Three-stage Kalman filter for state and fault estimation of linear stochastic systems with unknown inputs

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Abstract

The paper studies the problem of simultaneously estimating the state and the fault of linear stochastic discrete-time varying systems with unknown inputs. The fault and the unknown inputs affect both the system state and output. However, if the dynamical evolution models of the fault and the unknown inputs are available the filtering problem is solved by the Optimal Three-Stage Kalman Filter (OThSKF). The OThSKF is obtained after decoupling the covariance matrices of the Augmented state Kalman Filter (ASKF) using a Three-Stage U-V transformation. Nevertheless, if the fault and the unknown inputs models are not perfectly known the Robust Three-Stage Kalman Filter (RThSKF) is applied to give an unbiased minimum-variance estimation. Finally, a numerical example is given in order to illustrate the proposed filters.

Key words : stochastic systems, state estimation, fault estimation, unknown inputs decoupling, three-stage Kalman filter.

1 Introduction

Unknown input filtering (UIF) for linear stochastic systems has gained the interest of many researchers during the last decades. In this context, this problem has been extensively studied using the Kalman filtering approach, see e.g. [1]-[9].

When the model of the unknown inputs is available, it is possible to obtain an optimal estimation by using the Augmented State Kalman Filter (ASKF). To reduce computation costs of the ASKF, [2] has introduced the Two-Stage Kalman Filter (TSKF). His approach consists in decoupling the ASKF into the state subfilter and unknown inputs subfilter. Friedland's filter is only optimal for constant bias. Many authors have extended the Friedland's idea to treat the stochastic bias, e.g. [1, 10, 14, 15]. In the same context, [6] have generalized Friedland's filter by destroying the bias noise effect to obtain the Optimal Two-Stage Kalman Filter (OTSKF). [8] proposed a generalization of the OTSKF to get the Optimal Multi-Stage Kalman Filter (OMSKF). Recently, [11] have developed an adaptive version of TSKF noted ATSKF (Adaptive Two-Stage Kalman Filter) and they have analysed the stability of this filter in [12].

On the other hand, when the unknown inputs model is not available, the unbiased minimum variance (UMV) state estimations are insensitive with the unknown inputs. [16] has developed a Kalman filter with unknown inputs by minimizing the trace of the state error covariance matrix under an algebraic constraint. [13] have used a parameterizing technique as an extension of the Kitanidis's results to derive an UMV estimator. [5] has developed a robust filter in two-stage noted RTSKF (Robust Two-Stage Kalman Filter) equivalent to Kitanidis's filter. Next, the same author [3] has proposed an extension of the RTSKF (named ERTSKF) to solve the addressed general unknown-input filtering problem. For obtain ERTSKF, the author has introduced a new constrained relationship to modify the optimal unbiased minimum-variance filter (OUMVF) presented in [4]. Recently, [7] solved the problem of unbiased fault and state estimation for linear system with unknown disturbances in the case when we do not have a prior knowledge about the dynamical evolution of the fault and the unknown disturbances.

The main objective of this paper is to develop two new filter structures, that can solve the problem of simultaneously estimating the state and the fault in presence of the unknown inputs. In this case, when the dynamical evolutions of the fault and the unknown inputs are available, the Optimal Three-Stage Kalman Filter (OThSKF) is used. However, when the fault and the unknown inputs are not perfectly known, we develop the Robust Three-Stage Kalman Filter (RThSKF). This latter is obtained by using a modification in measurement update equations of the fault and the unknown inputs subfilters of the OThSKF. This idea constitutes an extension of RTSKF in [5].

The remainder of this paper is organized as follows. Section 2 states the problem of interest. In section 3, the design of the ASKF, OThSKF and RThSKF are developed. Finally, an illustrative example of the proposed approach techniques is presented.

2 Statement of the problem

The problem consists of designing a filter that gives a robust state and fault estimation for linear time-varying stochastic systems in the presence of unknown inputs. This problem is described by the bloc diagram of Figure 1.

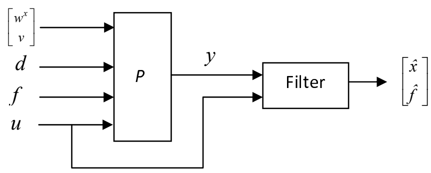

Figure 1: State and fault estimator filter

The plant P represents the linear time-varying discrete stochastic systems with unknown inputs and additive faults and is described by

$$P : \begin{cases} x_{k+1} & = A_k x_k + B_k u_k + F_k^x f_k + E_k^x d_k + w_k^x \\ y_k & = H_k x_k + F_k^y f_k + E_k^y d_k + v_k \end{cases} \tag{1}$$

where $x_k \in R^n$ is the state vector, $u_k \in R^r$ is the known control input, $f_k \in R^p$ is the additive fault vector, $d_k \in R^q$ is the unknown inputs and $y_k \in R^m$ is the observation vector. Matrices A_k, B_k, F_k^x, E_k^x, H_k, F_k^y and E_k^y are known and have appropriate dimensions.

We should treat d_k and f_k as a stochastic processes with given wide-sense representation.

Thus, the dynamics of d_k may be assumed as

$$d_{k+1} = d_k + w_k^d \tag{2}$$

The additive faults f_k is generated by

$$f_{k+1} = f_k + w_k^f \tag{3}$$

Assumptions :

• $\mathbf{A_1}$: the noises w_k^x et v_k are zero-mean white noise sequences with the following covariances :

$$\mathcal{E}(w_k^x w_\ell^{xT}) = Q_k^x \delta_{k\ell}$$
$$\mathcal{E}(v_k v_\ell^T) = R_k \delta_{k\ell}$$
$$\mathcal{E}(w_k v_\ell^T) = 0$$

where ^{T} denotes transpose and $\delta_{k\ell}$ denotes the Kronecker delta function.

- $\mathbf{A_2}$: the noises w_k^f and w_k^d are zero-mean white noise sequence with the following covariances:

$$\begin{cases} \mathcal{E}(w_k^f w_\ell^{fT}) = Q_k^f \delta_{k\ell} \\ \mathcal{E}(w_k^d w_\ell^{dT}) = Q_k^d \delta_{k\ell} \\ \mathcal{E}(w_k^x w_\ell^{fT}) = Q_k^{xf} \delta_{k\ell} \\ \mathcal{E}(w_k^x w_\ell^{dT}) = Q_k^{xd} \delta_{k\ell} \\ \mathcal{E}(w_k^f w_\ell^{dT}) = Q_k^{fd} \delta_{k\ell} \end{cases} \tag{4}$$

- $\mathbf{A_3}$: the initial state is a gaussian random variable and is uncorrelated with the white noise processes w_k^x and v_k :

$$\mathcal{E}(x_0) = \overline{x}_0 \text{ and } \mathcal{E}((x_0 - \overline{x}_0)(x_0 - \overline{x}_0)^T) = P_0^x.$$

- $\mathbf{A_4}$: the initial fault and unknown input satisfy the followings :

$$\begin{cases} \mathcal{E}(f_0) = \overline{f}_0 \\ \mathcal{E}(d_0) = \overline{d}_0 \\ \mathcal{E}((f_0 - \overline{f}_0)((f_0 - \overline{f}_0)^T) = P_0^f \\ \mathcal{E}((d_0 - \overline{d}_0)((d_0 - \overline{d}_0)^T) = P_0^d \\ \mathcal{E}((x_0 - f_0)((x_0 - f_0)^T) = P_0^{xf} \\ \mathcal{E}((x_0 - d_0)((x_0 - d_0)^T) = P_0^{xd} \\ \mathcal{E}((f_0 - d_0)((f_0 - d_0)^T) = P_0^{fd} \end{cases} \tag{5}$$

- $\mathbf{A_5}$: the first conditions on matrices ranks:

$$rank(H_{k+1}) = m(\geq max(p,q)),\ rank(F_k^x) = p,\ rank(H_{k+1}F_k^x) = p,\ rank(E_k^x) = q \text{ and } rank(H_{k+1}E_k^x) = q$$

- $\mathbf{A_6}$: the second conditions on matrices ranks:

$$rank(H_{k+1}F_k^x + F_{k+1}^y) = p \text{ and } rank(H_{k+1}E_k^x + E_{k+1}^y) = q$$

In this work, two cases will be considered:

- firstly, we assume that the noise statistical properties ($\mathbf{A_2}$) and the initial conditions of the fault and the unknown inputs ($\mathbf{A_4}$) are already known, so we will develop the OThSKF for state and fault optimal estimation,
- secondly, when the dynamical evolution models are not perfectly known, we implement the RThSKF for state and fault robust estimation.

3 Filters design

In this section, devoted to the state filter design, we first recall the structure of the augmented state Kalman filter, then the UV transformation is defined which is after used to decouple the augmented state Kalman filter equations into three subfilters. Finally, the three-stage Kalman filter (ThSKF) in two versions is developed: optimal (OThSKF) and robust (RThSKF).

3.1 Augmented State Kalman Filter (ASKF)

Treating x_k ,f_k and d_k as the augmented system state, the ASKF is described by

$$x^a_{k+1/k} = A^a_k x^a_{k/k} + B^a_k u_k \tag{6}$$

$$P^a_{k+1/k} = A^a_k P^a_{k/k} A^{aT}_k + Q_k \tag{7}$$

$$x^a_{k+1/k+1} = x^a_{k+1/k} + K^a_{k+1}(y_{k+1} - H^a_{k+1} x^a_{k+1/k}) \tag{8}$$

$$K^a_{k+1} = P^a_{k+1/k} H^{aT}_{k+1}(H^a_{k+1} P^a_{k+1/k} H^{aT}_{k+1} + R_{k+1})^{-1} \tag{9}$$

$$P^a_{k+1/k+1} = (I - K^a_{k+1} H^a_{k+1}) P^a_{k+1/k} \tag{10}$$

where

$$x^a_{(.)} = \begin{bmatrix} x_{(.)} \\ f_{(.)} \\ d_{(.)} \end{bmatrix}, A^a_k = \begin{bmatrix} A_k & F^x_k & E^x_k \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, B^a_k = \begin{bmatrix} B_k \\ 0 \\ 0 \end{bmatrix}, H^a_k = \begin{bmatrix} H_k & F^y_k & E^y_k \end{bmatrix}$$

$$P^a_{(.)} = \begin{bmatrix} P^x_{(.)} & P^{xf}_{(.)} & P^{xd}_{(.)} \\ P^{fx}_{(.)} & P^f_{(.)} & P^{fd}_{(.)} \\ P^{dx}_{(.)} & P^{df}_{(.)} & P^d_{(.)} \end{bmatrix}, Q_k = \begin{bmatrix} Q^x_k & Q^{xf}_k & Q^{xd}_k \\ Q^{fx}_k & Q^f_k & Q^{fd}_k \\ Q^{dx}_k & Q^{df}_k & Q^d_k \end{bmatrix}$$

The filter model (6)-(10) may be used to produce the optimal state estimate if the assumptions $\mathbf{A_1} - \mathbf{A_4}$ are checked. But, this filter has two main disadvantages: the increase of the computational cost with the augmentation of the state dimension and the rise of numerical problems during the implementation [5]. So, to solve these problems, we should use the three-stage Kalman filtering technique.

3.2 U-V transformations

According to [5] and [6], the ThSKF is obtained by the application of a three-stage U-V transformation in order to decouple the ASKF covariance matrices, i.e. $P^a_{k+1/k}$ and $P^a_{k+1/k+1}$. The aim is to find matrices U_{k+1} and V_{k+1} such that

$$P^a_{k+1/k} = U_{k+1}\overline{P}^a_{k+1/k}U^T_{k+1} \tag{11a}$$

$$P^a_{k+1/k+1} = V_{k+1}\overline{P}^a_{k+1/k+1}V^T_{k+1} \tag{11b}$$

with $\overline{P}^a_{(.)} = diag\left\{\overline{P}^x_{(.)}, \overline{P}^f_{(.)}, \overline{P}^d_{(.)}\right\}$ where $\overline{P}^x_{(.)}$,$\overline{P}^f_{(.)}$ and $\overline{P}^d_{(.)}$ denote the transformed covariance matrices.
We define the structures of the U_{k+1} and V_{k+1} matrices as follow:

$$U_{k+1} = \begin{bmatrix} I & U^{12}_{k+1} & U^{13}_{k+1} \\ 0 & I & U^{23}_{k+1} \\ 0 & 0 & I \end{bmatrix} \tag{12a}$$

$$V_{k+1} = \begin{bmatrix} I & V^{12}_{k+1} & V^{13}_{k+1} \\ 0 & I & V^{23}_{k+1} \\ 0 & 0 & I \end{bmatrix} \tag{12b}$$

$U^{ij}_{k+1}$and V^{ij}_{k+1} for $i = 1 \ or \ 2$ and $j = 2 \ or \ 3$ are to be determined later.

Using the transformations (12), the equations (6),(8) and (9) are transformed into

$$x^a_{k+1/k} = U_{k+1}\overline{x}^a_{k+1/k} \tag{13a}$$

$$x^a_{k+1/k+1} = V_{k+1}\overline{x}^a_{k+1/k+1} \tag{13b}$$

$$K^a_{k+1} = V_{k+1}\overline{K}^a_{k+1} \tag{13c}$$

The inverse transformations of U_{k+1} and V_{k+1} (12) will have this form

$$U_{k+1}^{-1} = \widetilde{U}_{k+1} = \begin{bmatrix} I & \widetilde{U}_{k+1}^{12} & \widetilde{U}_{k+1}^{13} \\ 0 & I & \widetilde{U}_{k+1}^{23} \\ 0 & 0 & I \end{bmatrix} \tag{14a}$$

$$V_{k+1}^{-1} = \widetilde{V}_{k+1} = \begin{bmatrix} I & \widetilde{V}_{k+1}^{12} & \widetilde{V}_{k+1}^{13} \\ 0 & I & \widetilde{V}_{k+1}^{23} \\ 0 & 0 & I \end{bmatrix} \tag{14b}$$

By direct computation, it is straigthforward to obtain

$$\widetilde{U}_{k+1}^{12} = -U_{k+1}^{12},\ \widetilde{U}_{k+1}^{13} = U_{k+1}^{12}U_{k+1}^{23} - U_{k+1}^{13} \text{ and } \widetilde{U}_{k+1}^{23} = -U_{k+1}^{23}$$
$$\widetilde{V}_{k+1}^{12} = -V_{k+1}^{12}, \widetilde{V}_{k+1}^{13} = V_{k+1}^{12}V_{k+1}^{23} - V_{k+1}^{13} \text{ and } \widetilde{V}_{k+1}^{23} = -V_{k+1}^{23}$$

Using these inverse transformations (14), we have

$$\overline{x}_{k+1/k}^{a} = \widetilde{U}_{k+1}x_{k+1/k}^{a} \tag{15a}$$
$$\overline{P}_{k+1/k}^{a} = \widetilde{U}_{k+1}P_{k+1/k}^{a}\widetilde{U}_{k+1}^{T} \tag{15b}$$

$$\overline{x}_{k+1/k+1}^{a} = \widetilde{V}_{k+1}x_{k+1/k+1}^{a} \tag{16a}$$
$$\overline{K}_{k+1}^{a} = \widetilde{V}_{k+1}K_{k+1}^{a} \tag{16b}$$
$$\overline{P}_{k+1/k+1}^{a} = \widetilde{V}_{k+1}P_{k+1/k+1}^{a}\widetilde{V}_{k+1}^{T} \tag{16c}$$

where

$$\overline{x}_{(.)}^{a} = \begin{bmatrix} \overline{x}_{(.)} \\ \overline{f}_{(.)} \\ \overline{d}_{(.)} \end{bmatrix}, \overline{P}_{(.)}^{a} = \begin{bmatrix} \overline{P}_{(.)}^{x} & 0 & 0 \\ 0 & \overline{P}_{(.)}^{f} & 0 \\ 0 & 0 & \overline{P}_{(.)}^{d} \end{bmatrix} \text{ and } \overline{K}_{(.)}^{a} = \begin{bmatrix} \overline{K}_{(.)}^{x} \\ \overline{K}_{(.)}^{f} \\ \overline{K}_{(.)}^{d} \end{bmatrix}$$

3.3 Decoupling

Using the two-step substitution method, the filter model (6)-(10) is transformed into

$$\overline{x}_{k+1/k}^{a} = \widetilde{U}_{k+1}\overline{U}_{k+1}\overline{x}_{k/k}^{a} + \widetilde{U}_{k+1}B_{k}^{a}u_{k} \tag{17}$$
$$\overline{P}_{k+1/k}^{a} = \widetilde{U}_{k+1}(\overline{U}_{k+1}P_{k/k}^{a}\overline{U}_{k+1}^{T} + Q_{k}^{a})\widetilde{U}_{k+1} \tag{18}$$
$$\overline{x}_{k+1/k+1}^{a} = \widetilde{V}_{k+1}\overline{U}_{k+1}\overline{x}_{k+1/k}^{a} + \overline{K}_{k+1}^{a}(y_{k+1} - S_{k+1}\overline{x}_{k+1/k}^{a}) \tag{19}$$
$$\overline{K}_{k+1}^{a} = \widetilde{V}_{k+1}\overline{U}_{k+1}\overline{P}_{k+1/k}^{a}S_{k+1}^{T}(S_{k+1}\overline{P}_{k+1/k}^{a}S_{k+1}^{T} + R_{k+1})^{-1} \tag{20}$$
$$\overline{P}_{k+1/k+1}^{a} = (\widetilde{V}_{k+1}\overline{U}_{k+1} - \overline{K}_{k+1}^{a}S_{k+1})\overline{P}_{k+1/k}^{a}(\widetilde{V}_{k+1}\overline{U}_{k+1})^{-1} \tag{21}$$

where

$$\overline{U}_{k+1} = A_{k}^{a}V_{k} = \begin{bmatrix} A_{k} & \overline{U}_{k+1}^{12} & \overline{U}_{k+1}^{13} \\ 0 & I & \overline{U}_{k+1}^{23} \\ 0 & 0 & I \end{bmatrix} \tag{22a}$$

$$S_{k+1} = \begin{bmatrix} S_{k+1}^{1} & S_{k+1}^{2} & S_{k+1}^{3} \end{bmatrix} \tag{22b}$$

with

$$\overline{U}_{k+1}^{12} = A_k V_k^{12} + F_k^x \tag{23a}$$
$$\overline{U}_{k+1}^{13} = A_k V_k^{13} + F_k^x V_k^{23} + E_k^x \tag{23b}$$
$$\overline{U}_{k+1}^{23} = V_k^{23} \tag{23c}$$

$$S_{k+1}^1 = H_{k+1} \tag{24a}$$
$$S_{k+1}^2 = H_{k+1} U_{k+1}^{12} + F_{k+1}^y \tag{24b}$$
$$S_{k+1}^3 = H_{k+1} U_{k+1}^{13} + F_{k+1}^y U_{k+1}^{23} + E_{k+1}^y \tag{24c}$$

Now, by developing the equations (18) we obtain respectively.

$$\overline{P}_{k+1/k}^x = A_k \overline{P}_{k/k}^x A_k^T + \overline{Q}_k^1 \tag{25}$$
$$\overline{P}_{k+1/k}^f = \overline{P}_{k/k}^f + \overline{Q}_k^2 \tag{26}$$
$$\overline{P}_{k+1/k}^d = \overline{P}_{k/k}^d + Q_k^d \tag{27}$$

$$0 = (\overline{U}_{k+1}^{23} - U_{k+1}^{23}) \overline{P}_{k/k}^d + Q^{fd} - U_{k+1}^{23} Q_k^d \tag{28}$$
$$0 = (\overline{U}_{k+1}^{13} - U_{k+1}^{13}) \overline{P}_{k/k}^d + Q^{xd} - U_{k+1}^{13} Q_k^d \tag{29}$$
$$0 = \overline{U}_{k+1}^{12} \overline{P}_{k/k}^f - U_{k+1}^{12} \overline{P}_{k+1/k}^f + \overline{U}_{k+1}^{13} \overline{P}_{k/k}^d \overline{U}_{k+1}^{23T} - U_{k+1}^{13} \overline{P}_{k+1/k}^d U_{k+1}^{23T} \tag{30}$$

where

$$\overline{Q}_k^1 = Q_k^x + \overline{U}_{k+1}^{12} \overline{P}_{k/k}^f \overline{U}_{k+1}^{12T} - U_{k+1}^{12} \overline{P}_{k+1/k}^f U_{k+1}^{12T} + \overline{U}_{k+1}^{13} \overline{P}_{k/k}^d \overline{U}_{k+1}^{13T} - U_{k+1}^{13} \overline{P}_{k+1/k}^d U_{k+1}^{13T} \tag{31}$$
$$\overline{Q}_k^2 = Q_k^f + \overline{U}_{k+1}^{23} \overline{P}_{k/k}^d \overline{U}_{k+1}^{23T} - U_{k+1}^{23} \overline{P}_{k+1/k}^d U_{k+1}^{23T} \tag{32}$$

Referring to (28)-(30), we obtain

$$U_{k+1}^{13} = (\overline{U}_{k+1}^{13} \overline{P}_{k/k}^d + Q_k^{xd})(\overline{P}_{k+1/k}^d)^{-1} \tag{33}$$
$$U_{k+1}^{23} = (\overline{U}_{k+1}^{23} \overline{P}_{k/k}^d + Q_k^{fd})(\overline{P}_{k+1/k}^d)^{-1} \tag{34}$$
$$U_{k+1}^{12} = (\overline{U}_{k+1}^{12} \overline{P}_{k/k}^f + \overline{U}_{k+1}^{13} \overline{P}_{k/k}^d \overline{U}_{k+1}^{23T} - U_{k+1}^{13} \overline{P}_{k/k}^f U_{k+1}^{23T} + Q_k^{xf})(\overline{P}_{k+1/k}^f)^{-1} \tag{35}$$

The development of (21), leads to

$$\overline{P}_{k+1/k+1}^x = (I - \overline{K}_{k+1}^x S_{k+1}^1) \overline{P}_{k+1/k}^x \tag{36}$$
$$\overline{P}_{k+1/k+1}^f = (I - \overline{K}_{k+1}^f S_{k+1}^2) \overline{P}_{k+1/k}^f \tag{37}$$
$$\overline{P}_{k+1/k+1}^d = (I - \overline{K}_{k+1}^d S_{k+1}^3) \overline{P}_{k+1/k}^d \tag{38}$$
$$0 = U_{k+1}^{12} - V_{k+1}^{12} - \overline{K}_{k+1}^x S_{k+1}^2 \tag{39}$$
$$0 = U_{k+1}^{13} - V_{k+1}^{13} - V_{k+1}^{12} U_{k+1}^{23} + V_{k+1}^{12} V_{k+1}^{23} - \overline{K}_{k+1}^x S_{k+1}^3 \tag{40}$$
$$0 = U_{k+1}^{23} - V_{k+1}^{23} - \overline{K}_{k+1}^f S_{k+1}^3 \tag{41}$$

Referring to (39)-(41), we obtain

$$V_{k+1}^{12} = U_{k+1}^{12} - \overline{K}_{k+1}^x S_{k+1}^2 \tag{42}$$
$$V_{k+1}^{13} = U_{k+1}^{13} - V_{k+1}^{12} \overline{K}_{k+1}^f S_{k+1}^3 - \overline{K}_{k+1}^x S_{k+1}^3 \tag{43}$$
$$V_{k+1}^{23} = U_{k+1}^{23} - \overline{K}_{k+1}^f S_{k+1}^3 \tag{44}$$

With reference to (17), (19) and (20), we obtain respectively

$$\begin{aligned}
\overline{x}_{k+1/k} &= A_k\overline{x}_{k/k} + B_k u_k + \overline{u}_k^1 && (45)\\
\overline{f}_{k+1/k} &= \overline{f}_{k/k} + \overline{u}_k^2 && (46)\\
\overline{d}_{k+1/k} &= \overline{d}_{k/k} && (47)\\
\overline{x}_{k+1/k+1} &= \overline{x}_{k+1/k} + \overline{K}^x_{k+1}(y_{k+1} - S^1_{k+1}\overline{x}_{k+1/k}) && (48)\\
\overline{f}_{k+1/k+1} &= \overline{f}_{k+1/k} + \overline{K}^f_{k+1}(y_{k+1} - S^1_{k+1}\overline{x}_{k+1/k} - S^2_{k+1}\overline{f}_{k+1/k}) && (49)\\
\overline{d}_{k+1/k+1} &= \overline{d}_{k+1/k} + \overline{K}^d_{k+1}(y_{k+1} - S^1_{k+1}\overline{x}_{k+1/k} - S^2_{k+1}\overline{f}_{k+1/k} - S^3_{k+1}\overline{d}_{k+1/k}) && (50)
\end{aligned}$$

$$\begin{aligned}
\overline{K}^x_{k+1} &= \overline{P}^x_{k+1/k}S^{1T}_{k+1}(S^1_{k+1}\overline{P}^x_{k+1/k}S^{1T}_{k+1} + R_{k+1})^{-1} && (51)\\
\overline{K}^f_{k+1} &= \overline{P}^f_{k+1/k}S^{2T}_{k+1}(S^2_{k+1}\overline{P}^f_{k+1/k}S^{2T}_{k+1} + S^1_{k+1}\overline{P}^x_{k+1/k}S^{1T}_{k+1} + R_{k+1})^{-1} && (52)\\
\overline{K}^d_{k+1} &= \overline{P}^d_{k+1/k}S^{3T}_{k+1}(S^3_{k+1}\overline{P}^d_{k+1/k}S^{3T}_{k+1} + S^2_{k+1}\overline{P}^f_{k+1/k}S^{2T}_{k+1} + S^1_{k+1}\overline{P}^x_{k+1/k}S^{1T}_{k+1} + R_{k+1})^{-1} && (53)
\end{aligned}$$

where

$$\begin{aligned}
\overline{u}_k^1 &= (\overline{U}^{12}_{k+1} - U^{12}_{k+1})\overline{f}_{k/k} + (\overline{U}^{13}_{k+1} - U^{13}_{k+1} - U^{12}_{k+1}(\overline{U}^{23}_{k+1} - U^{23}_{k+1}))\overline{d}_{k/k} && (54)\\
\overline{u}_k^2 &= (\overline{U}^{23}_{k+1} - U^{23}_{k+1})\overline{d}_{k/k} && (55)
\end{aligned}$$

3.4 Optimal Three-Stage Kalman filters (OThSKF)

To correct the estimation of the state and the fault, we should follow these equations

$$\begin{aligned}
\hat{x}_{k+1/k+1} &= \overline{x}_{k+1/k+1} + V^{12}_{k+1}\overline{f}_{k+1/k+1} + V^{13}_{k+1}\overline{d}_{k+1/k+1} && (56)\\
\hat{P}^x_{k+1/k+1} &= \overline{P}^x_{k+1/k+1} + V^{12}_{k+1}\overline{P}^f_{k+1/k+1}V^{12T}_{k+1} + V^{13}_{k+1}\overline{P}^d_{k+1/k+1}V^{13T}_{k+1} && (57)\\
\hat{f}_{k+1/k+1} &= \overline{f}_{k+1/k+1} + V^{23}_{k+1}\overline{d}_{k+1/k+1} && (58)\\
\hat{P}^f_{k+1/k+1} &= \overline{P}^f_{k+1/k+1} + V^{23}_{k+1}\overline{P}^d_{k+1/k+1}V^{23T}_{k+1} && (59)
\end{aligned}$$

Now, the state and the fault estimate can be obtained by OThSKF. To implement the OThSKF, we assume to know the following :

- Control input u_k
- Matrices A_k, B_k, H_k, F^x_k,F^y_k, E^x_k and E^y_k
- Covariance matrices : Q^x_k, R_k, Q^f_k, Q^d_k, Q^{xf}_k, Q^{xd}_k and Q^{fd}_k
- Initial values $\overline{x}_0$, $\overline{f}_0$, $\overline{d}_0$, P^x_0, P^f_0, P^d_0, P^{xf}_0, P^{xd}_0 and P^{fd}_0

Table 1 gathers the different steps of the filtering design and Figure 2 shows the interactions between the different blocks of calculus.

Table 1: OThSKF algorithm

Algorithme 1: state and fault estimation by OThSKF

- Step 0 : initialization

$k = 0$
$V^{13}_0 = P^{xd}_0(P^d_0)^{-1}$, $V^{23}_0 = P^{fd}_0(P^d_0)^{-1}$
$V^{12}_0 = (P^{xf}_0 - V^{13}_0 P^d_0 V^{23T}_0)(P^f_0 - V^{23}_0 P^d_0 V^{23T}_0)^{-1}$
$\overline{x}_{0/0} = \overline{x}_0 - V^{12}_0\overline{f}_0 - V^{13}_0\overline{d}_0$, $\overline{d}_{0/0} = \overline{d}_0$,
$\overline{f}_{0/0} = \overline{f}_0 - V^{23}_0\overline{d}_0$
$\overline{P}^d_{0/0} = P^d_0$, $\overline{P}^f_{0/0} = P^f_0 - V^{23}_0 P^d_0 V^{23T}_0$

$\overline{P}^x_{0/0} = P^x_0 - V^{12}_0 P^f_0 V^{12T}_0 - V^{13}_0 P^d_0 V^{13T}_0$

- Step 1 : preliminary
  To calculate $\overline{U}^{12}_{k+1}$, $\overline{U}^{13}_{k+1}$, $\overline{U}^{23}_{k+1}$ from (23)
  To calculate $\overline{P}^d_{k+1/k}$, U^{13}_{k+1}, U^{23}_{k+1}, $\overline{Q}^2_k$, $\overline{P}^f_{k+1/k}$, U^{12}_{k+1}, $\overline{u}^1_k$, $\overline{u}^2_k$ respectively from (27), (33), (34), (32), (26), (35), (31),(54) and (55)
  To calculate S^1_{k+1}, S^2_{k+1} and S^3_{k+1} from (24)
- Step 2 : state subfilter
  To calculate $\overline{x}_{k+1/k}$, $\overline{P}^x_{k+1/k}$, $\overline{K}^x_{k+1}$, $\overline{x}_{k+1/k+1}$, $\overline{P}^x_{k+1/k+1}$ respectively from (45), (25), (51), (48) and (36).
- Step 3 : fault subfilter
  To calculate $\overline{f}_{k+1/k}$, $\overline{K}^f_{k+1}$, $\overline{f}_{k+1/k+1}$ and $\overline{P}^f_{k+1/k+1}$ respectively from (46), (52), (49) and (37).
- Step 4 : unknown inputs subfilter
  To calculate $\overline{d}_{k+1/k}$, $\overline{K}^d_{k+1}$, $\overline{d}_{k+1/k+1}$ and $\overline{P}^d_{k+1/k+1}$ respectively from(47), (53), (50) and (38).
- Step 5 : the correction of the state and the fault estimations
  To update V^{12}_{k+1}, V^{13}_{k+1}, V^{23}_{k+1} respectively from (42), (43) and (44).
  To calculate $\hat{x}_{k+1/k}$, $\hat{P}^x_{k+1/k}$, $\hat{f}_{k+1/k+1}$ and $\hat{P}^f_{k+1/k+1}$ respectively from (56)-(59).
- Step 6 : $k = k + 1$ and return to step 1

We denote q^{-1} is a delay operator such that: $q^{-1}y_k = y_{k-1}$

3.5 Robust Three-Stage Kalman Filter (RThSKF)

The OThSKF is optimal in the minimum mean square error (MMSE) sense. However, this filter loses its optimality, when the statistical properties of models (2) and (3) are unknown or not perfectly known. So, it would be better to use a robust three-stage Kalman filter (RThKF) to get a good estimation of state and fault in presence of unknown inputs. This filter is obtained by modifying the measurement update equations of the unknown inputs subfilter and the fault subfilter of the OThSKF. The measurement update equations of the fault subfilter and the unknown inputs subfilter are rewritten as follow:

$$\begin{aligned}
\overline{f}_{k+1/k+1} &= (I - \overline{K}^f_{k+1}S^2_{k+1})\overline{f}_{k+1/k} + \overline{K}^f_{k+1}(y_{k+1} - S^1_{k+1}\overline{x}_{k+1/k}) && (60)\\
\overline{K}^f_{k+1} &= \overline{P}^f_{k+1/k+1}S^{2T}_{k+1}C^{-1}_{k+1} && (61)\\
\overline{d}_{k+1/k+1} &= (I - \overline{K}^d_{k+1}S^3_{k+1})\overline{d}_{k/k} + \overline{K}^d_{k+1}(y_{k+1} - S^1_{k+1}\overline{x}_{k+1/k} - S^2_{k+1}\overline{f}_{k+1/k}) && (62)\\
\overline{K}^d_{k+1} &= \overline{P}^d_{k+1/k+1}S^{3T}_{k+1}(S^2_{k+1}\overline{P}^f_{k+1/k}S^{2T}_{k+1} + C_{k+1})^{-1} && (63)
\end{aligned}$$

where $C_{k+1} = H_{k+1}\overline{P}^x_{k+1/k}H^T_{k+1} + R_{k+1}$

Firstly, to eliminate the two terms $\overline{f}_{k+1/k}$ and $\overline{d}_{k/k}$, we will choose the gain matrices $\overline{K}^f_{k+1}$ and $\overline{K}^d_{k+1}$ that can satisfy the followings algebraic constraints

$$\begin{aligned}
(I - \overline{K}^f_{k+1}S^2_{k+1}) &= 0 && (64)\\
(I - \overline{K}^d_{k+1}S^3_{k+1}) &= 0 && (65)\\
\overline{K}^d_{k+1}S^2_{k+1} &= 0 && (66)
\end{aligned}$$

In this case (60) and (62) become

$$\begin{aligned}
\overline{f}_{k+1/k+1} &= \overline{K}^f_{k+1}(y_{k+1} - S^1_{k+1}\overline{x}_{k+1/k}) && (67)\\
\overline{d}_{k+1/k+1} &= \overline{K}^d_{k+1}(y_{k+1} - S^1_{k+1}\overline{x}_{k+1/k}) && (68)
\end{aligned}$$

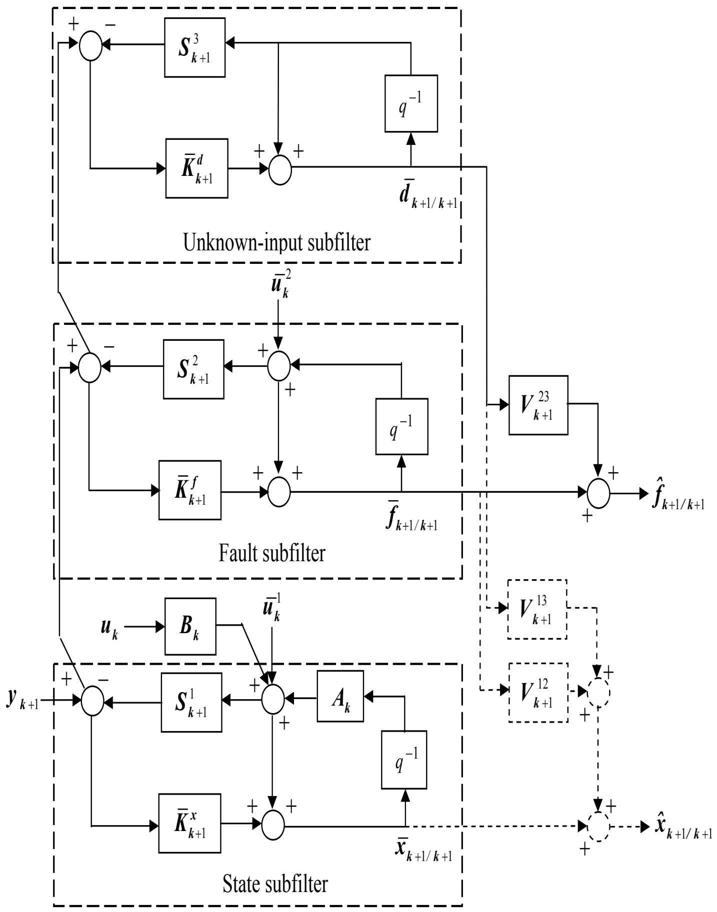

Figure 2: Bloc diagram of the OThSKF

Secondly, with substituting (61 and 63) into (64 and 65) and using (66), $\overline{P}^f_{k+1/k+1}$, $\overline{P}^d_{k+1/k+1}$ and $\overline{K}^d_{k+1}$ can be rewritten as

$$\overline{P}^f_{k+1/k+1} = (S^{2T}_{k+1}C^{-1}_{k+1}S^2_{k+1})^{-1} \tag{69}$$

$$\overline{P}^d_{k+1/k+1} = (S^{3T}_{k+1}C^{-1}_{k+1}S^3_{k+1})^{-1} \tag{70}$$

$$\overline{K}^d_{k+1} = \overline{P}^d_{k+1/k+1}S^{3T}_{k+1}C^{-1}_{k+1} \tag{71}$$

The equations (45) and (46) are rewritten, respectively, as follow:

$$\overline{x}_{k+1/k} = A_k\hat{x}_{k/k} + B_k u_k + \widetilde{u}^1_k \tag{72}$$

$$\overline{f}_{k+1/k} = \overline{f}_{k/k} + \widetilde{u}^2_k \tag{73}$$

where

$$\widetilde{u}^1_k = (F^x_k - U^{12}_{k+1})\overline{f}_{k/k} + (E^x_k - U^{13}_{k+1} + U^{12}_{k+1}U^{23}_{k+1})\overline{d}_{k/k} \tag{74}$$

$$\widetilde{u}^2_k = (\overline{U}^{23}_{k+1} - U^{23}_{k+1})\overline{d}_{k/k} \tag{75}$$

In order to return (72) and (73) robust against the fault and the unknown inputs we can choose that $\widetilde{u}^1_k = 0$ and $\widetilde{u}^2_k = 0$
In this case, the new matrices U^{12}_{k+1}, U^{13}_{k+1} and U^{23}_{k+1} are written as follow

$$U^{23}_{k+1} = \overline{U}^{23}_{k+1} = V^{23}_{k}$$
$$U^{12}_{k+1} = F^{x}_{k}$$
$$U^{13}_{k+1} = E^{x}_{k} + U^{12}_{k+1}U^{23}_{k+1} = E^{x}_{k} + F^{x}_{k}V^{23}_{k}$$

Finally, the robust three-stage Kalman filter (RThSKF) equations is summarized in Table 2. Figure 3 shows the interactions between the different blocks of calculus.

Table 2: RThSKF algorithm

Algorithme 2: state and fault estimation by RThSKF

- Step 0 : initialization

$$k = 0$$
$$\hat{x}_{0/0} = \overline{x}_0 \text{ , } \hat{P}^{x}_{0/0} = P^{x}_{0} \text{ and } V^{23}_{0}$$

- Step 1: state subfilter

$$\overline{x}_{k+1/k} = A_k \hat{x}_{k/k} + B_k u_k$$
$$\overline{P}^{x}_{k+1/k} = A_k \hat{P}^{x}_{k/k} A^{T}_{k} + Q^{x}_{k}$$
$$\overline{K}^{x}_{k+1} = \overline{P}^{x}_{k+1/k} S^{1T}_{k+1} C^{-1}_{k+1}$$
$$\overline{x}_{k+1/k+1} = \overline{x}_{k+1/k} + \overline{K}^{x}_{k+1}(y_{k+1} - S^{1}_{k+1}\overline{x}_{k+1/k})$$
$$\overline{P}^{x}_{k+1/k+1} = (I - \overline{K}^{x}_{k+1} S^{1}_{k+1})\overline{P}^{x}_{k+1/k}$$

- Step 2: fault subfilter

$$U^{12}_{k+1} = F^{x}_{k}$$
$$S^{2}_{k+1} = H_{k+1} U^{12}_{k+1} + F^{y}_{k+1}$$
$$\overline{P}^{f}_{k+1/k+1} = (S^{2T}_{k+1} C^{-1}_{k+1} S^{2}_{k+1})^{-1}$$
$$\overline{K}^{f}_{k+1} = \overline{P}^{f}_{k+1/k+1} S^{2T}_{k+1} C^{-1}_{k+1}$$
$$\overline{f}_{k+1/k+1} = \overline{K}^{f}_{k+1}(y_{k+1} - S^{1}_{k+1}\overline{x}_{k+1/k})$$

- Step 3: unknown input subfilter

$$U^{23}_{k+1} = V^{23}_{k}$$
$$U^{13}_{k+1} = E^{x}_{k} + F^{x}_{k} V^{23}_{k}$$
$$S^{3}_{k+1} = H_{k+1} U^{13}_{k+1} + F^{y}_{k+1} U^{23}_{k+1} + E^{y}_{k+1}$$
$$\overline{P}^{d}_{k+1/k+1} = (S^{3T}_{k+1} C^{-1}_{k+1} S^{3}_{k+1})^{-1}$$
$$\overline{K}^{d}_{k+1} = \overline{P}^{d}_{k+1/k+1} S^{3T}_{k+1} C^{-1}_{k+1}$$
$$\overline{d}_{k+1/k+1} = \overline{K}^{d}_{k+1}(y_{k+1} - S^{1}_{k+1}\overline{x}_{k+1/k})$$

- Step 4: the correction of the state and the fault estimations

$$V^{12}_{k+1} = U^{12}_{k+1} - \overline{K}^{x}_{k+1} S^{2}_{k+1}$$
$$V^{13}_{k+1} = U^{13}_{k+1} - V^{12}_{k+1}\overline{K}^{f}_{k+1} S^{3}_{k+1} - \overline{K}^{x}_{k+1} S^{3}_{k+1}$$
$$V^{23}_{k+1} = V^{23}_{k} - \overline{K}^{f}_{k+1} S^{3}_{k+1}$$
$$\hat{x}_{k+1/k+1} = \overline{x}_{k+1/k+1} + V^{12}_{k+1}\overline{f}_{k+1/k+1} + V^{13}_{k+1}\overline{d}_{k+1/k+1}$$
$$\hat{P}^{x}_{k+1/k+1} = \overline{P}^{x}_{k+1/k+1} + V^{12}_{k+1}\overline{P}^{f}_{k+1/k+1}V^{12T}_{k+1} + V^{13}_{k+1}\overline{P}^{d}_{k+1/k+1}V^{13T}_{k+1}$$
$$\hat{f}_{k+1/k+1} = \overline{f}_{k+1/k+1} + V^{23}_{k+1}\overline{d}_{k+1/k+1}$$
$$\hat{P}^{f}_{k+1/k+1} = \overline{P}^{f}_{k+1/k+1} + V^{23}_{k+1}\overline{P}^{d}_{k+1/k+1}V^{23T}_{k+1}$$

- Step 5: $k = k + 1$ and return to step 1.

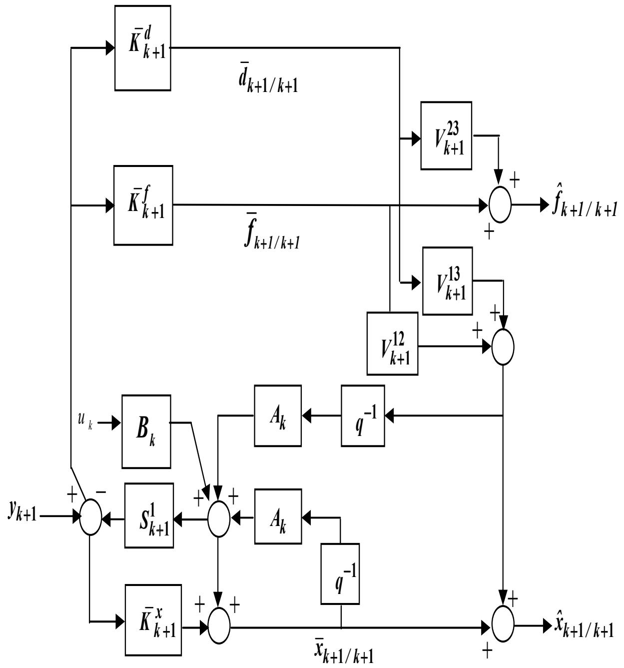

Figure 3: Bloc diagram of the RThSKF

4 Illustrative example

In this section, we will apply the proposed filters (OThSKF and RThSKF) to treat three different cases. The parameters of the system (1) and the models (2) and (3) are given by

$$x_k = \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \end{bmatrix}, A_k = \begin{bmatrix} a_k & 0.1 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.2 & 0.1 & 0.25 \end{bmatrix}, a_k = 0.5 + 0.35sin(k), B_k = \begin{bmatrix} 2 \\ -1.5 \\ 0.5 \end{bmatrix}$$

$$H_k = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \; F^x_k = \begin{bmatrix} 1 \\ 0.8 \\ -0.5 \end{bmatrix}, \; F^y_k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E^x_k = \begin{bmatrix} 0.1 \\ 0.2 \\ 1 \end{bmatrix}, \; E^y_k = \begin{bmatrix} 0.3 \\ 0.6 \end{bmatrix}$$

$$Q^x_k = 0.5 I_{3\times 3}, \quad R_k = 0.2 I_{2\times 2}, Q^f_k = 0.5, \quad Q^d_k = 0.5$$
$$Q^{xf}_k = (0.02 \quad 0.01 \quad 0.02)^T, Q^{xd}_k = (0.01 \quad 0.02 \quad 0.02)^T, \quad Q^{fd}_k = 0.01$$

The initial values of the state is $x_0 = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}^T$, the fault is $f_0 = 0$ and the unknown inputs is $d_0 = -1$.

All filters are initialized by taking the following values

$$\bar{x}_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \bar{f}_0 = 0, \bar{d}_0 = 0, P^x_0 = 20 I_{3\times 3}, P^f_0 = 20, P^d_0 = 20, P^{xf}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, P^{xd}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; and \; P^{fd} = 0$$

Figure 4 presents the input/output sequence of the system. The time of simulation is $N = 50$.

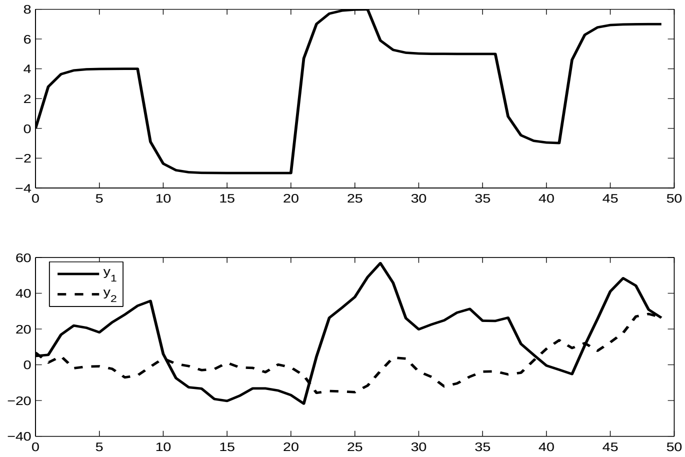

Figure 4: input/output of system

4.1 Covariance matrices and initial conditions of fault and unknown input are known

In this case, we take the exact values of covariance matrices of all noises used in (2) and (3) to implement ASKF and OThSKF

$$Q_k^f = 0.5, Q_k^d = 0.5, Q_k^{xf} = (0.02 \quad 0.01 \quad 0.02)^T, Q_k^{xd} = (0.01 \quad 0.02 \quad 0.02)^T \text{ and } Q_k^{fd} = 0.01.$$

Figure 5 presents the actual state vector first component ($x_{1,k}$), the fault (f_k) and the unknown inputs (d_k) and theirs estimated values obtained by the proposed filter OThSKF and RThSKF.

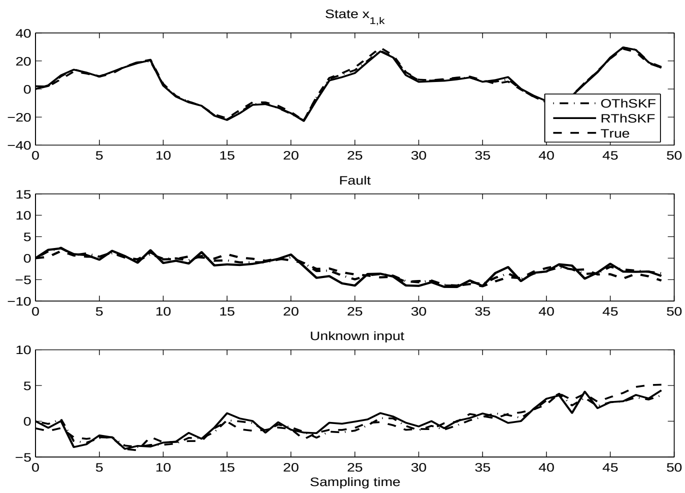

Figure 5: State, fault and unknown input

Convergence of the trace of the state covariance matrix $\hat{P}^x_{k+1/k+1}$ and fault covariance matrix $\hat{P}^f_{k+1/k+1}$ are shown in Figures 6 and 7 respectively.

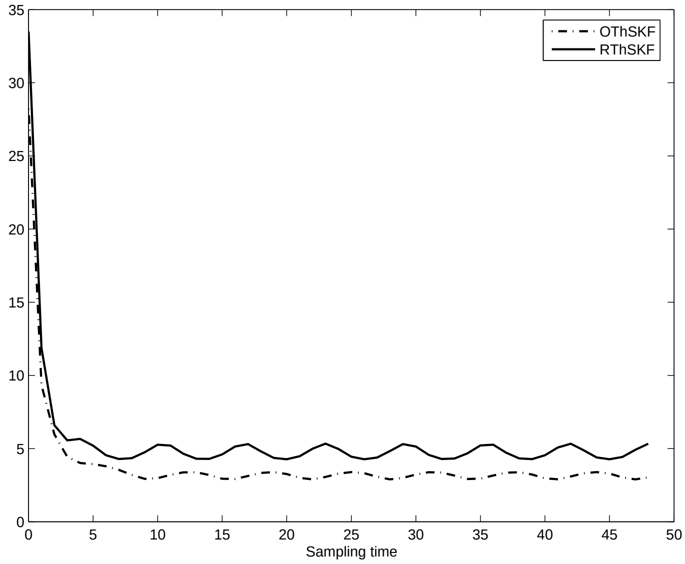

Figure 6: Trace of state covariance matrix

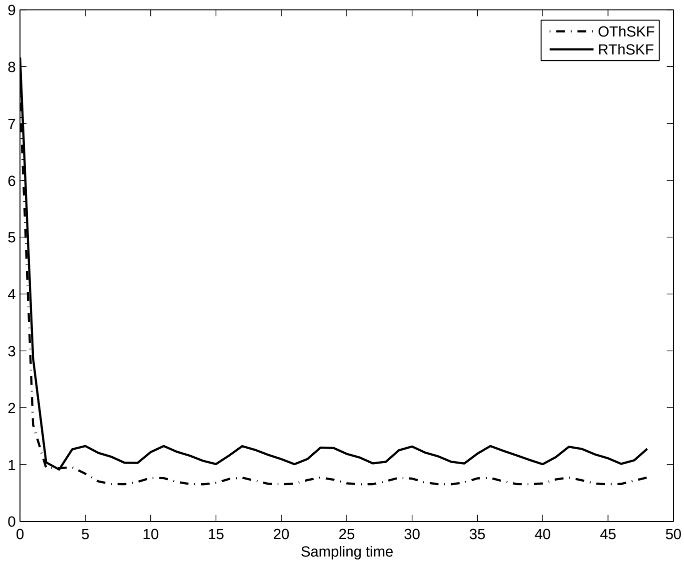

Figure 7: Fault covariance matrix

The simulation results in Tables 3-5, show the average root mean square errors (RMSE) in the estimated states, fault and unknown input. For example, the RMSE of the first component of state vector is calculated by

$$RMSE(x_{1,k}) = \sqrt{\frac{1}{N}\sum_{k=1}^{N}(x_{1,k} - \hat{x}_{1,k/k})^2}$$

In Table 3, it can be proved that the OThSKF and ASKF are equivalent where the demonstration has been made in [5]. The OThSKF and ASKF give the best estimations. However, the result obtained by RThSKF is not an optimal solution because this filter is not equivalent to the ASKF. The computational advantage of the OThSKF over the ASKF was demonstrated by using the floating-point operations or ”flops” in Matlab for one iteration as a measure of the computational complexity [6, 8]. Each multiplication and each addition contribute on flop count. According to the Table 3 , we note that the flops counted for

Table 3: Performances of the ASKF, OThSKF and RThSKF

RMSE	ASKF	OThSKF	RThSKF
$x_{1,k}$	1.02	1.02	1.56
$x_{2,k}$	1.30	1.30	2.11
$x_{3,k}$	0.97	0.97	1.50
f_k	0.78	0.78	1.27
d_k	0.75	0.75	0.90
$flops(one\ iteration)$	1340	1244	917

the OThSKF are fewer than that of the ASKF. On the other hand, the flops counted for the RThSKF are fewer than that of the OThSKF.

4.2 Covariance matrices of fault and unknown input are not perfectly known

Here, we assume that the covariance matrices of the fault and the unknown input are not perfectly known, so we take the following values

$$Q_k^f = 12.5, Q_k^d = 12.5, Q_k^{xf} = (0 \quad 0 \quad 0)^T, Q_k^{xd} = (0 \quad 0 \quad 0)^T \text{ and } \quad Q_k^{fd} = 0$$

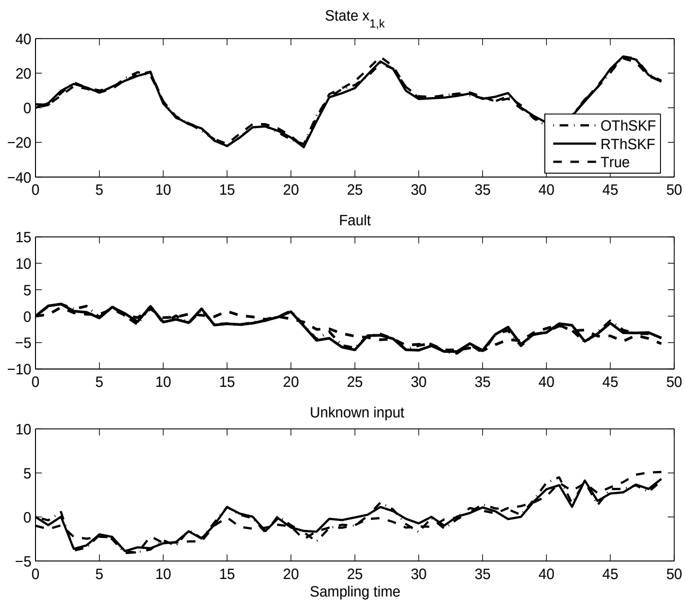

Figure 8: State, fault and unknown input

According to the Table 4 and the Figures 8-10, we note that OThSKF and ASKF lose theirs performances, but the performances of RTSKF remain unchangeable in spite of the significant error on the covariance matrices values.

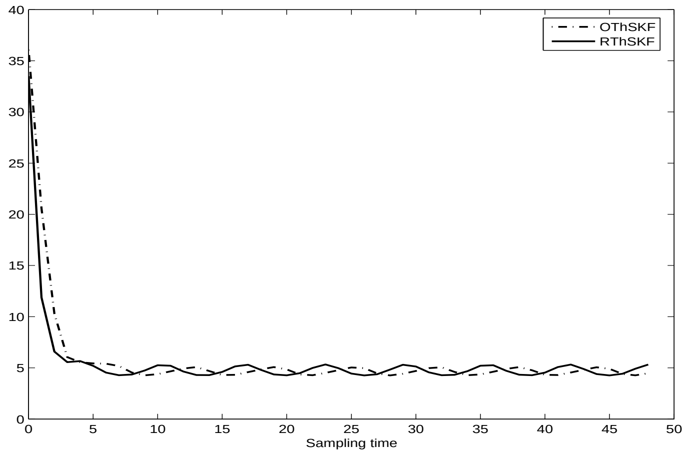

Figure 9: Trace of state covariance matrix

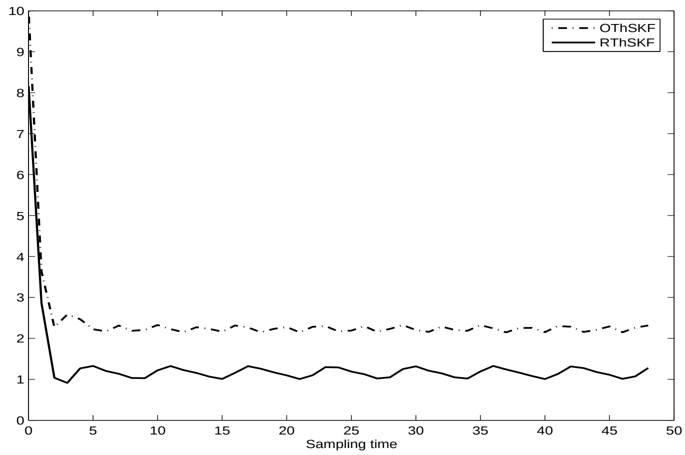

Figure 10: Fault covariance matrix

Table 4: Performances of the ASKF, OThSKF and RThSKF

RMSE	ASKF	OThSKF	RThSKF
$x_{1,k}$	1.49	1.49	1.56
$x_{2,k}$	1.66	1.66	2.11
$x_{3,k}$	1.21	1.21	1.50
f_k	1.23	1.23	1.27
d_k	0.95	0.95	0.90

4.3 Models of fault and unknown input are completely unknown

In this case, the fault and the unknown input are given by

$$f_k = 10u_s(k-15) - 10u_s(k-35) \text{ and } d_k = 6sin(0.5k)$$

where $u_s(k)$ is the unit-step function.
To implement OThSKF we take the following values as covariance matrices of the fault and the unknown input

$$Q_k^f = 0, \quad Q_k^d = 0, \quad Q_k^{xf} = (0 \quad 0 \quad 0)^T, Q_k^{xd} = (0 \quad 0 \quad 0)^T and \quad Q_k^{fd} = 0$$

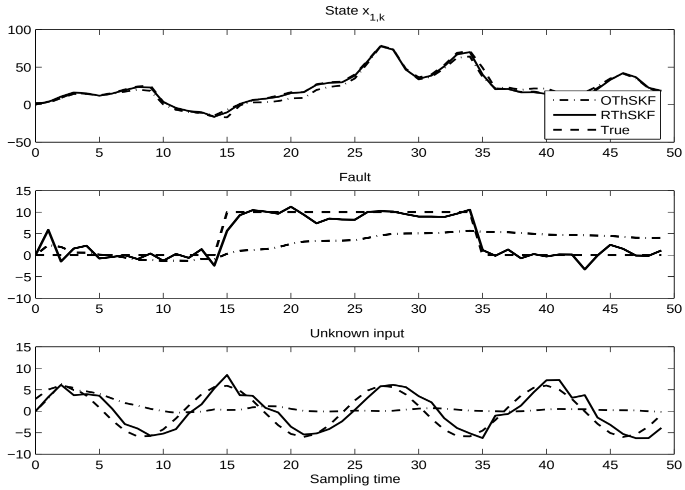

Figure 11: State, fault and unknown input

In Figure 11, we observe that the RThSKF gives the best state and fault estimation. Indeed, the evaluation of the RMSE

Table 5: Performances of the ASKF, OThSKF and RThSKF

RMSE	ASKF	OThSKF	RThSKF
$x_{1,k}$	3.62	3.62	2.00
$x_{2,k}$	6.71	6.71	2.22
$x_{3,k}$	5.39	5.39	1.53
f_k	4.91	4.91	1.54
d_k	1.38	1.38	1.24

presented in the Table 5 confirms this observation. But, the OThSKF completely loses its optimality.

5 Conclusion

In this paper, the robust three-stage Kalman filter is developedto obtain an effective state and fault estimation of linear stochastic system in the presence of unknown inputs. To achieve this aim, we had two cases ; in the first case, the OThSKF is used as the noise statistical properties of the fault and the unknown input were perfectly known. This filter is equivalent to ASKF and makes it possible to guarantee optimality of estimation. In the second case, the RThSKF is applied because the knowledge of fault and unknown input models was not completely or partially known. Indeed, the RThSKF remains powerful (Tables 4 and 5) in spite of the errors made on the covariance matrices characterizing the noise of fault and unknown input. Moreover, it is not necessary to know the initial values that are relatively related to the fault and the unknown input subfilters.

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