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State and Parameter Estimation for Time-varying Systems: a Takagi-Sugeno Approach

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1. Motivation and proposition

Motivation

To design an observer for time-varying linear systems

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To design an observer for time-varying linear systems

Proposition and outline

1. To describe the time-varying parameter behaviour using the sector nonlinearity approach and the convex polytopic transformation
2. To transform the original system into a Takagi-Sugeno model (with unmeasurable premise variables)
3. Establish the convergence conditions of the state and parameter estimation errors, which will be expressed in linear matrix inequalities (LMIs) formulation using the Lyapunov method.

Outline

1. Motivation and proposition
2. Problem statement
3. Observer design
4. Relaxed conditions
5. Noise measurement and filter synthesis
6. Illustrative example
7. Conclusions and perspectives

2. Problem statement

Time-varying linear system

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= Cx(t) \end{cases}$$

$$\begin{cases} A(t) &= A_0 + \theta(t)\overline{A}, \;\; \theta(t) \in [\underline{\theta}, \overline{\theta}] \\ B(t) &= B_0 + \theta(t)\overline{B} \end{cases}$$

$\theta(t) \in \mathbb{R}$ is a time-varying parameter, non measurable but bounded.

Polytopic decomposition

$$\theta(t) = \mu_1(\theta(t))\underline{\theta} + \mu_2(\theta(t))\overline{\theta}$$

$$\begin{cases} \mu_1(\theta(t)) &= \dfrac{\overline{\theta} - \theta(t)}{\overline{\theta} - \underline{\theta}} \\ \mu_2(\theta(t)) &= \dfrac{\theta(t) - \underline{\theta}}{\overline{\theta} - \underline{\theta}} \end{cases} \qquad \begin{cases} \sum_{i=1}^{2} \mu_i(\xi(t)) = 1 \\ 0 \leq \mu_i(\xi(t)) \leq 1, \quad i = 1, 2 \end{cases}$$

2. Problem statement

Time-varying linear system–> T-S (PLM) system

The initial linear system with the time-varying parameter $\theta(t)$ is expressed as a PLM :

$$\dot{x}(t) = \sum_{i=1}^{2} \mu_i(\theta(t))(A_i x(t) + B_i u(t))$$

$$\begin{cases} A_1 & = & A_0 + \underline{\theta}\,\overline{A}, \quad B_1 = B_0 + \underline{\theta}\,\overline{B} \\ A_2 & = & A_0 + \overline{\theta}\,\overline{A}, \quad \overline{B}_2 = B_0 + \overline{\theta}\,\overline{B} \end{cases}$$

Case of *n* parameters

The proposed writting is applicable to the vector case with n parameters $\theta_j(t)$ affecting the matrices $A(t)$ and $B(t)$. Using the PLM representation, each parameter is written under the polytopic form and then a compact T-S form is deduced.

3. Observer design

Joint state and time-varying parameter observer

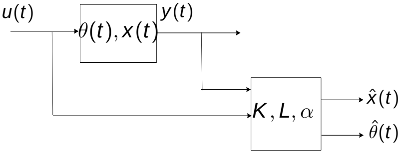

FIGURE : Joint state and time-varying parameter observer

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{2} \mu_i(\hat{\theta}(t))(A_i \hat{x}(t) + B_i u(t)) \\ +L_i(y(t) - \hat{y}(t)) \\ \dot{\hat{\theta}}(t) = \sum_{i=1}^{2} \mu_i(\hat{\theta}(t))(K_i(y(t) - \hat{y}(t)) \\ -\alpha_i \hat{\theta}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases}$$

Unknown gain matrices L_i, K_i and α_i to be computed to minimize the $\mathcal{L}_2$ gain from $\theta(t)$ to the state and parameter estimation errors :

- $e_x(t) = x(t) - \hat{x}(t)$ the state estimation error
- $e_\theta(t) = \theta(t) - \hat{\theta}(t)$ the time-varying parameter estimation error

3. Observer design

Difficulty

The estimation problem is not trivial since the activation functions in the system depend on $\theta(t)$, while those of the observer depend on its estimate $\hat{\theta}(t)$, then its dynamics cannot be directly computed.

Solution : Rewritting of the state estimation error

Based on the convex sum property of the weighting functions, rewrite the state equation as an uncertain-like system :

$$\dot{x}(t) = \sum_{i=1}^{2} \mu_i(\hat{\theta}(t)) \left((A_i + \Delta A(t))x(t) + (B_i + \Delta B(t))u(t) \right)$$

Rewritting of the state estimation error

$$\begin{aligned}\Delta A(t) &= \sum_{i=1}^{2}(\mu_i(\theta(t)) - \mu_i(\hat{\theta}(t)))\overline{A}_i \\ &= \mathcal{A}\Sigma_A(t)E_A \\ \Delta B(t) &= \sum_{i=1}^{2}(\mu_i(\theta(t)) - \mu_i(\hat{\theta}(t)))\overline{B}_i \\ &= \mathcal{B}\Sigma_B(t)E_B\end{aligned}$$

$$\mathcal{A} = \begin{bmatrix} A_1 & A_2 \end{bmatrix},\ \Sigma_A(t) = \begin{pmatrix} \delta_1(t)I_{n_x} & 0 \\ 0 & \delta_2(t)I_{n_x} \end{pmatrix},$$

$$\mathcal{B} = \begin{bmatrix} B_1 & B_2 \end{bmatrix},\ \Sigma_B(t) = \begin{pmatrix} \delta_1(t)I_{n_u} & 0 \\ 0 & \delta_2(t)I_{n_u} \end{pmatrix},$$

$$E_A = \begin{bmatrix} I_{n_x} & I_{n_x} \end{bmatrix}^T,\ E_B = \begin{bmatrix} I_{n_u} & I_{n_u} \end{bmatrix}^T$$

$$\Sigma_A^T(t)\Sigma_A(t) \leq I, \quad \Sigma_B^T(t)\Sigma_B(t) \leq I$$

3. Observer design

Estimation errors dynamic

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} \mu_i(\hat{\theta}(t)) \left((A_i + \Delta A(t))x(t) + (B_i + \Delta B(t))u(t)\right) \\ \dot{\hat{x}}(t) = \sum_{i=1}^{2} \mu_i(\hat{\theta}(t))(A_i\hat{x}(t) + B_i u(t)) + L_i(y(t) - \hat{y}(t)) \\ \dot{\hat{\theta}}(t) = \sum_{i=1}^{2} \mu_i(\hat{\theta}(t))(K_i(y(t) - \hat{y}(t)) - \alpha_i\hat{\theta}(t)) \end{cases}$$

$$\begin{aligned} \dot{e}_x(t) &= \sum_{i=1}^{2} \mu_i(\hat{\theta}(t)) \left((A_i - L_i C)e_x(t) + \Delta A(t)x(t) + \Delta B(t)u(t)\right) \\ \dot{e}_\theta(t) &= \sum_{i=1}^{2} \mu_i(\hat{\theta}(t)) \left(\dot{\theta}(t) - K_i C e_x(t) + \alpha_i\theta(t) - \alpha_i e_\theta(t)\right) \end{aligned}$$

3. Observer design

Augmented system dynamic

Let us consider the augmented vectors $e_a(t) = \begin{pmatrix} e_x^T(t) & e_\theta^T(t) \end{pmatrix}^T$ and $\omega(t) = \begin{pmatrix} x^T(t) & \theta^T(t) & \dot{\theta}^T(t) & u^T(t) \end{pmatrix}^T$.

$$\dot{e}_a(t) = \sum_{i=1}^{2} \mu_i(\hat{\theta}(t)) \left(\Phi_i e_a(t) + \Psi_i(t)\omega(t)\right)$$

$$\Phi_i = \begin{pmatrix} A_i - L_i C & 0 \\ -K_i C & -\alpha_i \end{pmatrix}, \quad \Psi_i(t) = \begin{pmatrix} \Delta A(t) & 0 & 0 & \Delta B(t) \\ 0 & \alpha_i & I & 0 \end{pmatrix}$$

Our objective is to guarantee the stability of the augmented system and the boundedness of the transfer from the input $\omega(t)$ to $e_a(t)$ (attenuate the effect of $\omega(t)$ on the estimation)

3. Observer design : Proceeding

$$\dot{e}_a(t) = \sum_{i=1}^{2} \mu_i(\hat{\theta}(t)) \left(\Phi_i e_a(t) + \Psi_i(t)\omega(t)\right)$$

Solution : the bounded real lemma is applied and the solution is expressed in terms of LMI

Considering the following quadratic Lyapunov function and the $\mathcal{L}_2$ criterion

$$V(e_a(t)) = e_a^T(t) P e_a(t),\ P = P^T > 0, P = \text{diag}(P_0, P_1)$$

$$\dot{V}(t) + e_a^T(t) e_a(t) - \omega^T(t) \Gamma_2 \omega(t) < 0$$

$$\Gamma_2 = \text{diag}(\Gamma_2^k),\ \Gamma_2^k < \beta I,\ \text{for } k = 0, 1, 2, 3$$

such that Γ_2 allows to attenuate the transfer of some $\omega(t)$ components to $e_a(t)$ components.

3. Observer design : Theorem 1/2

There exists a joint robust state and parameter observer for a linear time-varying parameter system with an $\mathcal{L}_2$-gain from $\omega(t)$ to $e_a(t)$, bounded by β, if there exist matrices $P_0 = P_0^T > 0$, $\Gamma_2^0 = (\Gamma_2^0)^T$, $\Gamma_2^3 = (\Gamma_2^3)^T > 0$, F_i and R_i and positive scalars P_1, β, λ_1, λ_2, Γ_2^1, Γ_2^2, and $\overline{\alpha}_i$, solution of the optimization problem (1) under LMI constraints (2) (next slide), for $i = 1, 2$:

$$\min_{P_0, P_1, R_i, F_i, \overline{\alpha}_i, \lambda_1, \lambda_2, \Gamma_2^0, \Gamma_2^1, \Gamma_2^2, \Gamma_2^3} \beta \tag{1}$$

$$\Gamma_2^k < \beta I, \text{ for } k = 0, 1, 2, 3$$

The observer gains are given by

$$\begin{cases} L_i = P_0^{-1} R_i \\ K_i = P_1^{-1} F_i \\ \alpha_i = P_1^{-1} \overline{\alpha}_i \end{cases}$$

3. Observer design : Theorem 2/2

$$\begin{pmatrix} M_{11} & -C^T F_i^T & 0 & 0 & 0 & 0 & P_0\mathcal{A} & P_0\mathcal{B} \\ * & -2\overline{\alpha}_i + 1 & 0 & \overline{\alpha}_i & P_1 & 0 & 0 & 0 \\ * & * & M_{33} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\Gamma_2^1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Gamma_2^2 & 0 & 0 & 0 \\ * & * & * & * & * & -\Gamma_2^3 + \lambda_2 E_B^T E_B & 0 & 0 \\ * & * & * & * & * & * & -\lambda_1 I & 0 \\ * & * & * & * & * & * & 0 & -\lambda_2 I \end{pmatrix} < 0 \quad (2)$$

$$\begin{cases} M_{11} = P_0 A_i + A_i^T P_0 - R_i C - C^T R_i^T + I_{n_x} \\ M_{33} = -\Gamma_2^0 + \lambda_1 E_A^T E_A \end{cases}$$

4. Relaxed conditions

Relaxed conditions

The LMIs to solve may be relaxed using the convex sum property of the weighting functions $\mu_i(t)$

$$\mu_2(t) = 1 - \mu_1(t)$$

$$\dot{x}(t) = \sum_{i=1}^{2} \mu_i(\hat{\theta}(t)) \left((\overline{A}_i + \Delta A(t))x(t) + (\overline{B}_i + \Delta B(t))u(t)\right)$$

$$\Delta A(t) = \delta_1(t)(A_1 - A_2) \qquad \text{instead of } \Delta A(t) = \sum_{i=1}^{2} (\mu_i(\theta(t)) - \mu_i(\hat{\theta}(t)))\overline{A}_i$$

$$\Delta B(t) = \delta_1(t)(B_1 - B_2) \qquad \text{instead of } \Delta B(t) = \sum_{i=1}^{2} (\mu_i(\theta(t)) - \mu_i(\hat{\theta}(t)))\overline{B}_i$$

$$\delta_1(t) = \mu_1(\theta(t)) - \mu_1(\hat{\theta}(t))$$

4. Relaxed conditions : new theorem 1/2

There exists a robust state and parameter observer for the linear time-varying parameter system with a bounded $\mathcal{L}_2$ gain $\beta > 0$ of the transfer from $\omega(t)$ to $e_a(t)$ if there exists matrices $P_0 = P_0^T > 0$, $\Lambda_1 = \Lambda_1^T$, $\Lambda_2 = \Lambda_2^T$, $\Gamma_2^0 = (\Gamma_2^0)^T$, $\Gamma_2^3 = (\Gamma_2^3)^T > 0$, F_i and R_i and positive scalars P_1, Γ_2^1, Γ_2^2 and $\overline{\alpha}_i$ solution of the optimization problem (3) under LMI constraints (4) (next slide), for $i = 1, 2$:

$$\min_{P_0,P_1,R_i,F_i,\overline{\alpha}_i,\Lambda_1,\Lambda_2,\Gamma_2^0,\Gamma_2^1,\Gamma_2^2,\Gamma_2^3} \beta \tag{3}$$

$$\Gamma_2^k < \beta I \text{ for } k = 0, 1, 2, 3$$

The observer gains are still given by

$$\begin{cases} L_i = P_0^{-1} R_i \\ K_i = P_1^{-1} F_i \\ \alpha_i = P_1^{-1} \overline{\alpha}_i \end{cases}$$

4. Relaxed conditions : new theorem 2/2

$$\begin{pmatrix} M_{11} & -C^T F_i^T & 0 & 0 & 0 & 0 & P_0 & P_0 \\ * & -2\overline{\alpha}_i + 1 & 0 & \overline{\alpha}_i & P_1 & 0 & 0 & 0 \\ * & * & -\Gamma_2^0 + \Lambda_A & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\Gamma_2^1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Gamma_2^2 & 0 & 0 & 0 \\ * & * & * & * & * & -\Gamma_2^3 + \Lambda_B & 0 & 0 \\ * & * & * & * & * & * & -\Lambda_1 & 0 \\ * & * & * & * & * & * & 0 & -\Lambda_2 \end{pmatrix} < 0 \tag{4}$$

where Λ_1 and Λ_2 are matrices (instead of scalars for the previous development) and where :

$$\Lambda_A = (\overline{A}_1 - \overline{A}_2)^T \Lambda_1 (\overline{A}_1 - \overline{A}_2)$$
$$\Lambda_B = (\overline{B}_1 - \overline{B}_2)^T \Lambda_2 (\overline{B}_1 - \overline{B}_2)$$

$$M_{11} = P_0 A_i + A_i^T P_0 - R_i C - C^T R_i^T + I_{n_x}$$

5. Noise measurement and filter synthesis

Time-varying linear system with measurement noise

$$\begin{cases} \dot{x}(t) & = & A(t)x(t) + B(t)u(t) \\ y(t) & = & Cx(t) + Gb(t) \end{cases}$$

where $b(t)$ is the noise measurement and matrices $A(t)$ and $B(t)$ have been already defined previuosly.

Objective

Our objective is to attenuate the effect of the parametric variation and the noise on the state and parameter estimations.

5. Noise measurement and filter synthesis

Augmented system dynamic

$$\dot{e}_a(t) = \sum_{i=1}^{2} \mu_i(\hat{\theta}(t)) \left(\Phi_i e_a(t) + \Psi_i(t)\omega(t)\right)$$

$$e_a(t) = \begin{pmatrix} e_x^T(t) & e_\theta^T(t) \end{pmatrix}^T, \; \omega(t) = \begin{pmatrix} x^T(t) & \theta^T(t) & \dot{\theta}^T(t) & u^T(t) & b^T(t) \end{pmatrix}^T$$

where $\omega(t)$ takes now into account the noise $b(t)$.

$$\Phi_i = \begin{pmatrix} A_i - L_i C & 0 \\ -K_i C & -\alpha_i \end{pmatrix}$$

$$\Psi_i(t) = \begin{pmatrix} \Delta A(t) & 0 & 0 & \Delta B(t) & -L_i G \\ 0 & \alpha_i & I & 0 & -K_i G \end{pmatrix}$$

Our objective is to design the joint state and parameter observer with a minimal $\mathcal{L}_2$-gain of the transfer from $\omega(t)$ to $e_a(t)$. The computation of the observer gains is detailed in the next theorem.

5. Noise measurement and filter synthesis

There exists a robust state and parameter observer for a linear time-varying parameters system subject to noise measurement with a bounded $\mathcal{L}_2$ gain β of the transfer from $\omega(t)$ to $e_a(t)$ ($\beta > 0$) if there exists matrices $P_0 = P_0^T > 0$, $\Lambda_1 = \Lambda_1^T$, $\Lambda_2 = \Lambda_2^T$, $\Gamma_2^0 = (\Gamma_2^0)^T$, $\Gamma_2^3 = (\Gamma_2^3)^T > 0$, $\Gamma_2^4 = (\Gamma_2^4)^T > 0$, F_i and R_i and positive scalars P_1, Γ_2^1, Γ_2^2 and $\overline{\alpha}_i$ solution of the optimization problem (5) under LMI constraints (6) (see next slide), for $i = 12$:

$$\min_{P_0, P_1, R_i, F_i, \overline{\alpha}_i, \Lambda_1, \Lambda_2,, \Gamma_2^0, \Gamma_2^1, \Gamma_2^2, \Gamma_2^3} \beta \tag{5}$$

$$\Gamma_2^k < \beta \text{ for } k = 0, 1, 2, 3, 4$$

The observer gains are still given by

$$\begin{cases} L_i = P_0^{-1} R_i \\ K_i = P_1^{-1} F_i \\ \alpha_i = P_1^{-1} \overline{\alpha}_i \end{cases}$$

5. Noise measurement and filter synthesis

$$\begin{pmatrix} M_{11} & -C^T F_i^T & 0 & 0 & 0 & 0 & -R_i G & P_0 & P_0 \\ * & -2\overline{\alpha}_i + 1 & 0 & \overline{\alpha}_i & P_1 & 0 & -F_i G & 0 & 0 \\ * & * & -\Gamma_2^0 + \Lambda_A & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\Gamma_2^1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\Gamma_2^2 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\Gamma_2^3 + \Lambda_B & 0 & 0 & 0 \\ * & * & * & * & * & * & -\Gamma_2^4 & 0 & 0 \\ * & * & * & * & * & * & * & -\Lambda_1 & 0 \\ * & * & * & * & * & * & * & 0 & -\Lambda_2 \end{pmatrix} < 0 \quad (6)$$

with :

$$\begin{aligned} \Lambda_A &= (A_1 - A_2)^T \Lambda_1 (A_1 - A_2) \\ \Lambda_B &= (B_1 - B_2)^T \Lambda_2 (B_1 - B_2) \\ M_{11} &= P_0 A_i + A_i^T P_0 - R_i C - C^T R_i^T + I_{n_x} \end{aligned}$$

6. Illustrative example

Let consider the linear time-varying system defined by :

$$\dot{x}(t) = (A_0 + \theta(t)\overline{A})x(t) + Bu(t)$$

$$y(t) = Cx(t) + Gb(t)$$

$$A_0 = \begin{pmatrix} -0.3 & -1 & -0.3 \\ 0.1 & -2 & -0.5 \\ -0.1 & 0 & -0.1 \end{pmatrix}, \overline{A} = \begin{pmatrix} 0 & -1.1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1.1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 \\ 0.5 \\ 0.25 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\theta(t)$ varies between $[0, 1]$.
$b(t)$ is a normal distribution with zero mean and standard deviation 15% of the output magnitude affecting the system ($G = I_2$).

6. Illustrative example

- Nominal state $x_n(t)$: $\dot{x}_n(t) = A_0 x_n(t) + Bu(t)$
- Time-varying system states $x_v(t)$: $\dot{x}(t) = (A_0 + \theta(t)\overline{A})x(t) + Bu(t)$

The following figure illustrates the state deviation caused by the time-varying parameter

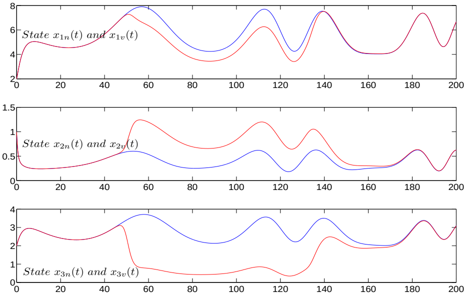

FIGURE : Nominal system (blue) -with parametric variation (red)

6. Illustrative example

To illustrate the actual and estimated states : $x_0 = \begin{pmatrix} 2 & 1 & 2 \end{pmatrix}$, $\hat{x}_0 = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$.

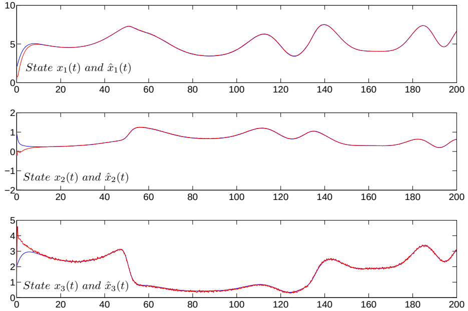

FIGURE : Actual and estimated states

Actual and estimated time-varying parameter

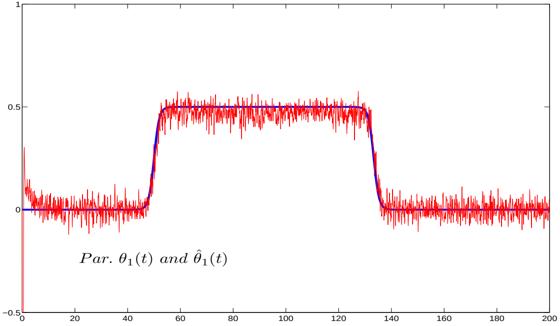

FIGURE : Time-varying parameter $\theta(t)$ and its estimate

Conclusions and perspectives

Conclusions

- A new systematic procedure was presented to deal with the state and parameter estimation for time-varying systems.
- Based on a T-S representation (by the sector nonlinearity approach and the convex polytopic transformation).
- The estimation problem and observer synthesis are expressed in terms of LMI optimization.
- First contribution where the time-varying problem is treated in such a way
- No assumption on the time-varying parameter and/or the system

Conclusions and perspectives

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- A new systematic procedure was presented to deal with the state and parameter estimation for time-varying systems.
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- The estimation problem and observer synthesis are expressed in terms of LMI optimization.
- First contribution where the time-varying problem is treated in such a way
- No assumption on the time-varying parameter and/or the system

Perspectives

- Application to nonlinear systems
- Use the results for Fault Tolerant Control (FTC)

Thanks

Thank you for your attention ! !

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Technical details

The bounded real lemma

Considering the system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{7}$$

System (7) is asymptotically stable with an attenuation of the transfer from $u(t)$ to $x(t)$ s.t. $\frac{||x||_2}{||u||_2} < \gamma$, if there exist a symmetric positive $P = P^T > 0$ and a positive scalar $\gamma > 0$ such that

$$\begin{pmatrix} A^T P + PA & PB \\ * & -\gamma I \end{pmatrix} < 0$$

where I is the identity matrix and $*$ denotes the symmetric item in a symmetric matrix.