Observer synthesis for uncertain nonlinear systems.
Application to waste-water treatment plants

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Abstract: The paper propose the observer synthesis for uncertain nonlinear systems affected by unknown inputs described by the multi-model formulation with unmeasurable decision variables. A proportional multi-integral observer is considered in order to estimate the state and the unknown inputs and minimize the influence of the model uncertainties. The stability analysis and the observer synthesis are expressed through linear matrix inequalities based on the Lyapunov method. The performances of the proposed observer synthesis method are highlighted through the application to a waste-water treatment plant model, which is an uncertain nonlinear system affected by unknown inputs.

1. INTRODUCTION

Environmental as technological systems have complex behavior caused by the non linearity of the relationship between the variables of the process, a large operating domain or variations of system parameters and external perturbations. On the other hand, in the field of the observer/controller synthesis for fault diagnosis and control purpose, the extension of linear methods to nonlinear systems is generally a difficult problem. Thus, it is a need to build systems that can operate over a wide range of operating conditions, such as models based on a decomposition of the system model into a number of simpler linear models. Multi-model (MM) has proven to be a powerful tool in the representation of nonlinear systems on a compact set of the state space, as mentioned in chap. 14 of Tanaka and Wang [2001] and also in the analysis and synthesis of a nonlinear control system Murray-Smith and Johansen [1997], Angelis [2001].

Several techniques for obtaining MM were developed Johansen et al. [2000], Tanaka and Wang [2001], Nagy et al. [2010]. The sector nonlinearity approach Tanaka and Wang [2001] allows to exactly rewrite a nonlinear system into a MM, but the choice of the decision variables has not been systematically realized. A systematic multi-modeling rewriting with a motivated choice of these variables is presented in Nagy et al. [2010] which also allows to avoid the model linearization and its drawbacks. This last method will be used here for obtaining the MM.

This paper mainly focuses on the use of multi-models for observer synthesis Bergsten et al. [2002], Marx et al. [2007]. The observer design represents, in the last decades, an active research field owing to its particular importance in observer-based control, fault diagnosis and fault tolerant control Tanaka and Wang [2001], Chen and Saif [2007], Koenig and Mammar [2002], Ichalal et al. [2010].

Most of the existing works, dedicated to observer design based on MM are established for MM with measurable decision variables (inputs/outputs), that represents a simplified situation

Tanaka and Wang [2001], Marx et al. [2007]. The MM under study in this paper involves unmeasurable decision variables depending on the state variables -frequently met in practical situations- that are not always accessible.

A proportional multi-integral (PMI) observer approach Nagy-Kiss et al. [2011] for uncertain nonlinear systems with unknown inputs presented under a MM form is proposed in this paper. The state and unknown input estimation given by this observer is made simultaneously and the influence of the model uncertainties is minimized through a $\mathcal{L}_2$ gain. The convergence conditions of the state and unknown input estimation errors are expressed through LMIs (Linear Matrix Inequalities) Boyd et al. [1994], Tanaka and Wang [2001] by using the Lyapunov method and the $\mathcal{L}_2$ approach. PMI observers were previously proposed by Jiang et al. [2000] for linear systems, by Koenig [2005], Gao and Ho [2004] in order to estimate a large class of polynomial signals for LTI descriptor systems and by Ichalal et al. [2009] to estimate state and unknown inputs of nonlinear systems expressed under Takagi-Sugeno form but without considering uncertainties affecting the system. Therefore, this motivates us to derive a novel PMI observer technique that handles the uncertainties influence on the estimation error.

The practical contribution of this paper is to apply the proposed modeling and observer method to the realistic model of a waste-water treatment process (WWTP) modeled by an ASM1 nonlinear model with ten states, that is equivalently rewritten as a MM. The measures used for simulation process are those of the European program benchmark Cost 624 Coop [2002]. The choice of the known/unknown inputs, the measures and the real conditions is made by taking into account the properties of the Bleesbruck treatment station from Luxembourg. The numerical simulation results for the proposed application show good state and unknown inputs estimation performances.

Section 2 presents the multi-modeling approach and give the problem statement. Section 3 gives the proposed observer synthesis. Some results and performances of the proposed observer

are illustrated and designed in section 4 through a complex model of WWTP. Conclusions and future works are given in the end of this proposal.

2. MULTI-MODELLING APPROACH

Generally, a nonlinear system can be described by:

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t), d(t)) \\ y(t) &= g(x(t), u(t), d(t))\end{aligned} \quad (1)$$

The MM approach allows to represent any nonlinear dynamic system with uncertainties and affected by unknown inputs in a compact set of the state space with a convex combination of linear sub-models Nagy et al. [2010]:

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(x, u) \left[(A_i + \Delta A_i(t)) x(t) + E_i d(t) \right.$$
$$\left. + (B_i + \Delta B_i(t)) u(t) \right] \quad (2a)$$
$$y(t) = C x(t) + G d(t) \quad (2b)$$
$$\sum_{i=1}^{r} \mu_i(x, u) = 1, \ \mu_i(x, u) \geq 0, \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^{n_u} \quad (2c)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^{n_u}$ is the known input, $d(t) \in \mathbb{R}^{n_d}$ is the unknown input, $y(t) \in \mathbb{R}^{n_y}$ is the measured output and the matrices of appropriate dimensions are known and constant excepted $\Delta A_i(t)$ and $\Delta B_i(t)$ that satisfy the following equations

$$\Delta A_i(t) = M_i^a F_a(t) N_i^a, \quad \text{with} \quad F_a^T(t) F_a(t) \leq I \quad (3a)$$
$$\Delta B_i(t) = M_i^b F_b(t) N_i^b, \quad \text{with} \quad F_b^T(t) F_b(t) \leq I \quad (3b)$$

where both $F_a(t) \in \mathbb{R}^{f_1 \times f_1}$ and $F_b(t) \in \mathbb{R}^{f_2 \times f_2}$ are unknown and time varying and M_i^a, M_i^b, N_i^a and N_i^b are known matrices of appropriate dimensions. One can note that the activating functions μ_i depend on the system state that is not available to the measurement.

In the sequel, the following assumption is made:

Hypothesis 1. The unknown input $d \in C^q$ is assumed to be a bounded time varying signal with null q^{th} derivative:

$$d^{(q)}(t) = 0 \quad (4)$$

In proportional integral (PI) observer design the assumption on the constancy of the unknown input ($\dot{d} = 0$) is needed for the theoretical proof of the estimation error convergence Koenig and Mammar [2002]. This first hypothesis still gives good result if the unknown inputs vary slowly. Although, for fast variations of the unknown input no good estimation are obtained. Then, PMI observer is more adequate for this problem, because the observer estimates the $(q - 1)^{th}$ derivatives of the unknown input and gives a good precision for the estimation of the unknown inputs as in Ichalal et al. [2009]. For instance, one will see, in section 4, that good estimation results are obtained with this last hypothesis 1.

3. OBSERVER DESIGN FOR UNCERTAIN NONLINEAR SYSTEMS

In order to estimate both the system state and the unknown input, the following PMI Observer is proposed:

$$\dot{\hat{x}}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t), u(t)) \left(A_i \hat{x}(t) + B_i u(t) + E_i \hat{d}(t) \right.$$
$$\left. + L_{Pi}(y(t) - \hat{y}(t)) \right) \quad (5a)$$
$$\hat{y}(t) = C \hat{x}(t) + G \hat{d}(t) \quad (5b)$$
$$\dot{\hat{d}}_0(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t), u(t)) L_{Ii}^0 (y(t) - \hat{y}(t)) + \hat{d}_1 \quad (5c)$$
$$\dot{\hat{d}}_j(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t), u(t)) L_{Ii}^j (y(t) - \hat{y}(t)) + \hat{d}_{j+1} \quad (5d)$$

for $j = 1, ..., q - 1$ and where $\hat{d}_i$, $i = 1, 2, ..., (q - 1)$ are the estimations of the $(q - 1)$ first derivatives of $d(t)$. The state and unknown inputs estimation errors are:

$$e = x - \hat{x}, \ e_0 = \dot{d} - \dot{\hat{d}}_0, \ ..., \ e_{q-1} = \dot{d}_{q-1} - \dot{\hat{d}}_{q-1}$$

The observer design reduces to find the gains $L_{Ii} = \begin{bmatrix} L_{Ii}^{0\,T} & L_{Ii}^{1\,T} & \cdots & L_{Ii}^{q-1} \end{bmatrix}$ and L_{Pi} s.t. the state and unknown input estimation error obey to a stable system.

Notation 3.1. The symbol $*$ in a block matrix denotes the blocks induced by symmetry. For any square matrix M, $\mathbb{S}(M)$ is defined by $\mathbb{S}(M) = M + M^T$.

Theorem 2. The observer (5) estimating the state and unknown input of the system (2) and minimizing the $\mathcal{L}_2$-gain γ of the known and unknown inputs on the state and unknown input estimation error is obtained by finding symmetric positive definite matrices $P_1 \in \mathbb{R}^{(n+q \cdot n_d) \times (n+q \cdot n_d)}$ and $P_2 \in \mathbb{R}^{n \times n}$, matrices $\overline{P}_j \in \mathbb{R}^{(n+q \cdot n_d) \times n_y}$ and positive scalars ε_{1i} and ε_{2i} for all $i = 1, \cdots, r$ that minimize the scalar $\overline{\gamma}$ under the following LMI constraints

$$\mathcal{M}_{ij} < 0, \ i, j = 1, \ldots, r \quad (6)$$

where $\mathcal{M}_{ij}$ is defined by

$$\mathcal{M}_{ij} = \begin{bmatrix} \Theta_{ij}^{11} & \Theta_{ij}^{12} & \Theta_{ij}^{13} & \Theta_{ij}^{14} & P_1 \overline{M}_i^a & P_1 \overline{M}_i^b \\ * & \Theta_{ij}^{22} & P_2 B_i & P_2 E_i & P_2 M_i^a & P_2 M_i^b \\ * & * & \Theta_{ij}^{33} & 0 & 0 & 0 \\ * & * & * & -\overline{\gamma} I_{n_d} & 0 & 0 \\ * & * & * & * & -\varepsilon_{1i} I_{f_1} & 0 \\ * & * & * & * & * & -\varepsilon_{2i} I_{f_2} \end{bmatrix}$$

$$\begin{aligned} \Theta_{ij}^{11} &= I_{n+q \cdot n_d} + \mathbb{S}(P_1 \overline{A}_j - \overline{P}_j \overline{C}) & \Theta_{ij}^{12} &= P_1 (\tilde{A}_i - \tilde{A}_j) \\ \Theta_{ij}^{13} &= P_1 (\tilde{B}_i - \tilde{B}_j) & \Theta_{ij}^{14} &= P_1 (\tilde{E}_i - \tilde{E}_j) \quad (7) \\ \Theta_{ij}^{22} &= \varepsilon_{1i} N_i^{aT} N_i^a + \mathbb{S}(P_2 A_i) & \Theta_{ij}^{33} &= \varepsilon_{2i} N_i^{bT} N_i^b - \overline{\gamma} I_{n_u} \end{aligned}$$

and where the over-lined and tilde matrices are defined by

$$\overline{C} = \begin{bmatrix} C \\ G \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T, \ \overline{A}_i = \begin{bmatrix} A_i & E_i & 0 & 0 & \cdots & 0 \\ 0 & 0 & I_{n_d} & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & & 0 & \cdots & I_{n_d} \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix},$$

$$\tilde{A}_i = \begin{bmatrix} A_i \\ 0 \\ . \\ \vdots \\ 0 \end{bmatrix}, \tilde{E}_i = \begin{bmatrix} E_i \\ 0 \\ . \\ \vdots \\ 0 \end{bmatrix}, \overline{M}_i^a = \begin{bmatrix} M_i^a(t) \\ 0 \\ . \\ \vdots \\ 0 \end{bmatrix}, \overline{M}_i^b = \begin{bmatrix} M_i^b(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (8)$$

The observer gains are then obtained by:

$$L_j = \begin{bmatrix} L_{Pj} \\ L_{Ij} \end{bmatrix} = P_1^{-1} \overline{P}_j \quad (9)$$

Proof. Let us define an augmented state and its estimate

$$x_a(t) = \begin{bmatrix} x(t)^T & d(t)^T & d_1(t)^T & \cdots & d_{q-1}(t)^T \end{bmatrix}^T \quad (10)$$

$$\hat{x}_a(t) = \begin{bmatrix} \hat{x}(t)^T & \hat{d}(t)^T & \hat{d}_1(t)^T & \cdots & \hat{d}_{q-1}(t)^T \end{bmatrix}^T \quad (11)$$

The augmented state estimation error is defined by $e_a(t) = x_a(t) - \hat{x}_a(t)$. Using (2a) and (4), the system and observer equations can be respectively written as

$$\dot{x}_a(t) = \sum_{i=1}^{r} \mu_i(x_a(t), u(t)) \left[(\overline{A}_i + \overline{\Delta A}_i(t)) x_a(t) \right.$$
$$\left. + (\overline{B}_i + \overline{\Delta B}_i(t)) u(t) \right] \quad (12a)$$

$$y(t) = \overline{C} x_a(t) \quad (12b)$$

with :

$$\overline{B}_i = \begin{bmatrix} B_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \overline{N}_i^{bT} = \begin{bmatrix} N_i^{bT} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \overline{N}_i^{aT} = \begin{bmatrix} N_i^{aT} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\overline{\Delta B}_i(t) = \overline{M}_i^b F^b(t) \overline{N}_i^b, \quad \overline{\Delta A}_i(t) = \overline{M}_i^a F^a(t) \overline{N}_i^a \quad (13)$$

and

$$\dot{\hat{x}}_a(t) = \sum_{j=1}^{r} \mu_j(\hat{x}_a(t), u(t)) \left[\overline{A}_j \hat{x}_a(t) + \overline{B}_j u(t) \right.$$
$$\left. + L_j (y(t) - \hat{y}(t)) \right] \quad (14a)$$

$$\hat{y}(t) = \overline{C} \hat{x}_a(t) \quad (14b)$$

One should note that in (12a) the activating functions depend on $x_a(t)$, whereas they depend on $\hat{x}_a(t)$ in (14a) and then the comparison of the state x_a (12b) and its reconstruction (14a) seems to be difficult. In order to cope with the difficulty of expressing the augmented state estimation error in a tractable way, (12b) is re-written, based on the property (2c). Consequently, the augmented state estimation error obeys to the following nonlinear system

$$\begin{bmatrix} \dot{e}_a(t) \\ \dot{x}(t) \end{bmatrix} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(x_a(t), u(t)) \mu_j(\hat{x}_a(t), u(t))$$
$$\cdot \left\{ \mathcal{A}_{ij}(t) \begin{bmatrix} e_a(t) \\ x(t) \end{bmatrix} + \mathcal{B}_{ij}(t) \begin{bmatrix} u(t) \\ d(t) \end{bmatrix} \right\} \quad (15a)$$

$$e_a(t) = [I_{n+n_d} \ 0] \begin{bmatrix} e_a(t) \\ x(t) \end{bmatrix} \quad (15b)$$

where

$$\mathcal{A}_{ij}(t) = \begin{bmatrix} \overline{A}_j - \overline{L}_j \overline{C} & \tilde{A}_i - \tilde{A}_j + \widetilde{\Delta A}_i(t) \\ 0 & A_i + \Delta A_i(t) \end{bmatrix} \quad (16)$$

$$\mathcal{B}_{ij}(t) = \begin{bmatrix} \overline{B}_i - \overline{B}_j + \overline{\Delta B}_i(t) & \tilde{E}_i - \tilde{E}_j \\ B_i + \Delta B_i(t) & E_i \end{bmatrix} \quad (17)$$

$$\widetilde{\Delta A}_i(t) = \begin{bmatrix} \Delta A_i(t)^T & 0 & \cdots & 0 \end{bmatrix}^T \quad (18)$$

The candidate Lyapunov function for (15) is

$$V(x_a(t), x(t)) = \begin{bmatrix} e_a(t) \\ x(t) \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} e_a(t) \\ x(t) \end{bmatrix} \quad (19)$$

where P_1 and P_2 are symmetric positive definite matrices. The objective is to find the gains L_j of the observer that minimize the $\mathcal{L}_2$-gain from the known and unknown inputs $u(t)$ and $d(t)$ to the state and unknown input estimation error $e_a(t)$. It is well known Boyd et al. [1994] that the $\mathcal{L}_2$-gain from $\begin{bmatrix} u(t) \\ d(t) \end{bmatrix}$ to $e_a(t)$ is bounded by γ if

$$\dot{V}(e_a(t), x(t)) + e_a^T(t) e_a(t) - \gamma^2 (u^T(t) u(t) + d^T(t) d(t)) < 0 \quad (20)$$

With some Schur complements and defining $\overline{P}_j = P_1 L_j$ and $\overline{\gamma} = \gamma^2$, the previous inequality becomes

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(x_a(t)) \mu_j(\hat{x}_a(t)) \mathcal{M}_{ij} < 0 \quad (21)$$

It follows that (20) is satisfied if the LMI (6) holds, which achieves the proof. For more details see Nagy-Kiss et al. [2011], where a PI observer was proposed for a reduced ASM1 model.

The proposed estimation method requires that the system must be quadratically robustly stable, as shown in the block (2,2) of the LMI $\mathcal{M}_{ij}$.

The performance analysis and the synthesis of observer for multi-model is based on the resolution of LMI by using the Lyapunov method. Because of the convex sum property of the weighting functions (2c), the LMI are only evaluated at the polytope vertices (A_i, B_i) (or in the uncertain case adding the terms ΔA_i and ΔB_i) and the weighting functions do not occur in the resolution of the LMIs. It is to mention that the LMI formulation of observer design for MM consists only in sufficient but not necessary conditions. That is the reason why it is essential to propose the appropriate MM structure. Detailed choice criteria for MM are discussed in Nagy et al. [2010]; it is to mention here the observability criteria, which means that MM structure with observable submodels are necessary. Thus, avoid submodel matrices B_i with null columns is an observability requirement. It should be also mentioned that a MM with a low number of submodels is preferred, which leads to less computational requirements by reducing the number of LMI to be solved.

Remark 3.1. If the system is also affected by unknown input which estimators are not needed, the system (2) becomes

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(x(t)) \left[(A_i + \Delta A_i(t)) x(t) \right.$$
$$\left. + (B_i + \Delta B_i(t)) u(t) + E_i d(t) + F_i w(t) \right] \quad (22a)$$

$$y(t) = Cx(t) + Gd(t) + Hw(t) \quad (22b)$$

where $w(t) \in \mathbb{R}^{n_w}$ denotes these type of unknown inputs. In this case, the previous result can readily be adapted in order to estimate both $x(t)$ and $d(t)$ while minimizing the influence of $w(t)$ on the estimation errors. One should see that the matrices $\mathcal{M}_{ij}$ in (6) should be replaced by $\tilde{\mathcal{M}}_{ij}$ defined by

$$\tilde{\mathcal{M}}_{ij} = \begin{bmatrix} \mathcal{M}_{ij} & \Psi_{ij} \\ * & -\overline{\gamma} I_{n_w} \end{bmatrix} \quad (23)$$

with $\Psi_{ij}^T = \begin{bmatrix} (\overline{P}_j \overline{H} - P_1 \overline{F}_i)^T & F_i^T P_2 & 0 & 0 \end{bmatrix}$, $\overline{H}^T = \begin{bmatrix} H^T & 0 \end{bmatrix}$ and $\overline{F}_i^T = \begin{bmatrix} F_i^T & 0 \end{bmatrix}$.

4. WASTE-WATER TREATMENT PLANT

4.1 Process description and ASM1 model

The widely used activated sludge based waste-water treatment consists in mixing waste waters with a rich mixture of bacteria in order to degrade the organic matter Olsson and Newell [1999].

In this work, a part of the COST Benchmark is considered. The Benchmark is based on the most common WWTP: a continuous flow activated sludge plant, performing nitrification and de-nitrification. A configuration with a single tank with a

settler/clarifier was developed. The objective of this study is to use the data generated by this benchmark.

For observer/controller design, models of reduced complexity are generally used. Nevertheless, this paper considers a quite complete ASM1 model for WWTP involving the following components: soluble carbon S_S, particulate X_S, dissolved oxygen S_O, heterotrophic biomass X_{BH}, ammonia S_{NH}, nitrate S_{NO}, autotrophic biomass X_{BA}, soluble inert S_I, suspended inert X_I, soluble organic nitrogen S_{ND} and suspended organic nitrogen X_{ND}. Only the following components are not considered in the ASM1 model: the inert component X_P and the alkalinity S_{alk}. As in practical situation, a single organic compound, denoted X_{DCO}, will be considered by adding the soluble part S_S and the particulate part X_S Smets et al. [2003]. The following state vector is taken:

$$x(t) = [X_{DCO}(t), S_O(t), S_{NH}(t), S_{NO}(t), X_{BH}(t) \cdots$$
$$\cdots X_{BA}(t), S_I(t), X_I(t), S_{ND}(t), X_{ND}(t)]^T \quad (24)$$

The following assumptions are considered: the dissolved oxygen concentration input $(S_{O,in})$ is null, $S_{NO,in} \cong 0$ and $X_{BA,in} \cong 0$, which is in conformity with the European Benchmark COST 624 Coop [2002].
In practice, the concentrations $X_{DCO,in}$, $S_{NH,in}$ and $X_{BH,in}$ are not measured online. Thus, these concentrations are replaced with their respective daily mean values. A daily mean value will be considered for $X_{DCO,in}$ and $X_{BH,in}$. The concentration $S_{NH,in}$ is considered as unknown input. The measurements of $(X_{DCO}, S_O, S_{NH}$ and $S_{NO})$ are considered to be available online. Consequently, the output y, the known input u and the unknown input d vectors are:

$$y(t) = [X_{DCO}(t), S_O(t), S_{NH}(t), S_{NO}(t)]^T \quad (25)$$
$$u(t) = [X_{DCO,in}(t), q_a(t), X_{BH,in}, S_{I,in}(t), \cdots$$
$$\cdots X_{I,in}(t), S_{ND,in}(t), X_{ND,in}(t)]^T \quad (26)$$
$$d(t) = S_{NH,in}(t) \quad (27)$$

Let us consider the dynamic ASM1 model with the state vector (24):

$$\dot{X}_{DCO}(t) = -\frac{1}{Y_h}[\varphi_1(t) + \varphi_2(t)] + (1 - f_p)[\varphi_4(t)$$
$$+ \varphi_5(t)] + D_1(t)$$
$$\dot{S}_O(t) = \frac{Y_h - 1}{Y_h}\varphi_1(t) + \frac{Y_a - 4.57}{Y_a}\varphi_3(t) + D_2(t)$$
$$\dot{S}_{NH}(t) = -i_{xb}[\varphi_1(t) + \varphi_2(t)] - \left[i_{xb} + \frac{1}{Y_a}\right]\varphi_3(t)$$
$$+ (i_{xb} - f_p i_{xp})[\varphi_4(t) + \varphi_5(t)] + D_3(t)$$
$$\dot{S}_{NO}(t) = \frac{Y_h - 1}{2.86 Y_h}\varphi_2(t) + \frac{1}{Y_a}\varphi_3(t) + D_4(t)$$
$$\dot{X}_{BH}(t) = \varphi_1(t) + \varphi_2(t) - \varphi_4(t) + D_5(t)$$
$$\dot{X}_{BA}(t) = \varphi_3(t) - \varphi_5(t) + D_6(t)$$
$$\dot{S}_I(t) = D_7(t)$$
$$\dot{X}_I(t) = f_p[\rho_4(t) + \rho_5(t)] + D_8(t)$$
$$\dot{S}_{ND}(t) = -\rho_6(t) + \rho_8(t) + D_9(t)$$
$$\dot{X}_{ND}(t) = (i_{xb} - f_p i_{xp})[\rho_4(t) + \rho_5(t)] - \rho_8(t) + D_{10}(t) \quad (28)$$

where Y_a, Y_h, f_p, i_{xb} and i_{xp} are constant coefficients and $\varphi_i(t)$, $i = 1, \cdots, 8$ are given by:

$$\varphi_1(t) = \mu_h \frac{X_{DCO}(t)}{K_{dco} + X_{DCO}(t)} \frac{S_O(t)}{K_{oh} + S_O(t)} X_{BH}(t)$$
$$\varphi_2(t) = \mu_h \eta_{NOg} \frac{X_{DCO}(t)}{K_{dco} + X_{DCO}(t)} \frac{S_{NO}(t)}{K_{no} + S_{NO}(t)} \frac{K_{oh} X_{BH}(t)}{K_{oh} + S_O(t)}$$
$$\varphi_3(t) = \mu_a \frac{S_{NH}(t)}{K_{nh,a} + S_{NH}(t)} \frac{S_O(t)}{K_{o,a} + S_O(t)} X_{BA}(t)$$
$$\varphi_4(t) = b_h X_{BH}(t)$$
$$\varphi_5(t) = b_a X_{BA}(t)$$
$$\varphi_6(t) = k_a S_{ND}(t) X_{BH}(t)$$
$$\varphi_7(t) = k_h X_{DCO}(t) \eta(t) X_{BH}(t)$$
$$\varphi_8(t) = k_h X_{ND}(t) \eta(t) X_{BH}(t)$$

$$\eta(t) = \frac{\frac{1}{X_{BH}(t)}}{K_{dco} + \frac{X_{DCO}(t)}{X_{BH}(t)} \left[\frac{S_O(t)}{K_{oh} + S_O(t)} + \frac{\eta_h K_{oh}}{K_{oh} + S_O(t)} \frac{S_{NO}(t)}{K_{no} + S_{NO}(t)}\right]}$$

The input/output balance is defined by:

$$D_1(t) = D_{in}(t) \left[X_{DCO,in}(t) - X_{DCO}(t)\right]$$
$$D_2(t) = D_{in}(t) \left[-S_O(t)\right] + K q_a(t) \left[S_{O,sat} - S_O(t)\right]$$
$$D_3(t) = D_{in}(t) \left[S_{NH,in}(t) - S_{NH}(t)\right]$$
$$D_4(t) = D_{in}(t) \left[-S_{NO}(t)\right]$$
$$D_5(t) = D_{in}(t) \left[X_{BH,in}(t) - X_{BH}(t) + \frac{f_r(1 - f_w)}{f_r + f_w} X_{BH}(t)\right]$$
$$D_6(t) = D_{in}(t) \left[-X_{BA}(t) + \frac{f_r(1 - f_w)}{f_r + f_w} X_{BA}(t)\right]$$
$$D_7(t) = D_{in}(t) \left[S_{I,in}(t) - S_I(t)\right]$$
$$D_8(t) = D_{in}(t) \left[X_{I,in}(t) - X_I(t) + \frac{f_r(1 - f_w)}{f_r + f_w}\right] X_I(t)$$
$$D_9(t) = D_{in}(t) \left[S_{ND,in}(t) - S_{ND}(t)\right]$$
$$D_{10}(t) = D_{in}(t) \left[X_{ND,in}(t) - X_{ND}(t) + \frac{f_r(1 - f_w)}{f_r + f_w}\right] X_{ND}(t) \quad (29)$$

where $D_{in}(t) = \frac{q_{in}(t)}{V}$. The following heterotrophic growth and decay kinetic parameters are used Olsson and Newell [1999]: $\mu_h = 3.733[1/24h]$, $\mu_a = 0.3[1/24h]$, $K_s = 20[g/m^3]$, $f_{ss} = 0.79$, $K_{oh} = 0.2[g/m^3]$, $K_{o,a} = 0.4[g/m^3]$, $K_{no} = 0.5[g/m^3]$, $K_{nh,a} = 1[g/m^3]$, $b_h = 0.3[1/24h]$, $b_a = 0.05[1/24h]$, $\eta_{NOg} = 0.8$. The stoichiometric parameters are $Y_h = 0.6[g \text{ cell}]$, $Y_a = 0.24[g \text{ cell}]$, $i_{xb} = 0.086[g$ N], $i_{xp} = 0.06[g$ N], $f_p = 0.1$ and the oxygen saturation concentration is $S_{O,sat} = 10[g/m^3]$. The fractions f_r and f_w: $f_r = 1.1$, $f_w = 0.04$ and the tank volume is $V = 1333[m^3]$.

4.2 Multi-model description for ASM1

For lack of space, only the essential points are given in the following. For more details the reader is referred to Nagy et al. [2010]. The idea is to equivalently rewrite the ASM1 model (28) under the MM form (2), e.e. to find r, the matrices A_i, B_i, E_i, ΔA_i and the weighting functions $\mu_i(x, u)$. First, the decision variables are defined as nonlinearities of the system (28):

$$z_1(x, u) = \frac{q_{in}(t)}{V}$$
$$z_2(x, u) = \frac{X_{DCO}(t)}{K_{dco} + X_{DCO}(t)} \frac{S_O(t)}{K_{oh} + S_O(t)}$$
$$z_3(x, u) = \frac{1}{K_{o,a} + S_O(t)} \frac{S_{NH}(t)}{K_{nh,a} + S_{NH}(t)} X_{BA}(t)$$
$$z_4(x, u) = S_{NH}(t)$$
$$z_5(x, u) = \frac{X_{DCO}(t)}{K_{dco} + X_{DCO}(t)} \frac{S_{NO}(t)}{K_{no} + S_{NO}(t)} \frac{K_{oh}}{K_{oh} + S_O(t)}$$
$$z_6(x, u) = X_{DCO}(t) \eta(t)$$

Remark 4.1. In order to avoid potential infeasible LMI solutions for the observer design, the number of decision variables should be reduced. Small dynamic variations and values can be observed for z_3, z_5 and z_6 compared to the other decision variables, which allows to consider their means $\tilde{z}_3$, $\tilde{z}_5$ and $\tilde{z}_6$ for the construction of the MM form (2).

A convex polytopic transformation is performed for all the decision variables ($j = 1, 2, 4$), as follows:

$$z_j(x, u) = F_{j,1}(z_j(x, u)) z_{j,1} + F_{j,2}(z_j(x, u)) z_{j,2} \quad (31)$$

where the scalars $z_{j,1}$, $z_{j,2}$ are respectively the maxima and minima of $z_j(x, u)$ and $F_{j,1}(z_j)$, $F_{j,2}(z_j)$ are defined by

$$F_{j,1}(z_j) = \frac{z_j(x, u) - z_{j,2}}{z_{j,1} - z_{j,2}}, \ F_{j,2}(z_j) = \frac{z_{j,1} - z_j(x, u)}{z_{j,1} - z_{j,2}} \quad (32)$$

By multiplying the functions $F_{j,\sigma_i^j}(z_j(x, u))$, the $r = 8$ weighting functions are obtained:

$$\mu_i(z) = F_{1,\sigma_i^1}(z_1(u)) F_{2,\sigma_i^2}(z_2(x, u)) F_{4,\sigma_i^4}(z_4(x, u)) \quad (33)$$

The constant matrices A_i, B_i and E_i defining the 8 sub-models are given by:

$$A_i = A(z_{1,\sigma_i^1}, z_{2,\sigma_i^2}, z_{4,\sigma_i^4}) \quad (34a)$$

$$B_i = B(z_{1,\sigma_i^1}) \quad (34b)$$

$$E_i = E(z_{1,\sigma_i^1}), i = 1, ..., 8, \ j = 1, 2, 4 \quad (34c)$$

where the matrices $A[a_{i,j}] \in \mathbb{R}^{10 \times 10}$, $B[b_{i,j}] \in \mathbb{R}^{10 \times 6}$ and $E[e_{i,j}] \in \mathbb{R}^{10 \times 1}$ are defined by the following compounds:
$a_{1,1}(x, u) = a_{3,3}(x, u) = a_{4,4}(x, u) = a_{7,7}(x, u) = a_{9,9}(x, u) = b_{1,1}(x, u) = b_{7,3}(x, u) = b_{8,4}(x, u) = b_{9,5}(x, u) = b_{10,6}(x, u) = e_{3,1}(x, u) = -z_1(u)$, $b_{2,2} = KS_{O,sat}$ and

$$a_{1,5}(x, u) = -\frac{\mu_h}{Y_h} z_2(x, u) + (1 - f_p) b_h - \frac{\mu_h \eta_{NOg}}{Y_h} \tilde{z}_5$$

$$a_{1,6}(x, u) = (1 - f_p) b_a$$

$$a_{2,2}(x, u) = -z_1(u) - K q_a - \frac{4.57 - Y_a}{Y_a} \mu_a \tilde{z}_3$$

$$a_{2,5}(x, u) = \frac{(Y_h - 1)\mu_h}{Y_h} z_2(x, u)$$

$$a_{3,2}(x, u) = -(i_{xb} + \frac{1}{Y_a})\mu_a \tilde{z}_3$$

$$a_{3,5}(x, u) = (i_{xb} - f_p i_{xp})b_h - i_{xb}\mu_h[z_2(x, u) + \eta_{NOg} \tilde{z}_5]$$

$$a_{3,6}(x, u) = (i_{xb} - f_p i_{xp}) b_a$$

$$a_{4,2}(x, u) = \frac{1}{Y_a}\mu_a \tilde{z}_3$$

$$a_{4,5}(x, u) = \frac{Y_h - 1}{2.86 Y_h} \mu_h \eta_{NOg} \tilde{z}_5$$

$$a_{5,5}(x, u) = \mu_h z_2(x, u) - b_h + \mu_h \eta_{NOg} \tilde{z}_5 + z_1(u)\tilde{f}$$

$$a_{6,2}(x, u) = \mu_a \tilde{z}_3$$

$$a_{6,6}(x, u) = z_1(u)\tilde{f} - b_a$$

$$a_{8,5}(x, u) = f_p b_h$$

$$a_{8,6}(x, u) = f_p b_a$$

$$a_{8,8}(x, u) = \tilde{f} z_1(u)$$

$$a_{9,5}(x, u) = -k_a z_4(x, u) + k_h \tilde{z}_6$$

$$a_{10,5}(x, u) = (i_{xb} - f_p i_{xp})b_h - k_h \tilde{z}_6$$

$$a_{10,6}(x, u) = (i_{xb} - f_p i_{xp})b_a$$

$$a_{10,10}(x, u) = \tilde{f} z_1(u) \quad (35)$$

where $\tilde{f} = \left[\frac{f_r(1 - f_w)}{f_r + f_w} - 1 \right]$. The rest of matrices compounds not mentioned here are zero. Using this, the reduced MM form (Remark 4.1) of the ten state reduced ASM1 is completed. It should be said that the reduced MM represent very good the reduced ASM1 (28), excepting the two concentrations S_{NO} and X_{ND} for which a quite good representation is nevertheless obtained, as seen in figure 1.
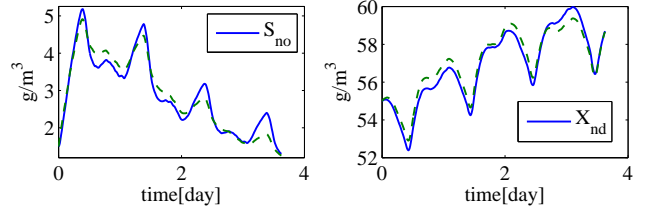

Fig. 1. Comparison between the reduced MM(dashed) and ASM1(solid)

In order to take into account parameter uncertainties on b_H and b_A, the MM structure is slightly modified. These parameters appear in the coefficients a_{15}, a_{16}, a_{35}, a_{36}, a_{55} and a_{66} in (35), allowing to separate the uncertain part $\Delta A(t)$ from the known one $A(t)$. The parameter variation on b_H (resp. b_A) is of 20% (resp. 25%) of its nominal value, i.e. $b_H \in [0.25 ; 0.35]$ (resp. $b_A \in [0.04 ; 0.06]$).The uncertain terms of ASM1 $\Delta B_i(t) = 0$ and $\Delta A_i(t) = \Delta A(t)$ are written under the form $\Delta A(t) = M^a F_a(t) N^a$ where:

$$M^a = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^T \quad (36)$$

$$F_a(t) = \begin{bmatrix} \Delta b_H(t) & 0 \\ 0 & \Delta b_A(t) \end{bmatrix} \quad (37)$$

$$N^a = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (38)$$

The data used for simulation are generated with the complete ASM1 model ($n = 13$) Henze et al. [1987], in order to represent a realistic behavior of a WWTP. Applying the Theorem 1 for $q = 4$ the observer (5) is designed by finding positive scalars ϵ_{1i}, ϵ_{2i}, positive definite matrices P_1 and P_2 and matrices $\bar{P}_i$ ($i = 1, \cdots, 8$) -that are not given here due to space limitation- such that the convergence conditions, given in Theorem 1 hold. The value of the attenuation rate from the known and unknown inputs $u(t)$ and $d(t)$ to the state and fault estimation error $e_a(t)$ is $\bar{\gamma} = 0.52$. The positive scalars regrouped in vectors are $\epsilon_1 = [0.3313, 0.3339, 0.3428, 0.3424, 0.3064, 0.3099, 0.3167, 0.3185]$, $\epsilon_1 = [0.2767, 0.2766, 0.2772, 0.2777, 0.2772, 0.2763, 0.2778, 0.2788]$. A comparison between the actual state variables, the unknown inputs and their respective estimates is depicted in figures 2 and 3. In fig. 2, the estimation errors for S_{NO} and X_{ND} are in part generated by the reduction made on the MM (see the Remark 4.1).

5. CONCLUSIONS AND FUTURE WORKS

5.1 Conclusions

A MM approach provides the state of the art solutions to many problems involving estimation, filtering, control, and/or modeling. As a major advantage of the MM against a general
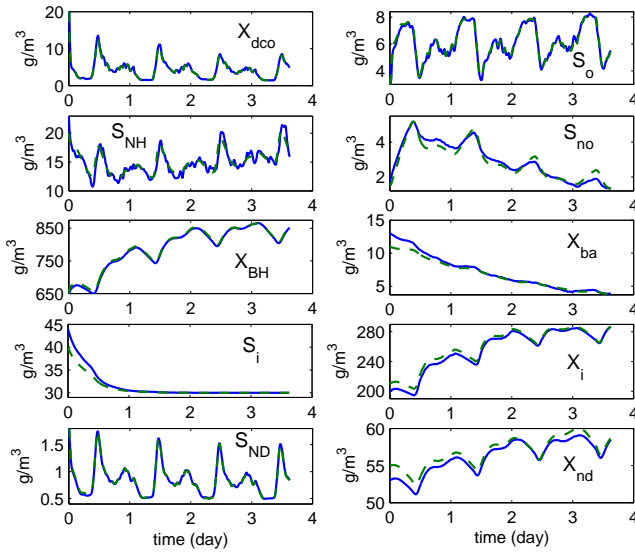
Fig. 2. State estimation using PMI observer (dashed) for ASM1 (solid)
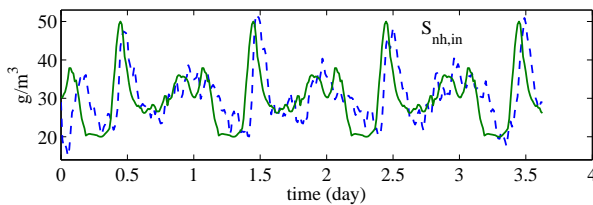

Fig. 3. Unknown input estimation using PMI observer (dashed) for complete ASM1 (solid)

nonlinear model there is the possibility to use many tools developed in the framework of linear system.

5.2 Future Works

The future developments are numerous and among them one can think of the choice of adapted structures to the local models with specific properties, such as diagnosability. Another important problem concerns the use of MM and PMI observers to detect and isolate sensor and actuator faults in the system.

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