

# Fault diagnosis in Takagi-Sugeno nonlinear systems

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**Abstract:** This paper addresses a new scheme for fault diagnosis in nonlinear systems described by a Takagi-Sugeno multiple models. Two cases are considered, the first one concern the T-S models with known premise variables (the input or the output of the system). The second case suppose that the weighting functions depend on unmeasurable premise variables (state of the system). The approach is based on the design of observer-based residual generator by minimization of the disturbances and maximizing the effects of the faults. The Lyapunov method is used to stability analysis and design of the residual generator. The convergence conditions are given in LMI formulation.

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## 1. INTRODUCTION

Linear models are largely studied and an important literature is devoted to this class of systems. Although they provide solutions for many problems, nonlinear behavior are often present in practical systems then reduces the domain of applicability and performances of tools developed for linear models. Indeed, linear models only represent the behavior of the system around a local operating point. It is known that nonlinear systems are complex and difficult to study, so all the works on the nonlinear systems concerned only specific classes but there is no a general framework like for linear systems. One of the more interesting classes of nonlinear systems is the Takagi-Sugeno (TS) multiple model form which is introduced in Takagi and Sugeno [1985]. It is proved in Tanaka and Wang [2001] that often nonlinear behaviors can be represented exactly or approximated by TS multiple models. The most advantage of these models is the ability to extend the tools designed in the linear system framework. Indeed, many topics of control are extended to TS systems, such as stability and stabilization in Tanaka et al. [1998], Guerra et al. [2006], Chadli et al. [2002], observers and state estimation in Akhenak et al. [2007], Bergsten et al. [2002].

Due to an increasing demand for higher performances, as well as for higher safety and reliability, the model-based approaches to fault diagnosis for dynamic systems have received more attention these last years Patton et al. [1989], Chen and Zhang [1991], Chen et al. [1996], Ding and Frank [1989], Marx et al. [2003]. Concerning the TS fuzzy systems few efforts have been made in fault detection and isolation. Nevertheless we can cite the method based on observers in Akhenak et al. [2007].

In this paper an observer-based approach is developed for robust residual generator and diagnosis which minimizes the sensitivity to the disturbances and maximizes the sensitivity to the faults. Two cases are studied. The first

case concern the T-S systems with measurable premise variables and the second one deals with the systems with unmeasurable premise variables. The paper is organized as follows, section 2 gives some notations and states the problem. Robust residual generation is tackled in section 3. An LMI-based design of the residual generator is proposed.

## 2. PROBLEM STATEMENT

Consider the following continuous-time TS nonlinear system subject to faults and disturbances given by

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \mu_i(\xi(t)) (A_i x(t) + B_i u(t) + E_i d(t) + F_i f(t)) \\ y(t) = \sum_{i=1}^r \mu_i(\xi(t)) (C_i x(t) + D_i u(t) + G_i d(t) + R_i f(t)) \end{cases} \quad (1)$$

where  $A_i \in \mathbb{R}^{n \times n}$ ,  $B_i \in \mathbb{R}^{n \times n_u}$ ,  $C_i \in \mathbb{R}^{n_y \times n}$ ,  $D_i \in \mathbb{R}^{n_y \times n_u}$ ,  $E_i \in \mathbb{R}^{n \times n_d}$ ,  $F_i \in \mathbb{R}^{n \times n_f}$  and  $G_i \in \mathbb{R}^{n_y \times n_d}$ , and  $R_i \in \mathbb{R}^{n_y \times n_f}$ .

The weighing functions  $\mu_i$  are nonlinear and depend on the decision variable  $\xi(t)$  which can be measurable like  $\{u(t), y(t)\}$  or not measurable like the state  $x(t)$  of the system. The weighing functions satisfy the following properties:

$$\begin{cases} 0 \leq \mu_i(\xi(t)) \leq 1 \\ \sum_{i=1}^r \mu_i(\xi(t)) = 1 \end{cases} \quad (2)$$

Thus the structure of the multiple model is simple and is considered as a universal approximator since it can represent any nonlinear behavior according to an adequate number  $r$  of the local models. The multiple model structure provides a mean to generalize the tools developed for linear systems to nonlinear systems due to the properties expressed in (2).

The input signals  $u(t)$ ,  $f(t)$  and  $d(t)$  belong in  $\mathcal{L}_2$  set. The  $\mathcal{L}_2$ -norm of  $u(t) \in \mathcal{L}_2$  is given by  $\|u\|_2 = \sqrt{\int_0^{+\infty} u^T u dt}$ .

In the field of observer design and diagnosis of nonlinear systems using multiple model approach, Patton et al. [1998] proposed an observer-based method to generate residual generator and using an observer bank in order to achieve isolation, an application to DC motor is proposed. In Akhenak et al. [2007], a sliding mode observer for TS systems is proposed to detect and estimate actuator faults. In these works, the authors assumed that the weighting functions depend on measurable premise variables (input or output) of the system. It is clear that the choice of measurable premise variables offers a good simplicity to generalize the methods already developed for linear systems. But in the case where the premise variables are not measurable, the problem becomes very hard. However, this formalism is very important in both the exact representation of the nonlinear behavior by multiple model (see the simulation example) and in diagnosis method based on observer banks to detect and isolate actuator and sensor faults. Indeed in this case, the use of measurable premise variables requires to develop two different multiple models, but using multiple models with unmeasurable premise variables allows to develop only one model of the system behavior to detect and isolate actuator and sensor faults using observer banks. In the literature, a few works are devoted to the case of unmeasurable decision variables, nevertheless, we can cite Bergsten et al. [2002], Palm and Bergsten [2000], where the authors proposed the fuzzy Thau-Luenberger observer which is an extension of the classical Luenberger observer. The main contribution of this paper is to propose a method for fault diagnosis of nonlinear systems described by TS models with measurable and unmeasurable premise variables using the standard  $H_\infty$  framework developed for linear systems.

### 3. RESIDUAL GENERATOR DESIGN

#### 3.1 case 1: measurable premise variables

Let consider the TS nonlinear system subject to disturbances and sensor and actuator faults modeled in (1) An observer-based residual generator is proposed in the following form

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^r \mu_i(\xi) (A_i \hat{x}(t) + B_i u(t) + L_i (y(t) - \hat{y}(t))) \\ \hat{y}(t) = \sum_{i=1}^r \mu_i(\xi) (C_i \hat{x}(t) + D_i u(t)) \\ r(t) = M(y(t) - \hat{y}(t)) \end{cases} \quad (3)$$

where  $\hat{x}(t) \in R^n$  is the estimated state vector and  $r(t) \in R^{n_f}$  is the residual signal. The matrices  $L_i \in R^{n \times n_y}$  and  $M \in R^{n_f \times n_y}$  are the residual generator gains. The objective is to design the gains  $L_i$  and  $M$  in order to minimize the transfer from the disturbances  $w(t)$  and to maximize the transfer of the faults  $f(t)$  to the residual signal  $r(t)$ . Let define the state estimation error  $e(t) = x(t) - \hat{x}(t)$ . Its dynamic is deduced from (1) and (3) as follows

$$\begin{cases} \dot{e}(t) = A_\xi e(t) + E_\xi d(t) + F_\xi f(t) \\ r(t) = C_\xi e(t) + G_\xi d(t) + R_\xi f(t) \end{cases} \quad (4)$$

where

$$\begin{aligned} A_\xi &= \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi) \mu_j(\xi) (A_i - L_i C_k) \\ E_\xi &= \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi) \mu_j(\xi) (E_i - L_i G_k) \\ F_\xi &= \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi) \mu_j(\xi) (F_i - L_i R_k) \\ C_\xi &= \sum_{i=1}^r \mu_i(\xi) M C_i, G_\xi = \sum_{i=1}^r \mu_i(\xi) M G_i \\ R_\xi &= \sum_{i=1}^r \mu_i(\xi) M R_i \end{aligned}$$

For convenience, the system (4) can be written as the following compact form

$$r = G_{rd}d + G_{rf}f \quad (5)$$

where  $G_{rd}$  represents the transfer from the disturbances  $d(t)$  to  $r(t)$  and defined by

$$G_{rd} := \left( \frac{A_\xi}{MC_i} \middle| \frac{E_\xi}{G_\xi} \right) \quad (6)$$

and  $G_{rf}$  is the transfer from  $f(t)$  to  $r(t)$  which is defined by

$$G_{rf} = \left( \frac{A_\xi}{C_\xi} \middle| \frac{F_\xi}{R_\xi} \right) \quad (7)$$

In standard  $H_\infty$  framework (see figure 1.), the maxi-

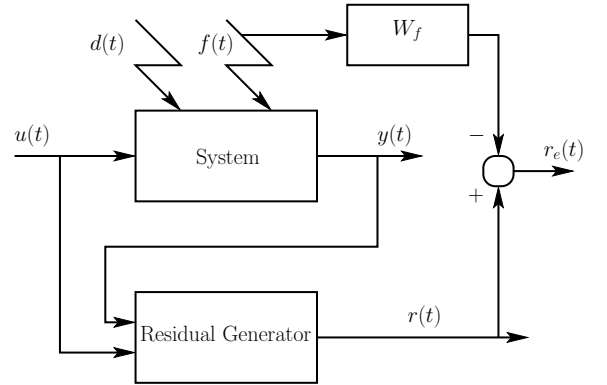


Fig. 1. Scheme of robust residual generation

mization of the effect of the faults  $f(t)$  on the residual  $r(t)$  can be expressed as a minimization problem. Indeed, by introducing a weighting parameter  $W_f$ , the problem is reduced to a minimization of the effect of the faults on the residual error  $r_e(t) = r(t) - W_f f(t)$ . As explained in Stoustrup and Niemann [2000] that the FDI problem depends on the selected structure of the weight parameter  $W_f$ . Indeed, The fault estimation problem is obtained when  $W_f = I$  and the detection problem is considered when  $W_f \in R^{1 \times n_f}$ . In addition,  $W_f$  can be chosen as a dynamic parameter. Consider the parameter  $W_f$  defined

$$W_f = \left( \frac{A_f}{C_f} \middle| \frac{B_f}{D_f} \right) \quad (8)$$

$W_f \in \mathcal{S}$  where  $\mathcal{S}$  is the set of stable filters which have been the following property

$$\|W_f\|_- = \inf_{w \in R} (\sigma(W_f(jw))) \geq 1 \quad (9)$$

(see Mazars et al. [2008] and Mazars et al. [2006] for more details). The interest in this kind of filters is that there is no attenuation of the faults but only an amplification on all frequency ranges which improves the problem of fault detection. The detection, isolation and estimation of the faults can be considered by an appropriate choice of the matrices  $A_f$ ,  $B_f$ ,  $C_f$  and  $D_f$ . The FDI problem is then formulated as the following multi-objective optimization problem

Obtain  $L_i$  and  $M$  which minimize  $a\gamma_f + (1-a)\gamma_d$  where  $a \in [0 \ 1]$  subject to the following constraints

$$\|G_{rf} - W_f\|_\infty < \gamma_f \quad (10)$$

$$\|G_{rd}\|_\infty < \gamma_d \quad (11)$$

$$A_i - L_i C_j \text{ is stable for } i, j = 1, \dots, r \quad (12)$$

The theorem 1 gives an LMI method to solve the optimization problem and provides the residual generator gains  $L_i$  and  $M$ .

**Theorem 1.** Given a positive parameter  $a$  and a weighting function  $W_f$ . The residual generator (3) exists if there exist matrices  $P_1 = P_1^T > 0$ ,  $P_2 = P_2^T > 0$  and gain matrices  $K_i$  and  $M$  and positive scalars  $\bar{\gamma}_f$  and  $\bar{\gamma}_d$  solution of the following optimization problem

$$\min_{L_i, M, P_1, P_2, K_i, \bar{\gamma}_f, \bar{\gamma}_d} a\bar{\gamma}_f + (1-a)\bar{\gamma}_d$$

s.t.

$$\begin{pmatrix} X_{ik}^1 & 0 & P_1 F_i - K_i R_k & C_k^T M^T \\ 0 & X_f^2 & P_2 B_f & -C_f^T \\ F_i^T P_1 - R_k^T K_i^T & B_f^T P_2 & -\bar{\gamma}_f I & R_k^T M^T - D_f^T \\ M C_k & -C_f & M R_k - D_f & -I \end{pmatrix} < 0 \quad (13)$$

$$\begin{pmatrix} X_{ik} & P_1 E_i - K_i G_k & C_k^T M^T \\ E_i^T P_1 - G_k^T K_i^T & -\bar{\gamma}_d I & G_k^T M^T \\ M C_k & M G_k & -I \end{pmatrix} < 0 \quad (14)$$

where

$$X_{ik}^1 = A_i^T P_1 + P_1 A_i - K_i C_k - C_k^T K_i^T \quad (15)$$

$$X_f^2 = A_f^T P_2 + P_2 A_f \quad (16)$$

$$\forall i, k = 1, \dots, r$$

The gains  $L_i$  are derived from

$$L_i = P_1^{-1} K_i \quad i = 1, \dots, r \quad (17)$$

and the attenuation levels are given by

$$\gamma_d = \sqrt{\bar{\gamma}_d} \quad \gamma_f = \sqrt{\bar{\gamma}_f} \quad (18)$$

**Proof.** In faulty case without disturbances the residual generator is reduced to  $r = G_{rf} f$ . In order to maximize the effects of faults on the residual we consider the weighting stable filter  $W_f(s)$  defined in (9). Then the maximization problem can be formulated as a minimization problem by solving (11).  $G_{rf} - W_f$  can be written in the following form

$$G_{rf} - W_f := \left( \begin{array}{c|c} A_\xi & 0 \\ \hline 0 & A_f \\ \hline C_\xi & -C_f \end{array} \middle| \begin{array}{c} F_\xi \\ B_f \\ R_\xi - D_f \end{array} \right) \quad (19)$$

Let define a positive and symmetric bloc diagonal matrix

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \quad (20)$$

Using the bounded real lemma Boyd et al. [1994], the condition (11) is formulated as follows

$$\begin{pmatrix} A_\xi^T P_1 + P_1 A_\xi^T & 0 & P_1 F_\xi & C_\xi^T \\ 0 & A_f^T P_2 + P_2 A_f^T & P_2 B_f & -C_f^T \\ F_\xi^T P_1 & B_f^T P_2 & -\gamma_f^2 I & R_\xi^T - D_f^T \\ C_\xi & -C_f & R_\xi - D_f & -I \end{pmatrix} < 0 \quad (21)$$

Using the definitions of the matrices  $A_\xi$ ,  $F_\xi$ ,  $C_\xi$  and  $R_\xi$  and the convex property of the weighing function, the following matrices are obtained

$$\begin{pmatrix} X_{ik}^1 & 0 & P_1 F_i - P_1 L_i R_k & C_k^T M^T \\ 0 & X_f^2 & P_2 B_f & -C_f^T \\ F_i^T P_1 - R_k^T K_i^T & B_f^T P_2 & -\gamma_f^2 I & R_k^T M^T - D_f^T \\ M C_k & -C_f & M R_k - D_f & -I \end{pmatrix} < 0 \quad (22)$$

where

$$X_{ik}^1 = A_i^T P_1 + P_1 A_i - P_1 L_i C_k - C_k^T L_i^T P_1 \quad (23)$$

$$X_f^2 = A_f^T P_2 + P_2 A_f^T \quad (24)$$

$$i, k = 1, \dots, r$$

In order to obtain the linear matrix inequality (14), we use the change of variables  $K_i = P L_i$  and  $\bar{\gamma}_f = \gamma_f^2$  and  $\bar{\gamma}_d = \gamma_d^2$ .

In fault-free case with disturbances, a similar way, by using the bounded real lemma, allows to obtain the LMI (15). The LMI (15) ensure the stability of the observer (i.e.  $A_i - L_i C_k$  are stable  $\forall i, k = 1, \dots, r$ ) and the robustness against disturbances.

Now, in the faulty case with disturbances, the relative importance of minimizing the effects of the disturbances and maximizing the effects of the faults on the residual signal can be expressed as a minimization of the linear combination  $a\gamma_f + (1-a)\gamma_d$  where  $a \in [0 \ 1]$ .

### 3.2 case 2: unmeasurable premise variables

In this section, it is assumed that the weighting functions of the TS nonlinear system (1) depend on the unmeasurable state  $x(t)$  of the system. The weighting function of the residual generator then depend on the estimated state  $\hat{x}(t)$  as follows

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^r \mu_i(\hat{\xi})(A_i \hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t))) \\ \hat{y}(t) = \sum_{i=1}^r \mu_i(\hat{\xi})(C_i \hat{x}(t) + D_i u(t)) \\ r(t) = M(y(t) - \hat{y}(t)) \end{cases} \quad (25)$$

By adding and subtracting the term

$$\sum_{j=1}^r \mu_j(\hat{\xi}(t)) (A_j x(t) + B_j u(t) + E_j d(t) + F_j f(t))$$

in state equation of the system (1) and the term

$$\sum_{j=1}^r \mu_j(\xi(t)) (C_j x(t) + D_j u(t) + G_j d(t) + R_j f(t))$$

in the output equation of (1) and by some manipulations using convex property of the weighting function the following equivalent system is obtained

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi) \mu_j(\hat{\xi}) (\tilde{A}_{ij} x(t) + \tilde{B}_{ij} u(t) + E_i d(t) + F_i f(t)) \\ y(t) = \sum_{i=1}^r \sum_{j=1}^r \mu_i(\xi) \mu_j(\hat{\xi}) (\tilde{C}_{ij} x(t) + \tilde{D}_{ij} u(t) + G_i d(t) + R_i f(t)) \end{cases} \quad (26)$$

where

$$\begin{aligned} \tilde{A}_{ij} &= A_i + \Delta A_{ij}, \tilde{C}_{ij} = C_i + \Delta C_{ij} \\ \tilde{B}_{ij} &= B_i + \Delta B_{ij}, \tilde{D}_{ij} = D_i + \Delta D_{ij} \end{aligned}$$

and

$$\Delta X_{ij} = X_i - X_j, \quad X_i \in \{A_i, B_i, C_i, D_i\} \\ i, j = 1, \dots, r$$

After calculating the dynamic of the state estimation error, the following is obtained

$$\begin{cases} e(t) = \tilde{A}_{\xi\hat{\xi}} e(t) + \Delta \tilde{A}_{\xi\hat{\xi}} x(t) + \tilde{B}_{\xi\hat{\xi}} \tilde{d}(t) + \tilde{F}_{\xi\hat{\xi}} f(t) \\ r(t) = \tilde{C}_{\xi\hat{\xi}} e(t) + \Delta \tilde{C}_{\xi\hat{\xi}} x(t) + \tilde{G}_{\xi\hat{\xi}} \tilde{d}(t) + \tilde{R}_{\xi\hat{\xi}} f(t) \end{cases} \quad (27)$$

By adopting the following writing

$$\sum_{i=1}^r \sum_{j=1}^r \sum_{k=1}^r \sum_{l=1}^r \mu_i(\xi) \mu_j(\hat{\xi}) \mu_k(\xi) \mu_l(\hat{\xi}) \Leftrightarrow \sum_{i,j,k,l=1}^r \mu_i \hat{\mu}_j \mu_k \hat{\mu}_l$$

The matrices of (28) are defined by

$$\begin{aligned} \tilde{A}_{\xi\hat{\xi}} &= \sum_{i,j,k,l=1}^r \mu_i \hat{\mu}_j \mu_k \hat{\mu}_l (A_j - L_j C_l) \\ \tilde{B}_{\xi\hat{\xi}} &= \sum_{i,j,k,l=1}^r \mu_i \hat{\mu}_j \mu_k \hat{\mu}_l [\Delta B_{ij} - L_j \Delta D_{kl} \quad E_i - L_j G_k] \\ \tilde{F}_{\xi\hat{\xi}} &= \sum_{i,j,k,l=1}^r \mu_i \hat{\mu}_j \mu_k \hat{\mu}_l (F_i - L_j R_k) \\ \tilde{C}_{\xi\hat{\xi}} &= \sum_{i,j,k,l=1}^r \mu_i \hat{\mu}_j \mu_k \hat{\mu}_l M C_l, \\ \tilde{G}_{\xi\hat{\xi}} &= \sum_{i,j,k,l=1}^r \mu_i \hat{\mu}_j \mu_k \hat{\mu}_l [M \Delta D_{kl} \quad M G_k] \\ \tilde{R}_{\xi\hat{\xi}} &= \sum_{i,j,k,l=1}^r \mu_i \hat{\mu}_j \mu_k \hat{\mu}_l M R_k \\ \Delta \tilde{A}_{\xi\hat{\xi}} &= \sum_{i,j,k,l=1}^r \mu_i \hat{\mu}_j \mu_k \hat{\mu}_l (\Delta A_{ij} - L_j \Delta C_{kl}) \\ \Delta \tilde{C}_{\xi\hat{\xi}} &= \sum_{i,j,k,l=1}^r \mu_i \hat{\mu}_j \mu_k \hat{\mu}_l \Delta C_{kl} \\ \tilde{d}(t) &= [u(t)^T \quad d(t)^T]^T \end{aligned}$$

Let define the augmented state vector  $\tilde{x} = [e^T \quad x^T]^T$ . The residual vector  $r$  is then given by the equation

$$r = G_{rd} \tilde{d} + G_{rf} f \quad (28)$$

where

$$G_{rd} = \left( \begin{array}{cc|c} \tilde{A}_{\xi\hat{\xi}} & \Delta \tilde{A}_{\xi\hat{\xi}} & \tilde{B}_{\xi\hat{\xi}} \\ 0 & A_{\xi} & \tilde{B}_{\xi} \\ \hline \tilde{C}_{\xi\hat{\xi}} & C_{\xi} & \tilde{G}_{\xi\hat{\xi}} \end{array} \right) \quad (29)$$

and

$$G_{rf} = \left( \begin{array}{cc|c} \tilde{A}_{\xi\hat{\xi}} & \Delta \tilde{A}_{\xi\hat{\xi}} & \tilde{B}_{\xi\hat{\xi}} \\ 0 & A_{\xi} & \tilde{B}_{\xi} \\ \hline \tilde{C}_{\xi\hat{\xi}} & C_{\xi} & \tilde{G}_{\xi\hat{\xi}} \end{array} \right) \quad (30)$$

$$A_{\xi} = \sum_{i=1}^r \mu_i(\xi) A_i, F_{\xi} = \sum_{i=1}^r \mu_i(\xi) F_i$$

$$\tilde{B}_{\xi} = \sum_{i=1}^r \mu_i(\xi) [B_i \quad E_i]$$

The FDI problem is the same as the problem given in (11)-(13). In order to determine the gains  $L_i$  and  $M$  of the residual generator (26), the theorem 2 gives an LMI solution of the problem (11)-(13) for TS nonlinear systems with unmeasurable premise variables.

**Theorem 2.** Given a positive parameter  $a$  and a weighting function  $W_f$ . The residual generator (3) exists if there exist matrices  $P_1 = P_1^T > 0$ ,  $P_2 = P_2^T > 0$  and gain matrices  $K_i$  and  $M$  and positive scalars  $\bar{\gamma}_1$  and  $\bar{\gamma}_2$  solution of the following optimization problem

$$\min_{L_i, M, P_1, P_2, K_i, \bar{\gamma}_f, \bar{\gamma}_d} a \bar{\gamma}_f + (1-a) \bar{\gamma}_d$$

s.t.

$$\left( \begin{array}{ccccc} X_{jl}^1 & \Xi_{ijkl} & 0 & P_1 F_i - K_j R_k & C_l^T M^T \\ * & X_i^2 & 0 & P_2 F_i & \Delta C_{kl}^T M^T \\ * & * & X_f^3 & P_3 B_f & -C_f^T \\ * & * & * & -\bar{\gamma}_f I & (M R_k - D_f)^T \\ * & * & * & * & -I \end{array} \right) < 0 \quad (31)$$

$$\left( \begin{array}{ccccc} X_{jl}^1 & \Xi_{ijkl} & P_1 \Delta B_{ij} - K_j \Delta D_{ij} & P_1 E_i - K_j G_k & C_l^T M^T \\ * & X_i^2 & P_2 B_i & P_2 E_i & \Delta C_{kl}^T M^T \\ * & * & -\bar{\gamma}_d I & 0 & 0 \\ * & * & * & -\bar{\gamma}_d I & G_k^T M^T \\ * & * & * & * & -I \end{array} \right) < 0 \quad (32)$$

where

$$X_{jl}^1 = A_j^T P_1 + P_1 A_j - K_j C_l - C_l^T K_j^T \quad (33)$$

$$X_i^2 = A_i^T P_2 + P_2 A_i \quad (34)$$

$$X_f^3 = A_f^T P_3 + P_3 A_f \quad (35)$$

$$\Xi_{ijkl} = P_1 \Delta A_{ij} - K_j \Delta C_{kl} \quad (36)$$

$$\forall i, j, k, l = 1, \dots, r$$

The gains  $L_i$  are derived from

$$L_i = P_1^{-1} K_i \quad i = 1, \dots, r \quad (37)$$

and the attenuation levels are given by

$$\gamma_d = \sqrt{\bar{\gamma}_d} \quad \gamma_f = \sqrt{\bar{\gamma}_f} \quad (38)$$

**Proof.** After calculating the augmented system with  $\tilde{x} = [e^T \quad x^T \quad x_f^T]^T$  by including the filter  $W_f$  which has  $x_f$  as a state vector and calculating  $r_e(t) = r(t) - r_f(t)$  where  $r_f(t)$  is the output of the filter  $W_f$  (see figure 1). The proof follows exactly the steps which have been given for the proof of the theorem 1.

**Remark 1.** Note that the theorem 2 is more general than the theorem 1. Indeed, if the weighting functions  $\mu_i$  of the system (1) depend on measurable premise variables, the problem given in the theorem 1 can be deduced from the theorem 2 by taking  $i = j$  and  $k = l$ .

#### 4. ROBUST FAULT DIAGNOSIS

Due to the presence of exogenous disturbances, the residual signals are different from zero even in the fault-free case. In the framework of fault detection, a threshold based on the obtained attenuation levels  $\gamma_f$  and  $\gamma_d$  is generated. An alarm is generated by comparison between the residual signals  $r(t)$  and the threshold. A fixed threshold is determined as follows

$$J_{th} = \gamma_d \rho \quad (39)$$

where  $\rho$  is the bound of  $d(t)$  in the measurable premise variables case and it represent the bound the bound of  $\tilde{d}(t)$  in the unmeasurable premise variables. The decision logic is given by

$$\begin{cases} |r_i(t)| < J_{th} \Rightarrow \text{no fault} \\ |r_i(t)| > J_{th} \Rightarrow \text{fault} \end{cases} \quad (40)$$

#### 5. DISCUSSION AND REMARKS

*Remark 2.* In order to improve the fault detection, residual generator is constructed for each fault separately. Each residual generator is designed to minimize the transfer from  $f_i$  to  $r_{ei} = r_i - W_{fi}f_i$ ,  $i = 1, \dots, n_f$ .

*Remark 3.* In the unmeasurable premise variables case, the system is seen as an uncertain system. The input  $u(t)$  then appear in the dynamic of state estimation error. The method proposed in this paper considers the input  $u(t)$  as a perturbation as  $d(t)$  and by considering the new perturbation vector  $\tilde{d}(t) = [u(t)^T \ d(t)^T]^T$  the problem is solved. It is clear that considering the input  $u(t)$  as a perturbation penalizes the fault detection because the computed threshold depends on the upper bound of  $\tilde{d}(t)$ . Using the method proposed in Casavola et al. [2008] for linear systems with polytopic uncertainties where  $u(t)$  is considered as a perturbation to minimize separately from  $d(t)$ . Indeed, instead of minimizing the index  $(a\bar{\gamma}_f + (1 - a)\bar{\gamma}_d)$  under the LMI constraints, the index which has been used in Casavola et al. [2008] described by  $(a\bar{\gamma}_f + b\bar{\gamma}_d + c\bar{\gamma}_u)$  can be used. An adaptive threshold can be then generated using a time-windowed rms-norm (see Casavola et al. [2008], Frank and Ding [1997]).

*Remark 4.* It is often considered that the fault vector  $f(t)$  has two components, the first one noted  $f_a(t)$  represent the vector of the faults affecting only the actuator, thus, they appear in the state equation. The second component noted  $f_s(t)$  are the vector of the faults affecting only the sensors. The output of the system is given by

$$y(t) = \sum_{i=1}^r \mu_i(\xi) (C_i x(t) + D_i u(t) + G_i d(t) + R_i f(t)) \quad (41)$$

In the case where the faults  $f_a(t)$  do not affect the output of the system, the matrices  $R_i$  are not full rank. As pointed out in (Stoustrup and Niemann [2000]), in this case, where  $W_f = I$  the attenuation level  $\gamma_f \geq 1$  or the problems in theorem 1 and 2 have not solutions. In order to avoid this problem, a perturbation term is added in the output equation as follows

$$y(t) = \sum_{i=1}^r \mu_i(\xi) \left( C_i x + D_i u + G_i d + \begin{bmatrix} \varepsilon_i & R_i^1 \end{bmatrix} \begin{bmatrix} f_a \\ f_s \end{bmatrix} \right) \quad (42)$$

where  $\varepsilon_i$  are the matrices of distribution of the actuator faults  $f_a(t)$  in the output equation and are chosen as small

as possible. In the context of fault isolation, this approach may generate a false alarms. To improve the isolation results, we propose to add and subtract the perturbation term and make the added term to ensure the full rank of  $R_i$  and consider the subtract term as a perturbation to minimize

$$y(t) = \sum_{i=1}^r \mu_i(\xi) \left( C_i x + D_i u + \bar{G}_i \bar{d} + \bar{R}_i \begin{bmatrix} f_a \\ f_s \end{bmatrix} \right) \quad (43)$$

where

$$\bar{G}_i = \begin{bmatrix} G_i & b\varepsilon_i \end{bmatrix}, \bar{R}_i = \begin{bmatrix} \varepsilon_i & R_i^1 \end{bmatrix}, \bar{d} = \begin{bmatrix} d \\ -\frac{f_a}{b} \end{bmatrix}$$

where  $b$  is a positive real parameter. Using this second approach, the threshold  $J_{th}$  is calculated by using the bound of the new perturbation vector  $\bar{d}(t)$ , thus the fault isolation is improved.

#### 6. NUMERICAL EXAMPLE

The proposed algorithm of robust diagnosis is illustrated by an academic example. Let consider the nonlinear system (1) defined by

$$\begin{aligned} A_1 &= \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 1 & -8 \end{bmatrix}, A_2 = \begin{bmatrix} -3 & 2 & -2 \\ 5 & -3 & 0 \\ 1 & 2 & -4 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 1 \\ 5 \\ 0.5 \end{bmatrix}, B_2 = \begin{bmatrix} 3 \\ 1 \\ -7 \end{bmatrix}, E_1 = \begin{bmatrix} 0 & 7 \\ 0 & 5 \\ 0 & 2 \end{bmatrix}, F_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ F_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 6 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

and

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 5 & 0 \\ 1 & 0 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The unknown inputs vector  $d(t)$  is made up of  $d_1(t)$  which affects only the outputs of the system and  $d_2(t)$  affecting only the dynamic of the system (see the matrices  $E_1$ ,  $E_2$  and  $G$ ). The first component of the vector  $f(t)$  is a sensor fault and the second component is an actuator fault.  $W_f$  is chosen to be a diagonal of first order low-pass filters. The minimization of  $\gamma$  results in  $\gamma = 0.1392$ , the obtained residuals are displays on figure 3.

A second simulation is performed in order to estimate the faults.  $W_f$  is then chosen an identity matrix. The original and estimated faults are depicted in figure 4.

#### 7. CONCLUSION

Considering nonlinear systems represented by TS systems, two methods for observer-based residual generator (RG) design are proposed. One is devoted to the systems where the premise variables depend on the measured variables such as the input or the output of the system which are available, the other one concerns the systems which the premise variables depend on the unmeasured state variables. Sufficient conditions for the existence of RG were established in the LMI formalism in order to ease RG design.

#### REFERENCES

A. Akhenak, M. Chadli, J. Ragot, and D. Maquin. Design of sliding mode unknown input observer for uncertain

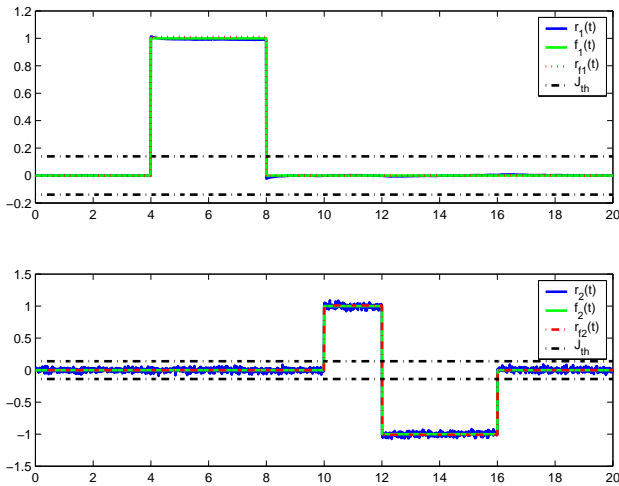


Fig. 2. Faults and corresponding residual signals

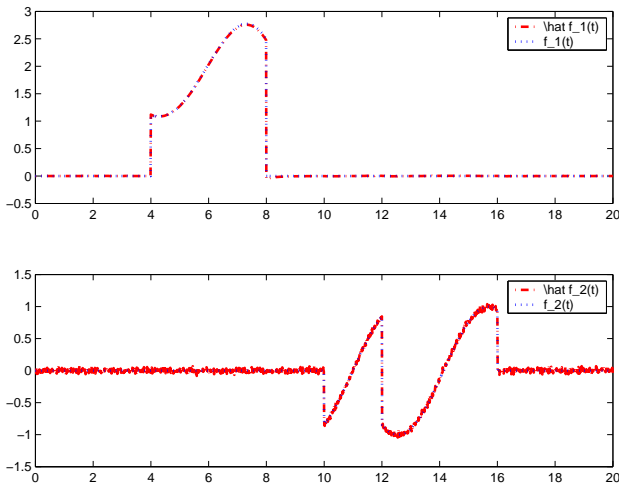


Fig. 3. Comparison of the faults (dashed lines) and residual signals (solid lines)

Takagi-Sugeno model. In *15th Mediterranean Conference on Control and Automation, MED'07*, Athens, Greece, 2007.

- P. Bergsten, R. Palm, and D. Driankov. Observers for Takagi-Sugeno fuzzy systems. *IEEE Transactions on Systems, Man, and Cybernetics - Part B: Cybernetics*, 32(1):114–121, 2002.
- S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. 1994.
- Alessandro Casavola, Domenico Famularo, and Giuseppe Franz. Robust fault detection of uncertain linear systems via quasi-lmis. *Automatica*, 44(1):289–295, January 2008.
- M. Chadli, D. Maquin, and J. Ragot. Non quadratic stability analysis of Takagi-Sugeno systems. In *IEEE Conference on Decision and Control, CDC'2002, Las Vegas, Nevada, USA*, 2002.
- J. Chen and H. Zhang. Robust detection of faulty actuators via unknown input observers. *International Journal of Systems Science*, 22(10):1829–1839, 1991.
- J. Chen, R.J. Patton, and H.Y. Zhang. Design of unknown input observers and robust fault detection filters. *International Journal of Control*, 63(1):85–105, January

1996.

- X. Ding and P.M. Frank. Fault detection via optimally robust detection filters. In *28th IEEE Conference on Decision and Control*, 1989.
- P. M. Frank and X. Ding. Survey of robust residual generation and evaluation methods in observer-based fault detection systems. *Journal of Process Control*, 7(6):403–424, December 1997.
- T.M. Guerra, A. Kruszewski, L. Vermeiren, and H. Tirmant. Conditions of output stabilization for nonlinear models in the Takagi-Sugeno's form. *Fuzzy Sets and Systems*, 157(9):1248–1259, May 2006.
- B. Marx, D. Koenig, and D. Georges. Robust fault diagnosis for descriptor systems-a coprime factorization approach. In *Proceedings of the 6th IFAC Symposium on Fault Detection, Supervision and Safety of Technical Processes*, pages 507–512, 2003.
- E. Mazars, Z. Li, and I.M. Jaimoukha. A QMI approach to the robust fault detection and isolation problem. In *Proceedings of the 6th IFAC Symposium on Fault Detection, Supervision and Safety of Technical Processes*, Beijing, China, 2006.
- E. Mazars, I. Jaimoukha, and L. Zhenhai. Computation of a reference model for robust fault detection and isolation residual generation. *Journal of Control Science and Engineering*, 2008. doi:10.1155/2008/790893.
- R. Palm and P. Bergsten. Sliding mode observers for Takagi-Sugeno fuzzy systems. *Ninth IEEE International Conference on Fuzzy Systems, FUZZ'IEEE*, 2000.
- R. Patton, P. Frank, and R. Clark. *Fault diagnosis in dynamic systems: Theory and application*. Prentice Hall international, 1989.
- R.J. Patton, J. Chen, and C.J. Lopez-Toribio. Fuzzy observers for non-linear dynamic systems fault diagnosis. In *37th IEEE Conference on Decision and Control, Tampa, Florida USA*, 1998.
- J. Stoustrup and H. Niemann. Application of an  $H_\infty$  based FDI and control scheme for the three tank system. In *Proceedings of the 4th IFAC Symposium on Fault Detection, Supervision and Safety for Technical Processes*, Budapest, Hungary, June 2000.
- T. Takagi and M. Sugeno. Fuzzy identification of systems and its applications to modeling and control. *IEEE Transactions on Systems, Man, and Cybernetics*, 15: 116–132, 1985.
- K. Tanaka and H.O. Wang. *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*. Hardcover, 2001.
- K. Tanaka, T. Ikeda, and H.O. Wang. Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs. *IEEE Transactions on Fuzzy Systems*, 6(2):250–265, 1998.