On the Unknown Input Observer Design : a Decoupling Class Approach

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Abstract— This paper deals with state estimation for linear discrete-time systems subject to unknown input. Although many papers have dealt with the problem of Unknown Input Observer design, state decoupling and reconstruction ; the goal is to present a new method allowing to characterize a class of unknown inputs to which the estimation error is decoupled. This contribution considers two problems : exact decoupling and $\mathscr{L}_2$-attenuation of the unknown input to the state estimation error.

I. INTRODUCTION

The Unknown Input Observer (UIO) design has received considerable interest due to its importance and connection with fault detection problem, since in many cases a part of the system's input is inaccessible (e.g. plant disturbance or actuator failure). Under such circumstances, a conventional observer that requires knowledge of all inputs cannot be used directly, then the UIO was developed to estimate the state of an uncertain system despite the existance of unknown inputs or uncertain disturbances [4], [2], [7], [9], [11], [12].

Several researches were achieved concerning the state estimation in the presence of unknown inputs. They can be gathered into two categories. The first one supposes an a priori knowledge on these nonmeasurable inputs ; in particular, Johnson [16] proposes a polynomial approach and Hostetter and Meditch [11] suggest approximating the unknown inputs by the response of a known dynamic system [1].
The second category proceeds either by estimation of the unknown inputs [17] or by their complete elimination from the system equation. However, some of these methods require differentiation of the measured outputs which can amplify the effect of the measurement's noise.

One of the most successful robust observer design methods ressorts to the disturbance decoupling principle [25] [12] [7]. The problem of UIO has been initialised by Basile and Marro [2], Guidorzi and Marro [9]. Since then, several contributions for UIOs have been proposed [4], [10]. For these methods, a rank condition relating the output distribution matrix and the input distribution matrices must be satisfied which is sometimes difficult and might be quite restrictive.
An approach to simultaneously estimate the unknown input and the system state using the PI observer has been

proposed by [23], [13] and [14]. However, this observer concerns the case of constant unknown inputs. In [18], a model-based oserver design in the presence of polynomial unknown inputs for TS fuzzy systems has been investigated. In [20], the authors present a method for state-estimation for Takagi-Sugeno descriptor systems affected by UI. Sufficient existence conditions of the unknown inputs decoupling observers are given and strict linear matrix inequalities are solved to determine the gain of the observers so that the estimated state asymptotically tends to the real one.

Summarizing, the UIO design is based either on the decoupling such that the estimation error must not depend on the the UI, or on the synthesis of an Integral Observer for the detection (estimation) of disturbances. But, as much as the authors know, all the proposed strategies impose structural and rank constraints.

This work advantage is to propose an exact decoupling instead of an attenuation or a decoupling for a constant UI. Moreover, it is proposed to decompose any UI into two terms. The first one is a sum of exponential functions from which the state estimates can be exactly decoupled. For a given system, the class of the UI satisfying that property is clearly satisfied. The effect of the remaining part of the UI on the state estimates is then attenuated in an $\mathscr{L}_2$ framework. Two cases will be treated, the exact decoupling case and the almost-exact case which consists in decoupling the estimation to a subset of the UI, while attenuing the $\mathscr{L}_2$ gain from the other UI to the estimation.

This paper is organised as follows : Section II presents a second order system in order to introduce the decoupling strategy and the *Uknown Input Class for exact decoupling* notion and how to generate this class. Section III is a generalization of the second section. In section IV, we introduce the notion of partial decoupling and the linear matrix inequalities conditions to ensure the $\mathscr{L}_2$ attenuation of the UI effect on the system. In order to impove the obtained results, a pole placement will also be applied.
However, the usual linearization approaches are not suitable to the present problem since BMIs (Bilinear Matrix Inequalities) are to be dealt with. A gain adjustment technique is then applied. This synthesis linearize the inequalities by fixing one of the unknown variable [19]. This kind of procedure can be found in the centrage-XY procedure [15], the D-K iteration mentionned in [22] or Yamada's approach [24]. However, there is no guarantee that the proposed structure assures the algorithm convergence.
Finally, in the last section, simulations are presented to show

the efficiency of the proposed approach.

II. ILLUSTRATIVE EXAMPLE

To begin with, the procedure is introduced with the help of a simple example being a second order system. The different steps leading to the *Uknown Input Class* for exact decoupling are detailed. Consider a second order system described by :

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k + D\eta_{k-1} \\ y_k &= Cx_k + e\eta_{k-1} \end{cases} \tag{1}$$

Vectors $x_k \in \mathbb{R}^2$, $u_k \in \mathbb{R}$, $\eta_k \in \mathbb{R}$ and $y_k \in \mathbb{R}$ are the system state, input, unknown input and the output vector respectively. The system matrices are real valued, constant and of appropriate dimensions :

$$x_k = \begin{pmatrix} x_{1k} \\ x_{2k} \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$C = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \quad D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \tag{2}$$

The proposed Proportional Integral Observer (PIO) of gain K and the UIO depending on an auxiliary variable $z_k \in \mathbb{R}$ are respectivelly given by the following equations :

$$\begin{cases} \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + D\hat{\eta}_{k-1} + K\tilde{y}_k \\ \hat{y}_k &= C\hat{x}_k + e\hat{\eta}_{k-1} \\ \tilde{y}_k &= y_k - \hat{y}_k \end{cases} \tag{3}$$

$$\begin{cases} z_{k+1} &= \gamma_1 \tilde{y}_k + \lambda_1 z_k \\ \hat{\eta}_{k+1} &= \gamma_2 z_k + \lambda_2 \hat{\eta}_k \end{cases} \tag{4}$$

with :

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

It can be noted that the UIO has a filter structure with as an input the output reconstruction error $\tilde{y}_k$. This filter parameters γ_1, γ_2, λ_1 and λ_2 allow to modify the gain and time constant of the UIO.

In (4), only one integrator is used but this structure can be generalized to multiple integrator observer. However, the choice $z_{k+1} = \gamma_1 \tilde{y}_{k+1} + \lambda_1 z_k$ with $\gamma_1 = 1$ and $\lambda_1 = 0$ leads to $z_{k+1} = \tilde{y}_k$ which is equivalent to a first order filter for the UI estimation. The system observer is described by equations (3) and (4). The gains K, γ_1, γ_2, λ_1 and λ_2 are choosen according to the state and UI reconstruction specifications.

In the following, the state and UI estimation errors are expressed as function of the UI. Since the system and its observer are linear, the time operator q, is adequate to express the errors :

$$\begin{aligned} \tilde{x}_k &= x_k - \hat{x}_k \\ \tilde{\eta}_k &= \eta_k - \hat{\eta}_k \end{aligned}$$

From equations (1), (3) and (4), the state reconstruction error function of the UI is given by :

$$\begin{cases} \tilde{x}_{1k} = \frac{N_1(q)}{D(q)} \eta_k \\ \\ \tilde{x}_{2k} = \frac{N_2(q)}{D(q)} \eta_k \end{cases} \tag{5}$$

$$\begin{cases} N_1(q) = (q - \lambda_1)(q - \lambda_2)(\tilde{a}_{12}\tilde{d}_2 - \tilde{a}_{22}\tilde{d}_1 + q\tilde{d}_1) \\ N_2(q) = (q - \lambda_1)(q - \lambda_2)(\tilde{a}_{21}\tilde{d}_1 - \tilde{a}_{11}\tilde{d}_2 + q\tilde{d}_2) \\ D(q) = ((q - \lambda_1)(q - \lambda_2) + \gamma e)((q - \tilde{a}_{11})(q - \tilde{a}_{22}) - \tilde{a}_{12}\tilde{a}_{21}) \\ \quad + \gamma c_1(\tilde{a}_{12}\tilde{d}_2 - \tilde{a}_{22}\tilde{d}_1 + q\tilde{d}_1) + \gamma c_2(\tilde{a}_{21}\tilde{d}_1 - \tilde{a}_{11}\tilde{d}_2 + q\tilde{d}_2) \end{cases} \tag{6}$$

with :

$$\begin{cases} \tilde{a}_{11} = a_{11} - k_1 c_1 \\ \tilde{a}_{12} = a_{12} - k_1 c_2 \\ \tilde{a}_{21} = a_{21} - k_2 c_1 \\ \tilde{a}_{22} = a_{22} - k_2 c_2 \\ \tilde{d}_1 = d_1 - k_1 e \\ \tilde{d}_2 = d_2 - k_2 e \end{cases} \tag{7}$$

From (5), conditions for the estimation errors to be independent from the UI can easily be derived. Then, the UI family satisfying an exact decoupling is solution of :

$$\begin{cases} \frac{N_1(q)}{D(q)} \eta_k = 0 \\ \\ \frac{N_2(q)}{D(q)} \eta_k = 0 \end{cases} \tag{8}$$

In order to find the solution η_k assuring the precedent condition, it is imposed that $N_1(q)$ and $N_2(q)$ have the same roots. It should also be checked if some solutions are common to $D(q)$ and $N_1(q)$ (or $N_2(q)$). For the first point, we have the following condition :

$$\frac{\tilde{a}_{11}\tilde{d}_2 - \tilde{a}_{22}\tilde{d}_1}{\tilde{a}_{21}\tilde{d}_1 - \tilde{a}_{11}\tilde{d}_2} = \frac{\tilde{d}_1}{\tilde{d}_2} \tag{9}$$

which imposes a constraint on the observer coefficients k_1 and k_2. That leads to :

$$q_0 = \frac{\tilde{a}_{21}\tilde{d}_1 - \tilde{a}_{11}\tilde{d}_2}{\tilde{d}_2} \tag{10}$$

is a common root between $N_1(q)$, $N_2(q)$ and $D(q)$.
Then (8) is written as :

$$(q - \lambda_1)(q - \lambda_2)\eta_k = 0 \tag{11}$$

The solution is given by an UI being the sum of two exponential functions :

$$\eta_k = A_1 \lambda_1^k + A_2 \lambda_2^k \tag{12}$$

where coefficients A_1 and A_2 are arbitrarily set. Finally, the choice of the observer values λ_1 and λ_2 gives the UI class assuring the exact decoupling of the state error toward the UIO for any values of the coefficients A_1 and A_2.

III. RECONSTRUCTION ERRORS : DISTURBANCES DECOUPLING

Let us define the following system equations :

$$\begin{cases} x_{k+1} &= Ax_k + Bu_k + D\eta_{k-1} \\ y_k &= Cx_k + e\eta_{k-1} \end{cases} \tag{13}$$

Vectors $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $\eta_k \in \mathbb{R}$ and $y_k \in \mathbb{R}^p$ are the system state, input, unknown input and the output vectors respectively. The system matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $D \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{p \times n}$ et $e \in \mathbb{R}^{p \times 1}$ are real valued and known.

The proposed system observer of gain K and the UIO are respectively given by the following equations :

$$\begin{cases} \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + D\hat{\eta}_{k-1} + K\tilde{y}_k \\ \hat{y}_k &= C\hat{x}_k + e\hat{\eta}_{k-1} \\ \tilde{y}_k &= y_k - \hat{y}_k \end{cases} \tag{14}$$

$$\begin{cases} z_{k+1} &= \Gamma\tilde{y}_k + \Lambda z_k \\ \hat{\eta}_{k+1} &= \gamma z_k + \lambda\hat{\eta}_k \end{cases} \tag{15}$$

with appropriate dimensions : $z_k \in \mathbb{R}^q$, $K \in \mathbb{R}^{n\times p}$, $\Gamma \in \mathbb{R}^{q\times p}$, $\gamma \in \mathbb{R}^{1\times q}$, $\Lambda \in \mathbb{R}^{q\times q}$ and $\lambda \in \mathbb{R}$.

Depending on the value of the UIO parameters, we can have either a proportional observer, an integral or a multiple integral observer. In this particular case, the choice $\lambda_1 = 1$ or $\lambda_2 = 1$ introduces the two integrators in this filter structure. By following the same steps as in the previous section, the state and UI reconstruction errors are expressed ; we get from equation (15) with the time operator q :

$$\hat{\eta}_k = (q-\lambda)^{-1}\gamma z_k \tag{16}$$

$$(qI_q - \Lambda)z_k = \Gamma\tilde{y}_k \tag{17}$$

From (16), (17), (13) and (14), we have :

$$(qI_q - \Lambda)z_k = \Gamma C\tilde{x}_k + \Gamma e(\eta_{k-1} - \hat{\eta}_{k-1})$$

which leads to :

$$[(qI_q - \Lambda) + \Gamma e(q-\lambda)^{-1}q^{-1}\gamma]z_k = \Gamma C\tilde{x}_k + \Gamma eq^{-1}\eta_k \tag{18}$$

The state error $\tilde{x}_k$ dynamic is obtained from (13) and (14) :

$$\begin{aligned} \tilde{x}_{k+1} &= \overline{A}\tilde{x}_k + \overline{D}\tilde{\eta}_{k-1} \\ \overline{A} &= A - KC \\ \overline{D} &= D - Ke \end{aligned} \tag{19}$$

From (19) and (17), we have :

$$\begin{aligned} (qI_n - \overline{A})\tilde{x}_k &= \overline{D}\tilde{\eta}_{k-1} \\ &= \overline{D}q^{-1}\eta_k - \overline{D}q^{-1}(q-\lambda)^{-1}\gamma z_k \end{aligned}$$

That gives the state estimation error :

$$\tilde{x}_k = (qI_n - \overline{A})^{-1}\overline{D}q^{-1}\eta_k - (qI_n - \overline{A})^{-1}\overline{D}q^{-1}(q-\lambda)^{-1}\gamma z_k \tag{20}$$

By replacing this expression in (18), we have :

$$z_k = \overline{Z}^{-1}\overline{\Lambda}\eta_k \tag{21}$$

with :

$$\begin{aligned} \overline{\Lambda} &= \Gamma C(qI_n - \overline{A})^{-1}\overline{D} + \Gamma e \\ \overline{Z} &= q(qI_q - \Lambda) + \overline{\Lambda}(q-\lambda)^{-1}\gamma \end{aligned} \tag{22}$$

Finally, replacing (21) in (16) and (20) to have :

$$\begin{cases} \hat{\eta}_k = (q-\lambda)^{-1}\gamma\overline{Z}^{-1}\overline{\Lambda}\eta_k \\ \tilde{x}_k = (qI_n - \overline{A})^{-1}\overline{D}q^{-1}\left[1 - (q-\lambda)^{-1}\gamma\overline{Z}^{-1}\overline{\Lambda}\right]\eta_k \end{cases} \tag{23}$$

the UI estimation error becomes :

$$\tilde{\eta}_k = \left[1 - (q-\lambda)^{-1}\gamma\overline{Z}^{-1}\overline{\Lambda}\right]\eta_k$$

From (23) we have the state estimation error decoupling condition toward the UI :

$$(qI_n - \overline{A})^{-1}\overline{D}q^{-1}\left[1 - (q-\lambda)^{-1}\gamma\overline{Z}^{-1}\overline{\Lambda}\right]\eta_k = 0 \tag{24}$$

In order to decouple the state from the UI and assure an exact estimation of it, the following condition has to be verified :

$$\left[1 - (q-\lambda)^{-1}\gamma\overline{Z}^{-1}\overline{\Lambda}\right]\eta_k = 0 \tag{25}$$

Equation (25) may be extended as $\frac{N(q)}{D(q)}\eta_k = 0$. From solving this last equation, we have the roots that define the UI class that ensure an exact decoupling of the estimation error toward the UI. This class is written as : $\sum_i A_i\lambda_i$ where the λ_i correspond to the roots of (25) and A_i are totally free parameters.

IV. PARTIAL DECOUPLING OBSERVER

In the previous section, was detailed how to find the class of UI ensuring an exact decoupling of the UI in respect to the state estimation error. In the following section, a general case with an UI that does not respond to the decoupling condition is considered. In this case, the problem is solved by attenuating the effect (transfer) of the UI to the estimation error and propose linear matrix inequalities to determinate the observer gain so that the estimated state asymptotically tends to the real one.

In addition to the two previous cases (exact and partial decoupling), we also have a third one, which is a mixt between the two solutions. In fact, any UI may be decomposed into a sum of two terms $\eta_k = \eta_k^d + \eta_k^a$. The first term corresponds to the decoupling term and the second one is the approximaion term onto $\mathcal{L}_2$ attenuation is applied. In subsection A, we only present the attenuation approach ; but, in the simulation section the combined approach will be illustrated.

A. $\mathcal{L}_2$ Attenuation

System and observer equations are given by :

$$\begin{cases} \tilde{x}_{k+1} &= \overline{A}\tilde{x}_k + \overline{D}\tilde{\eta}_{k-1} \\ \tilde{\eta}_k &= \eta_k - \lambda\eta_{k-1} - \gamma z_{k-1} + \lambda\tilde{\eta}_{k-1} \\ z_{k+1} &= \Gamma C\tilde{x}_k + \Gamma e\tilde{\eta}_{k-1} + \Lambda z_k \end{cases} \tag{26}$$

The corresponding matrix form is given by :

$$e_{k+1} = A_1 e_k + B_1 \eta_k^a \tag{27}$$

with :

$$A_1 = \begin{pmatrix} \overline{A} & \overline{D} & 0 & 0 \\ 0 & \lambda & 0 & -\gamma \\ \Gamma C & \Gamma e & \Lambda & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} 0 & 0 \\ 1 & -\lambda \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$e_k = \begin{pmatrix} \tilde{x}_k \\ \tilde{\eta}_{k-1} \\ z_k \\ z_{k-1} \end{pmatrix}, \eta_k^a = \begin{pmatrix} \eta_k \\ \eta_{k-1} \end{pmatrix} \tag{28}$$

In particular, (27) gives the UI influence on the estimation errors. To focus on the impact of the UI on the state estimation, a new observer output is considered :

$$g_k = C_1 e_k \tag{29}$$

with : $C_1 = \begin{pmatrix} I & 0 & 0 & 0 \end{pmatrix}$.

Considering the Real Bounded Lemma [3], the system (27),

it is stable and the $\mathscr{L}_2$ gain is bounded by $\frac{\|g_k\|_2}{\|\eta_k\|_2} < \mu$ if there exists a positive symmetric matrice P and a positive scalar μ such that the following condition holds :

$$\begin{bmatrix} A_1^T P A_1 - P & A_1^T P B_1 & C_1^T \\ B_1^T P A_1 & B_1^T P B_1 - \mu^2 I & 0 \\ C_1 & 0 & -\mu^2 I \end{bmatrix} < 0 \qquad (30)$$

According to [8] and [21], the previous problem can be reformulated by searching a positive symetric definite matrice P, gains K and G such that :

$$\begin{bmatrix} -P & A_1^T P B_1 & C_1^T & A_1^T G^T \\ B_1^T P A_1 & B_1^T P B_1 - \mu^2 I & 0 & 0 \\ C_1 & 0 & -\mu^2 I & 0 \\ G A_1 & 0 & 0 & -G - G^T + P \end{bmatrix} < 0 \qquad (31)$$

where A_1 defined in (28) with the help of (19), depends on K. Let us remark that inequality (31) is not linear. For that reason some transformations are needed to obtain LMIs. Let us write the matrix A_1 such that :

$$A_1 = \overline{A}_1 - R K \overline{B}_1 \qquad (32)$$

with :

$$\overline{A}_1 = \begin{pmatrix} A & D & 0 & 0 \\ 0 & \lambda & 0 & -\gamma \\ \Gamma C & \Gamma e & \Lambda & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad R = \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \overline{B}_1 = \begin{pmatrix} C & e & 0 & 0 \end{pmatrix} \qquad (33)$$

Replacing A_1 by (32) in (31), we have :

$$\begin{bmatrix} -P & \overline{A}_1^T P B_1 & C_1^T & \overline{A}_1^T G^T \\ B_1^T P \overline{A}_1 & B_1^T P B_1 - \overline{\mu} I & 0 & 0 \\ C_1 & 0 & -\overline{\mu} I & 0 \\ G \overline{A}_1 & 0 & 0 & P - G - G^T \end{bmatrix} + M^T N + N^T M < 0 \qquad (34)$$

with $M = \begin{bmatrix} -\overline{B}_1^T K^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$, $N = \begin{bmatrix} 0 \\ B_1^T P R \\ 0 \\ G R \end{bmatrix}^T$ and $\overline{\mu} = \mu^2$

Let us recall the following lemma [26]. Consider two real matrices Π and Λ with appropriate dimensions, for any positive matrix Σ the following inequality holds :

$$\Pi^T \Lambda + \Lambda^T \Pi \leq \Pi^T \Sigma \Pi + \Lambda^T \Sigma^{-1} \Lambda \qquad (35)$$

Applying this lemma, (34) becomes :

$$\begin{bmatrix} -P & \overline{A}_1^T P B_1 & C_1^T & \overline{A}_1^T G^T \\ B_1^T P \overline{A}_1 & B_1^T P B_1 - \overline{\mu} I & 0 & 0 \\ C_1 & 0 & -\overline{\mu} I & 0 \\ G \overline{A}_1 & 0 & 0 & P - G - G^T \end{bmatrix} +$$

$$M^T \Sigma M + N^T \Sigma^{-1} N < 0 \qquad (36)$$

Applying Schur's complement, we get :

$$\begin{bmatrix} -P & \overline{A}_1^T P B_1 & C_1^T & \overline{A}_1^T G^T & \overline{B}_1^T K^T & 0 \\ B_1^T P & B_1^T P B_1 - \overline{\mu} I & 0 & 0 & 0 & B_1^T P R \\ C_1 & 0 & -\overline{\mu} & 0 & 0 & 0 \\ G \overline{A}_1 & 0 & 0 & -G - G^T + P & 0 & G R \\ K \overline{B}_1 & 0 & 0 & 0 & -\Sigma^{-1} & 0 \\ 0 & R^T P B_1 & 0 & R^T G^T & 0 & -\Sigma \end{bmatrix} < 0 \qquad (37)$$

At last, (37) becomes :

$$\begin{bmatrix} -P & \overline{A}_1^T P B_1 & C_1^T & \overline{A}_1^T G^T & \overline{B}_1^T F^T & 0 \\ B_1^T P & B_1^T P B_1 - \overline{\mu} I & 0 & 0 & 0 & B_1^T P R \\ C_1 & 0 & -\overline{\mu} & 0 & 0 & 0 \\ G \overline{A}_1 & 0 & 0 & -G - G^T + P & 0 & G R \\ F \overline{B}_1 & 0 & 0 & 0 & -\Sigma^T & 0 \\ 0 & R^T P B_1 & 0 & R^T G^T & 0 & -\Sigma \end{bmatrix} < 0 \qquad (38)$$

with $F = \Sigma K$. The LMI must be solved in P, G, F and the gain K is obtained by $K = \Sigma^{-1} F$.

B. Pole Assignment

The minimization of the attenuation factor μ may result in slow dynamics of the state estimation error. This problem can be solved by pole assignment of the closed loop system in a specified region. The considered region is a disk centred at $(q, 0)$ with radius α. Thus, the condition to answer this constraint is given by the following : find $P = P^T > 0$ and $Q = Q^T > 0$ such that the following LMI [6] occurs :

$$\begin{bmatrix} -\alpha Q & -q Q + Q A - G C \\ (-q Q + Q A - G C)^T & -\alpha P \end{bmatrix} < 0 \qquad (39)$$

with $G = Q K$. We have to solve this LMI regarding to Q and G then we deduce K. Thus, to ensure the stabilty and pole assignment, the conditions (38) and (39) must be fulfilled simultaneously.

C. Gain Ajustement

From matrices F and G definitions, there is a dependence between the two LMIS (38) and (39). Then we have to solve simultaneously these two conditions. It can be noted $LMI1(P, K)$ and $LMI2(Q, K)$. The proposed method is based on an ajustment technique allowing to set some variables and calculate others in an iterative way. More precisely, if the gain K is fixed, we solve $LMI1(P, K)$ regarding to P. Then we solve $LMI2(Q, K)$ regarding to Q and K and we use the obtained result K for another cycle (see table 1). This procedure was choosen in reason of its simplicity, but one should be aware that no optimality or convergence guarantee is given. However, since our study goal is to find a solution to the given conditions, an optimal solution is not a necessity.

Iterative optimisation for gain K :
1) Set $i = 0$. Chose a stabilisable value K_0. Put $K^{(i)} = K_0$.
2) $\mathscr{L}_2$ attenuation : Find $P^{(i+1)} > 0$ solution of $LMI1(P, K^{(i)})$.
3) Pole assignment : Find $Q^{(i+1)}$ and $K^{(i+1)}$ solution of $LMI2(Q, K)$.
4) Stopping condition :
   – If $||K^{(i+1)} - K^{(i)}|| < \varepsilon$ stop the algorithm : $K_{final} = K^{(i+1)}$.
   – Else, set $i = i + 1$ and go back to step 2.

Table1 : Adjustment algorithm

V. SIMULATIONS

Consider the following system described by :

$$A = \begin{pmatrix} 0.6 & -0.2 & -0.1 & 0.1 \\ -0.1 & 0.7 & -0.1 & 0.1 \\ 0.4 & 0 & 0.9 & -0.1 \\ 0 & 0.2 & 0 & 0.8 \end{pmatrix} \quad B = \begin{pmatrix} -0.3 & -0.4 \\ 0.5 & -0.4 \\ -0.1 & 0.6 \\ -0.2 & 0.7 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 0.2 \\ -0.3 \\ 0.1 \\ 0.1 \end{pmatrix} \quad E = -4.5$$

with the observer parameters :

$$\Lambda = 1; \lambda = 0.7 \text{ et } \Gamma = 0.2; \gamma = -0.4$$

At the first step, let us determine the observer gain K with the proposed iterative algorithm. The obtained gain K and attenuation μ for a pole assignment in a disk centred at $(0.3,0)$ with radius 0.2 are :

$$K = \begin{pmatrix} 0.1467 \\ 4.6498 \\ -7.7861 \\ 4.4794 \end{pmatrix} \quad \mu = 4.78 \tag{40}$$

The second step consists of finding the UI class for an exact decoupling. Let us recall that the state decoupling condition towards the UI with an exact estimation of the UI is given by (25). In this example, it corresponds to an UI composed of a linear combination of six exponential functions : two roots correspond to Λ and λ, the others are two complex conjugate and two real values given by :

$$\lambda_1 = \Lambda = 1 \, ; \, \lambda_2 = \lambda = 0.7 \, ; \, \lambda_3 = 0.36 \, ; \, \lambda_4 = 0.29$$
$$\lambda_{5,6} = 0.43 \pm 0.02i$$

Then, the class of UI for an exact decoupling is given by :

$$\eta_k = A_1\lambda_1^k + A_2\lambda_2^k + A_3\lambda_3^k + A_4\lambda_4^k + A_5 a^k \cos(\phi k + \psi) \tag{41}$$

with :
$a = \sqrt{Re(\lambda_5)^2 + Im(\lambda_5)^2}$ and $\cos(\phi) = \dfrac{Re(\lambda_5)^2}{2\sqrt{Re(\lambda_5)^2+Im(\lambda_5)^2}}$

The UI is defined by :

$$\eta_k = 0.1 - 0.2(0.7)^k + 0.4(0.36)^k +$$
$$0.2(0.29)^k - 0.5(0.93)^k cos(2.05k) \tag{42}$$

Finally, the considered UI η_k can be written as $\eta_k = \eta_k^d + \eta_k^a$ with η_k^d corresponds to the UI for exact decoupling and η_k^a the approximation error. The following figures are obtained for the initial conditions $x_0 = \begin{pmatrix} 0.5 & 0.1 & 0.2 & -0.1 \end{pmatrix}^T$ and $\hat{x}_0 = \begin{pmatrix} -0.5 & 0.5 & -0.4 & 0.2 \end{pmatrix}^T$. Fig.1 shows the system inputs. Fig.2 represents the UIs (for the exact $\eta_k = \eta_k^d$ and partial decoupling cases $\eta_k = \eta_k^d + \eta_k^a$) and their estimates and Fig 3. represents the state system and their estimate for both situations of exact and partial decoupling. In both situations, the state estimate is satisfactory.

In the case where a noise r affects the output measurment, we considere the new output defined as :

$$y_k = Cx_k + e\eta_{k-1} + Wr_k \tag{43}$$

Our goal is to attenuate the influence of the UI and the measurment noise on the state estimation error. To minimize their effect, we filter the output reconstruction error $\tilde{y}$ with a Low Pass Filter with a constant time of $(1-\alpha)$:

$$\tilde{y}_{f,k} = \alpha\tilde{y}_k + (1-\alpha)\tilde{y}_{f,k-1} \tag{44}$$

Fig. 1. System inputs

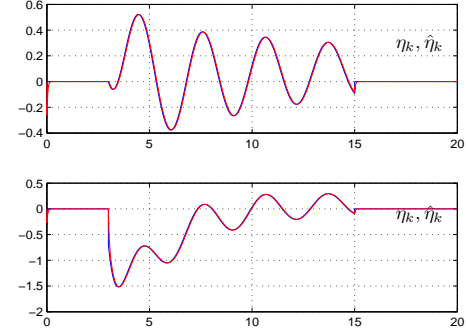Fig. 2. UI and its estimate :exact decoupling (top) $\mathscr{L}_2$-attenuation (bottom)

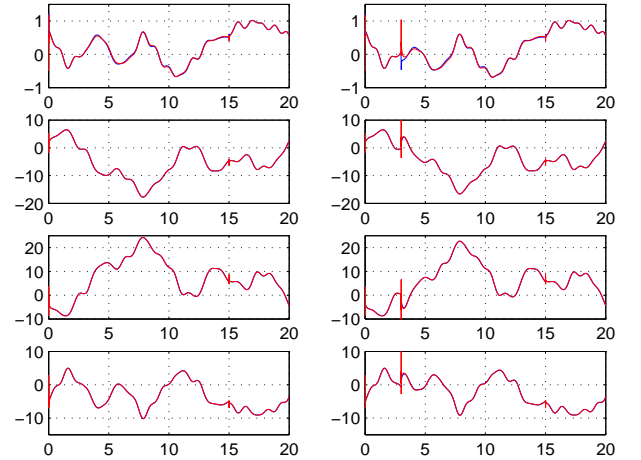

Fig. 3. System states and their estimates : exact decoupling (left) $\mathscr{L}_2$-attenuation (right)

By replacing this equation in the observer equations, we get :

$$\begin{cases} \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + D\hat{\eta}_{k-1} + K\tilde{y}_{f,k} \\ z_{k+1} &= \Gamma\tilde{y}_{f,k} + \lambda z_k \end{cases} \tag{45}$$

The different output and UI estimate (with and without filter) obtained for a weightning matrix of 10% and a coefficient filter $\alpha = 0.25$ are presented in the figures 4 and 5 :
The simulation results of the UI estimation obtained by the proposed observer are displayed on the previous figures. Solving the LMIs (38) may cause slow dynamics of the observer, so an eigenvalue assignment in a D-region allows to increase the performances of
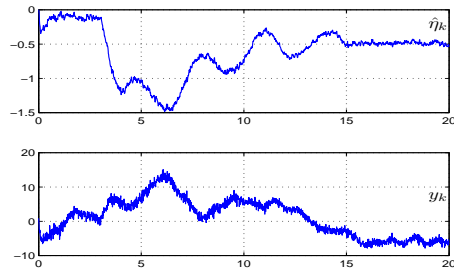
Fig. 4. UI estimate and system output without filter

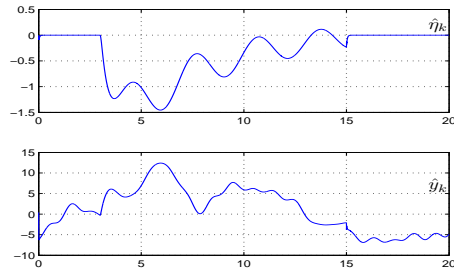Fig. 5. UI estimate and system output with filter

the observer. Also, in case of measurment noise, a filter is added at the output to improve the performances.

VI. CONCLUSION AND PERSPECTIVES

This paper addresses new method to design observers with unknown inputs. The proposed approach is based on a decoupling and estimation procedure such that we decouple the state system towards exponential UI without any rank constraints on the system's matrix. The main result is about the way to find the UI class ensuring an exact decoupling. The proposed work can be extended to the nonlinear case, in particular, systems with Takagi-Sugeno representation.

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