n_x} & P\Psi_i(t)\\ \hline \Psi_i^T(t)P & -\Gamma\end{array}\right)\right)\begin{pmatrix} e_a(t)\\ \omega(t)\end{pmatrix}<0 \tag{41}$$

The Lyapunov matrix P is chosen as

$$P=\mathrm{diag}(P_1,P_2) \tag{42}$$

From (15), (30), (40) and (42), (41) holds if

$$\mu_i(\hat{x}(t))\widetilde{\mu}_j(\hat{f}(t))\left(Q_{ij}+\mathscr{Q}(t)+\mathscr{Q}^T(t)\right)<0 \tag{43}$$

with:

$$
Q_{ij}=\left(\begin{array}{cc|cccc}
Q_{ij}^{11} & -\widetilde{C}_j^T\overline{K}_i^T & 0 & 0 & 0 & 0\\
* & -\overline{\alpha}_i-\overline{\alpha}_i^T+I_m & 0 & \overline{\alpha}_i & P_2 & 0\\ \hline
* & * & Q^{33} & 0 & 0 & 0\\
* & * & * & -\Gamma_2 & 0 & 0\\
* & * & * & * & -\Gamma_3 & 0\\
* & * & * & * & * & -\Gamma_4
\end{array}\right) \tag{44}
$$

$$Q_{ij}^{11}=P_1A_i+A_i^TP_1-R_i\widetilde{C}_j-\widetilde{C}_j^TR_i^T+I_{n_x} \tag{45}$$

As (43) is time depending, it is convenient to look for bound to the matrix $\mathscr{Q}(t)$. For that purpose, based on (17) and (18), the time-varying term of (43) can be expressed as:

$$
\begin{aligned}
\mathscr{Q}(t)=&\begin{pmatrix}P_1\mathscr{A}\\0\\0\\0\\0\\0\end{pmatrix}\Sigma_A(t)\begin{pmatrix}0 & 0 & E_A & 0 & 0 & 0\end{pmatrix}\\
&+\begin{pmatrix}P_1\mathscr{B}\\0\\0\\0\\0\\0\end{pmatrix}\Sigma_B(t)\begin{pmatrix}0 & 0 & 0 & 0 & 0 & E_B\end{pmatrix}\\
&+\begin{pmatrix}P_1L_i\mathscr{C}\\P_2K_i\mathscr{C}\\0\\0\\0\\0\end{pmatrix}\Sigma_C(t)\begin{pmatrix}0 & 0 & -E_c & 0 & 0 & 0\end{pmatrix}
\end{aligned} \tag{46}
$$

Using lemma 1 and property (21), there exists positive scalars λ_1, λ_B, λ_{1C} and λ_{2C}, such that

$$
\mathscr{Q}(t)+\mathscr{Q}^T(t)<\begin{pmatrix}
\mathscr{Q}^1 & 0 & 0 & 0 & 0 & 0\\
0 & \mathscr{Q}^2 & 0 & 0 & 0 & 0\\
0 & 0 & \mathscr{Q}^3 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \lambda_BE_B^TE_B
\end{pmatrix} \tag{47}
$$

with:

$$
\begin{aligned}
\mathscr{Q}^1 &= \lambda_1^{-1}P_1\mathscr{A}\mathscr{A}^TP_1+\lambda_B^{-1}P_1\mathscr{B}\mathscr{B}^TP_1+\\
&\quad(\lambda_{1C})^{-1}P_1L_i\mathscr{C}\mathscr{C}^TL_i^TP_1\\
\mathscr{Q}^2 &= \lambda_{2C}^{-1}P_2K_i\mathscr{C}\mathscr{C}^TK_i^TP_2\\
\mathscr{Q}^3 &= \lambda_1E_A^TE_A+(\lambda_{1C}+\lambda_{2C})E_C^TE_C
\end{aligned} \tag{48}
$$

for $i=1,\ldots,r$ and $j=1,\ldots,2^m$.

From inequality (47), since $\mu_i(\hat{x}(t))$ and $\widetilde{\mu}_j(\hat{f}(t))$ satisfy the convex sum property, with the variable changes (35), the LMI (33) implies (43) and (39). As a consequence, the $\mathscr{L}_2$-gain of the transfer from $\omega(t)$ to $e_a(t)$ is bounded by β, which achieves the proof. ■

IV. Numerical example

In this section, the proposed approach is applied to a biological wastewater treatment plant. A reduced form of an activated sludge reactor model is considered with only the carbon pollution and two state variables.

Starting from the nonlinear equations of the system, a T-S representation is given. Multiplicative sensor fault are considered. The objective is to synthesize an observer in order to simultaneously estimate the system states and the fault.

The process consists in mixing used waters with a rich mixture of bacteria in order to degrade the organic matter [9]. Under specific assumptions, some simplifications can be made and the nonlinear system can be represented with the following equations [5]:

$$\begin{cases} \dot{x}_1(t) = \frac{a x_1(t) x_2(t)}{x_2(t)+b} - x_1(t)u(t) \\ \\ \dot{x}_2(t) = -\frac{c a x_1(t) x_2(t)}{x_2(t)+b} + (d - x_2(t))u(t) \end{cases} \tag{49}$$

with $x_1(t)$ and $x_2(t)$, the biomass and substrat concentration respectively, and where a, b and d are known parameters. The input $u(t)$ represents the dwell-time in the treatment plant. The measured output is the biomass concentration ($y(t) = x_1(t)$).

It is assumed that a bounded multiplicative sensor fault $f_1(t)$ affects the output $y(t)$ such that:

$$y(t) = (1 + f_1(t))x_1(t) \tag{50}$$

As previously explained, $f_1(t)$ can also be written as:

$$f_1(t) = \widetilde{\mu}_1^1(f_1(t)) f_1^1 + \widetilde{\mu}_1^2(f_1(t)) f_1^2, \;\; f_1(t) \in [f_1^2, f_1^1] \tag{51}$$

with $f_1^2 = 0.125, f_1^1 = 0.625$, $\widetilde{\mu}_1^1(f_1(t))$ and $\widetilde{\mu}_1^2(f_1(t))$ are defined by (7). Parameters b, c, d have been identified and set to $b = 0.07$, $c = 0.7$ and $d = 2.5$.

From the system nonlinearities, let us consider the following premise variables:

$$z_1(t) = -u(t), \quad z_2(t) = \frac{a x_1(t)}{x_2(t)+b} \tag{52}$$

From (49) and (52), the following quasi-LPV form is obtained:

$$\dot{x}(t) = \begin{pmatrix} z_1(t) & z_2(t) \\ 0 & -c z_2(t) + z_1(t) \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ d \end{pmatrix} u(t) \tag{53}$$

Since a T-S model is obtained in a compact set of the state space, maximum and minimum values that occur in $z_1(t)$ and $z_2(t)$ may be calculated using the knowledge of the domain of variation of $u(t)$: $z_1(t) \in [-1, -0.2]$ and $z_2(t) \in [0.004, 15]$. Using the convex polytopic transformation, two partitions for each premise variable are constructed as follows:

$$\begin{cases} z_1(t) = F_{11}(z_1) z_1^2 + F_{12}(z_1) z_1^1 \\ z_2(t) = F_{21}(z_2) z_2^2 + F_{22}(z_2) z_2^1 \end{cases} \tag{54}$$

$$\text{with} \quad F_{11}(z_1) = \frac{z_1(t) - z_1^2}{z_1^1 - z_1^2}, \; F_{12}(z_1) = \frac{z_1^1 - z_1(t)}{z_1^1 - z_1^2}$$
$$F_{21}(z_2) = \frac{z_2(t) - z_2^2}{z_2^1 - z_2^2}, \; F_{22}(z_2) = \frac{z_2^1 - z_2(t)}{z_2^1 - z_2^2} \tag{55}$$

where the scalars z_1^1, z_1^2, z_2^1 and z_2^2 are defined as

$$\begin{aligned} z_1^1 &= \max_u z_1(t), \; z_1^2 = \min_u z_1(t) \\ z_2^1 &= \max_x z_2(t), \; z_2^2 = \min_x z_2(t) \end{aligned} \tag{56}$$

The submodels are defined by the pairs (A_i, B_i) with $i = 1, \ldots, 4$. Due to the choice of premise variables, all the B_i matrices are equal to $B^T = \begin{bmatrix} 0 & d \end{bmatrix}$. The matrices A_i are given by:

$$A_1 = \begin{pmatrix} z_1^1 & z_2^1 \\ 0 & -c z_2^1 + z_1^1 \end{pmatrix}, \; A_2 = \begin{pmatrix} z_1^1 & z_2^2 \\ 0 & -c z_2^2 + z_1^1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} z_1^2 & z_2^1 \\ 0 & -c z_2^1 + z_1^2 \end{pmatrix}, \; A_4 = \begin{pmatrix} z_1^2 & z_2^2 \\ 0 & -c z_2^2 + z_1^2 \end{pmatrix}$$

The T-S model of the system with the multiplicative sensor fault is obtained by an interpolation of the four previous submodels for the state and two submodels for the output equations:

$$\dot{x}(t) = \sum_{i=1}^{4} \mu_i(z(t))(A_i x(t) + B u(t)); \, y(t) = \sum_{j=1}^{2} \widetilde{\mu}_j(f_1(t)) \widetilde{C}_j x(t) \tag{57}$$

with $\widetilde{C}_1 = \begin{pmatrix} 1 + f_1^2 & 0 \end{pmatrix}$, $\widetilde{C}_2 = \begin{pmatrix} 1 + f_1^1 & 0 \end{pmatrix}$.

The weighting functions $\mu_j(f(t))$ are calculated from (7) and $\mu_i(z(t))$ as the following:

$$\begin{aligned} \mu_1(z(t)) &= F_{11}(z_1(t)) \; F_{21}(z_2(t)) \\ \mu_2(z(t)) &= F_{11}(z_1(t)) \; F_{22}(z_2(t)) \\ \mu_3(z(t)) &= F_{12}(z_1(t)) \; F_{21}(z_2(t)) \\ \mu_4(z(t)) &= F_{12}(z_1(t)) \; F_{22}(z_2(t)) \end{aligned} \tag{58}$$

As mentioned at the begining of this section, the objective is to synthesize a robust state and fault observer applying the proposed approach. To illustrate the time varying fault effect on the system, figure 1 depicts the output with and without the sensor fault. The system input, the state variables and their estimates, the time-varying fault and its estimate are depicted in the figures 2, 3 and 4 respectively. The initial conditions are taken as $x(0) = \begin{pmatrix} 0.1 & 1.5 \end{pmatrix}$ for the system and $\hat{x}(0) = \begin{pmatrix} 0.09 & 2.3 \end{pmatrix}$, $\hat{f}_1(0) = 0$ for the state and fault observer respectively. From the depicted figures, one can conclude on the efficiency of the synthetized state observer, since the two states are perfectly estimated as well as the time-varying multiplicative sensor fault $f_1(t)$.

V. Conclusion

In the present paper, a new systematic procedure is presented to deal with the state and multiplicative sensor fault estimation for nonlinear systems. It consists in transforming the original system into a Takagi-Sugeno model, based on the sector nonlinearity approach and the convex polytopic transformation. This transformation has the major interest to exactly represent the system without any loss of informations. The considered procedure is the following: from the nonlinear time-varying equations of the process, a global

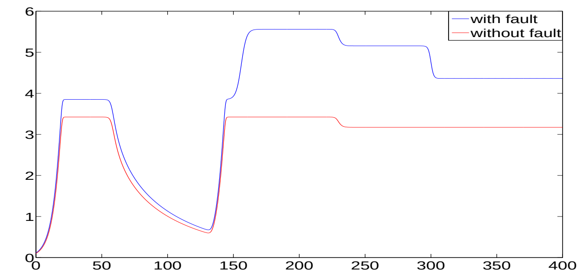

Fig. 1. Output with and without $f_1(t)$

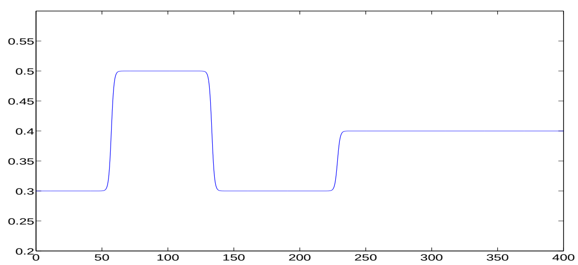

Fig. 2. System input $u(t)$

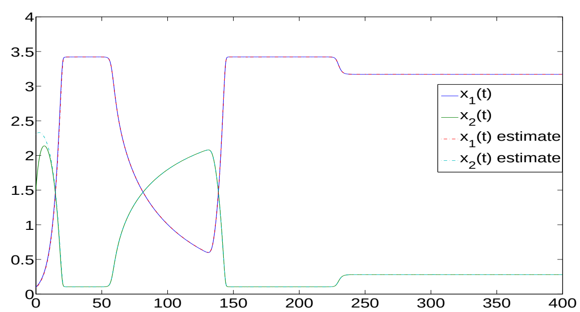

Fig. 3. System states and their estimates

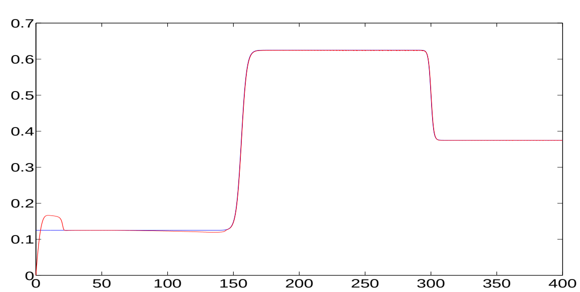

Fig. 4. Time-varying fault $f_1(t)$ (blue) and its estimate (red)

T-S model of the system is proposed. The proposed state and fault observer is then designed by solving an LMI optimization problem, i.e. by minimizing the $\mathscr{L}_2$ gain from the augmented input to the estimation errors. The chosen application example is an activated sludge reactor with multiplicative sensor fault on the output. From the nonlinear equations of the system, a T-S model of the system is derived. The proposed observer is synthetized and the obtained results illustrate its performance.

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