Generalised hold functions for fast sampling rates

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***Abstract*— It is well-known that generalised hold functions can be used to shift the zeros of sampled-data models for continuous-time systems. In this paper we consider the use of generalised holds to deal with sampling zeros only. We propose a hold design that places the sampling zeros asymptotically to the origin, when the sampling period tends to zero. The resulting generalised hold is only a function of the process relative degree. We also investigate the robustness of the procedure with respect to both finite sample periods and unmodelled plant dynamics.**

I. INTRODUCTION

The sampling process is a key element to represent continuous-time systems using discrete-time models, both for estimation and control purposes [1]. The poles of a sampled-data model are known to depend only on the sampling period and the poles of the underlying continuous-time system [2]. However, there is no simple relationship between the zeros of the continuous-time system and the zeros of the discrete-time model. For example, when a Zero Order Hold (ZOH) is used to generate the continuous-time input of a system, the discrete-time model will have *sampling zeros*, which converge to specific locations as the sampling period is reduced [3]. In the latter work it was shown that, for a rational system with relative degree higher than 2, the resulting sampled system is non-minimum phase (NMP) for small sampling intervals. These NMP zeros impose extra limitations on the achievable continuous-time performance and may also lead to poorly conditioned discretised systems [4], [5].

The *sampling zeros* are known to be a function of the hold used to generate the continuous-time input from the discrete sequence [6], [7], and [8]. However, the use of the so-called Generalised Hold Functions (GHF) [9] can give misleading results [10], [11]. For example, essential characteristics of the *continuous-time* system, such as NMP behaviour, can be *artificially removed* in the discrete-time transfer function [6].

For the stochastic modelling case there are also *sampling zeros* involved, which depend on the prefilter used prior to sampling (instantaneously) the system output [12].

In this paper we show how a GHF can be designed to remove only the effects of the sampling process by placing the *sampling zeros* of the discrete-time system asymptotically to the origin. The proposed hold depends **only** on the *relative degree* of the plant model. The robustness to this assumption is also explored.

The proposed sampling strategy avoids the presence of unstable sampling zeros due to the discretisation process. Therefore, it allows one to apply discrete-time control methods, such as adaptive control [13], [14] or Internal Model Control [15], where the presence of NMP zeros would impose additional performance limitations. In [16] fractional-order holds are used with the same purpose for the multivariable case.

II. REVIEW OF ASYMPTOTIC ZEROS FOR A ZERO ORDER HOLD

In this section, we recall the main results presented in [3] for the ZOH case.

Lemma 1: For a given sampling period Δ, the pulse transfer function corresponding to $G(s) = s^{-n}$, is given by:

$$H_{ZOH}(z) = \frac{\Delta^n}{n!}\,\frac{B_n(z)}{(z-1)^n} \qquad (1)$$

where:

$$B_n(z) = b_1^n z^{n-1} + b_2^n z^{n-2} + \ldots + b_n^n \qquad (2)$$

$$b_k^n = \sum_{\ell=1}^{k} (-1)^{k-\ell} \ell^n \binom{n+1}{k-\ell} \qquad (3)$$

Theorem 1: Let $G(s)$ be a rational function:

$$G(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \qquad (4)$$

and $H_{ZOH}(z)$ the corresponding pulse transfer function. Assume that $m < n$. Then as the sampling period Δ goes to 0, m zeros of $H_{ZOH}(z)$ go

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to 1 as $e^{z_i\Delta}$, and the remaining $n-m-1$ zeros of $H_{ZOH}(z)$ go to the zeros of the polynomial $B_{n-m}(z)$ defined in Lemma 1. $\square$

The polynomials defined in (2)–(3) correspond in fact to the Euler-Fröbenius polynomials and are known to satisfy several properties [17], [18].

III. Generalised Hold Functions

The use of GHF allows one to arbitrarily assign the zeros of the sampled model of a continuous-time system. Any GHF can be characterised by its *impulse response* $h_g(t)$, when a unitary discrete-time impulse is used as input, i.e., $u[k]=\delta_K[k]$.

Lemma 2: Let us consider the state-space continuous-time model:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5}$$

$$y(t) = Cx(t) \tag{6}$$

If we use a GHF with impulse response $h_g(t)$ to generate the input $u(t)$, then the equivalent discrete-time model is given by:

$$x_{k+1} = A_q x_k + B_{GHF}\, u_k \tag{7}$$

$$y_k = Cx_k \tag{8}$$

where $A_q = e^{A\Delta}$, and:

$$B_{GHF} = \int_0^\Delta e^{A(\Delta-\tau)}Bh_g(\tau)d\tau \tag{9}$$

Corollary 1: The zeros of the equivalent discrete-time system (7)–(8) are given by the solutions of the equation:

$$C\,\mathrm{adj}(zI - A_q)B_{GHF} = 0 \tag{10}$$

where $\mathrm{adj}(\cdot)$ denotes the adjoint matrix. $\square$

In [6], the case of GHFs defined by piecewise constant impulse response is also considered:

$$h_g(t) = f_N(t) = \begin{cases} g_1 & ; 0 \le t < \frac{\Delta}{N} \\ \vdots & \\ g_N & ; \frac{(N-1)\Delta}{N} \le t < \Delta \end{cases} \tag{11}$$

In this case, replacing in (9), we have:

$$B_{GHF} = \sum_{\ell=1}^{N} g_\ell \int_{\frac{(\ell-1)\Delta}{N}}^{\frac{\ell\Delta}{N}} e^{A(\Delta-\tau)}B\, d\tau \tag{12}$$

Remark 1: In [6] it is shown that (generically) the controllability of the pair (A,B) is enough to arbitrarily assign the roots of (10) by choosing the *weights* $\{g_\ell\}$ in (12).

IV. Asymptotic Zeros for Generalised Holds

We next investigate the asymptotic zeros obtained when a GHF is utilised. First, we extend Lemma 1 to the generalised hold case.

Lemma 3: Consider the n-th order integrator $G(s) = s^{-n}$. When a piecewise constant GHF, with n different subintervals, is used to generate its continuous-time input, then the corresponding discrete-time transfer function is given by:

$$H_{GHF} = \frac{\Delta^n}{n!(z-1)^n}\sum_{p=0}^{n-1} zB_p(z)(z-1)^{n-p-1}C_{n,p} \tag{13}$$

where $B_p(z)$ are defined in (2)–(3), and:

$$C_{n,p} = \binom{n}{p}\left(\tfrac{-1}{n}\right)^{n-p}\sum_{\ell=1}^{n} g_\ell\left[(\ell-1)^{n-p} - \ell^{n-p}\right] \tag{14}$$

Proof: See Appendix. $\square$

Note that (13) can be always rewritten as:

$$H_{GHF}(z) = \frac{\Delta^n}{n!(z-1)^n}\sum_{p=0}^{n-1}\alpha_p z^p \tag{15}$$

In Section V we show how to choose a piecewise constant GHF with n stages to assign arbitrarily the sampling zeros of the n-th order integrator.

Next we extend Theorem 1 to the case of GHF.

Theorem 2: Let $G(s)$ be a rational function as in (4), with relative degree $r = n-m$. Let $H_{GHF}(z)$ be the corresponding sampled transfer function obtained using a piecewise constant GHF with r stages.

Assume $m < n$ (or, equivalently, $r > 0$). Then, as the sampling period Δ goes to 0, m zeros of $H_{GHF}(z)$ go to 1 as $e^{z_i\Delta}$, and the remaining $r-1$ zeros of $H_{GHF}(z)$ (the *sampling zeros*) go to the roots of the polynomial:

$$\sum_{p=0}^{r-1} zB_p(z)(z-1)^{r-p-1}C_{r,p} = \sum_{p=0}^{r-1}\alpha_p z^p \tag{16}$$

Proof: The proof is similar to the proof of Theorem 1 (see [3]). First we note that the discrete-time model can be obtained as:

$$H_{GHF}(z) = \mathcal{Z}\left\{\mathcal{L}^{-1}\{G(s)F(s)\}\big|_{t=k\Delta}\right\} \tag{17}$$

where $G(s)$ and $F(s)$ are the Laplace transforms of the plant and generalised hold impulse responses, respectively. In particular, for a piecewise constant GHF with r subintervals:

$$F(s) = \frac{1}{s}\sum_{\ell=1}^{r} g_\ell\left(e^{-s\frac{(\ell-1)\Delta}{r}} - e^{-s\frac{\ell\Delta}{r}}\right) \tag{18}$$

Equation (17) can be rewritten as:

$$H_{GHF}(z) = \sum_{k=0}^{\infty} \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} G(s)F(s)e^{sk\Delta} ds\, z^{-k}$$
$$= \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} G(s)F(s) \left(\sum_{k=1}^{\infty} e^{sk\Delta} z^{-k} \right) ds$$
$$= \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} G(s)F(s) \frac{e^{s\Delta}}{z - e^{s\Delta}}\, ds \quad (19)$$

where γ is such that $G(s)/s$ has all its poles to the left of $\Re\{s\} = \gamma$. Now, if we change variables $w = s\Delta$ in the integral, and replace $F(s)$ and $G(s)$, we have:

$$\lim_{\Delta\to 0} \Delta^{-r} H_{GHF}(z) = \frac{K}{2\pi j} \times$$
$$\int_{\gamma\Delta-j\infty}^{\gamma\Delta+j\infty} \frac{e^w \sum_{\ell=1}^{r} g_\ell \left(e^{-\frac{(\ell-1)w}{r}} - e^{-\frac{\ell w}{r}} \right)}{w^{r+1}(z - e^w)} dw \quad (20)$$

It is readily shown that this expression corresponds to replace $G(s)$ by an r-th order integrator in (17). Thus:

$$\lim_{\Delta\to 0} \Delta^{-r} H_{GHF}(z) =$$
$$= \frac{K \sum_{p=0}^{r-1} z B_p(z)(z-1)^{r-p-1} C_{r,p}}{r!(z-1)^r}$$
$$= \frac{K(z-1)^m}{r!(z-1)^n} \sum_{p=0}^{r-1} \alpha_p z^p \quad (21)$$

□

V. A Generalised Hold to assign the Asymptotic Sampling Zeros

In this section, we present our main result, namely, that we can design a GHF to asymptotically assign (as Δ goes to 0) the sampling zeros to the origin.

Theorem 3: The coefficients $\{g_\ell\}_{\ell=1,\ldots,r}$ of the GHF in (18) can be chosen in such a way that the sampling zeros of the discrete-time model (16) converge asymptotically to $z = 0$.

Proof: So as to assign the sampling zeros to the origin, it follows from (16) that the following condition must hold:

$$\alpha_p = 0 \quad, \forall p \in \{0, \ldots, r-2\} \quad (22)$$

This is equivalent to having $r - 1$ linear equations on the coefficients $C_{r,p}$, and thus on the weights $\{g_\ell\}$. Moreover, the GHF must satisfy an extra condition to ensure unitary gain at frequency zero, i.e.:

$$\frac{1}{r} \sum_{\ell=1}^{r} g_\ell = 1 \quad (23)$$

Equations (22) and (23) define r conditions on the coefficients $\{g_\ell\}_{\ell=1,\ldots,r}$, which are (generically) linearly independent provided (A, B) is controllable (see Remark 1). □

Remark 2: A key observation in the previous result is that the GHF obtained by solving (22)–(23) depends on the plant relative degree only.

Remark 3: In Theorem 3 we chose to assign the asymptotic sampling zeros to the origin. This implies that, by a continuity argument, there is a finite sampling period $\Delta > 0$ such that they all lie *inside* the unit circle. The choice of this asymptotic location for the zeros will prove also to be robust to possible plant undermodelling (see Section VI).

Example 1: Consider the third order system:

$$G(s) = \frac{1}{(s+1)^3} \quad (24)$$

By Theorem 1, when we use a ZOH to generate its input and as the sampling period Δ tends to zero, the associated sampled-data transfer function is given by:

$$H_{ZOH}(z) \xrightarrow{\Delta\approx 0} \frac{\Delta^3(z+3.732)(z+0.268)}{(z-1)^3} \quad (25)$$

Note that the resulting discrete-time model is NMP, even though the continuous-time system has no finite zeros.

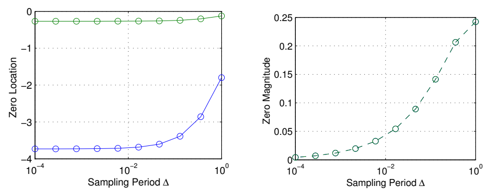
Using (16), (22), and (23), we now design a GHF given by the impulse response:

$$h_g(t) = \begin{cases} 29/2 & ;\ 0 \le t < \frac{\Delta}{3} \\ -17 & ;\ \frac{\Delta}{3} \le t < \frac{2\Delta}{3} \\ 11/2 & ;\ \frac{2\Delta}{3} \le t < \Delta \end{cases} \quad (26)$$

Note that this assigns the limiting zeros asymptotically to the origin, i.e., the combined hold and plant discrete-time model is, as Δ goes to 0:

$$H_{GHF}(z) \xrightarrow{\Delta\approx 0} \frac{\Delta^3 z^2}{(z-1)^3} \quad (27)$$

Figure 1(a) shows the sampling zeros as a function of Δ for the ZOH-case, whereas Figure 1(b) shows the (complex) sampling zeros magnitude when the GHF (26) is used. We see that the zeros are very close to their asymptotic values if the usual *rule of thumb* is employed, i.e., sampling 10 times faster than the fastest system pole.



(a) Zero Order Hold (b) Generalised Hold

Fig. 1. Sampling zeros versus sampling period in Example 1.

Furthermore, Figure 2 shows the zero and pole locations for the sampled version of the system using the fixed GHF (26), for sampling periods from 1 to 10^{-4}. All the resulting discrete-time models are minimum phase.

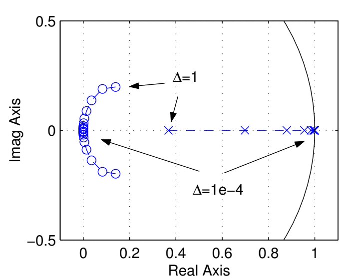


Fig. 2. Zero ('o') and pole ('x') locations for different sampling periods Δ for GHF in Example 1

Example 2: In this example we compare control loop performances for the system in Example 1, using a ZOH and the GHF in (26). We use Youla-parameterisation [15], as is schematically shown in Figure 3. $Q(z)$ is designed as an *approximate inverse* of the plant model. At this point one needs to be careful, in particular, avoiding NMP zero cancellations.

We consider a sampling period $\Delta = 0.1$. From (9), the discrete-time poles are $e^{p_i \Delta} = e^{-0.1} = 0.9048$. The (sampled) zeros depend on the hold used to generate the continuous-time input. Equations (25) and (27) define the zeros for ZOH and GHF cases, respectively. Based on this, we compare the performance of two control loops, one using ZOH and the controller:

$$Q_{ZOH}(z) = \frac{(1+0.268)(z-0.9048)^3}{(1-0.9048)^3(z+0.268)z^2} \qquad (28)$$

and the other one using the GHF defined by (26) and the controller:

$$Q_{GHF}(z) = \frac{(z-0.9048)^3}{(1-0.9048)^3 z^3} \qquad (29)$$

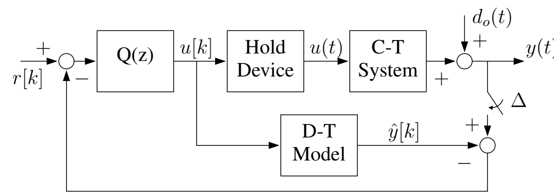


Fig. 3. Control loop using Youla parameterisation (Example 2).

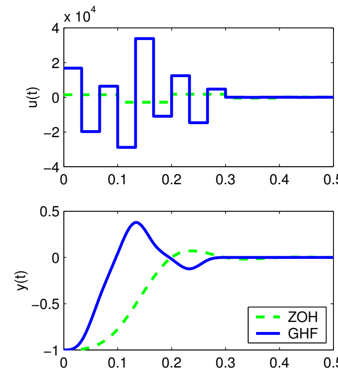
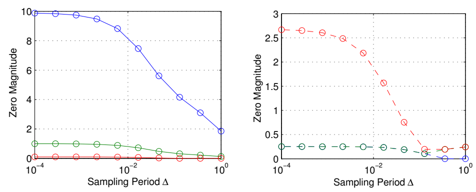


Fig. 4. Simulation results for Example 2.

Figure 4 shows the control signal $u(t)$ and system output $y(t)$ obtained for a unitary step output disturbance, $d_o(t) = -\mu(t)$. The magnitude of the control signal $u(t)$ is high because the controller we have chosen tries to achieve perfect output disturbance rejection. We can see that, even though the response using the GHF gives overshoot, the settling time is smaller than for the ZOH case. Indeed, the discrete *nominal* transfer function of the control loop for the GHF case is simply z^{-1}, whereas for the ZOH case it is $0.211z^{-1} + 0.789z^{-2}$. These discrete responses are evident in Figure 4 at the sampling instants. If we consider the intersample response, this is also improved by the use of the GHF. The integral of the output $y(t)$ squared is reduced from 0.1088 to 0.0557, when the ZOH and $Q_{ZOH}(z)$ are replaced by the GHF (26) and $Q_{GHF}(z)$.

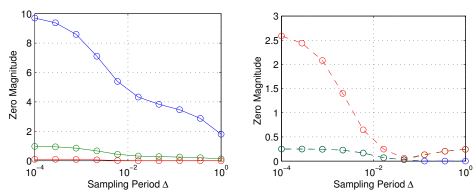
VI. Robustness Issues

The concept of relative degree is not robustly defined for continuous-time systems, since it can be affected, for example, by high frequency unmodelled poles. Thus, to use the GHF described above, one needs to define a *bandwidth of validity* for the model relative degree. If the sampling period is chosen to be fast relative to the nominal poles (say 10 times, as for the usual *rule of thumb*) but slow relative to possible unmodelled poles (say 10 times) then one can heuristically expect the



(a) Zero Order Hold (b) Generalised Hold

Fig. 5. Zero magnitudes for Example 3: fast pole at $s = -10^2$.



(a) Zero Order Hold (b) Generalised Hold

Fig. 6. Zero magnitudes for Example 3: fast pole at $s = -10^3$.

GHF design to perform as roughly as expected. On the other hand, if very fast sampling rates are used, then one can expect the relative degree to be *ill-defined*, and the GHF design may then fail to perform as desired. These ideas are illustrated and confirmed by the following example.

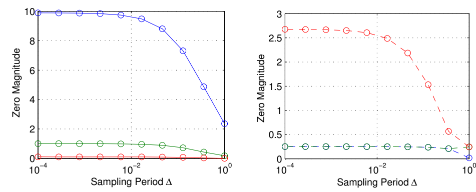
Example 3: Let us consider the system in Example 1, but now with an *unmodelled* fast pole:

$$G(s) = \frac{1}{(s+1)^3(0.01s+1)} \tag{30}$$

For the ZOH-case, Theorem 1 predicts that the asymptotic sampling zeros are $\{-3.732, -0.268\}$ for the *nominal model* of relative degree 3, and $\{-9.899, -1, -0.101\}$ for the *true model* of relative degree 4. We can see in Figure 5(a) that, as Δ decreases, the zeros first approach those corresponding to the *nominal* model, but then move to those corresponding the *true* model. For this case, we see that model (25) is basically reached for a sampling period $\Delta \approx 0.2$ but is not valid for $\Delta < 0.1$.

Similarly, we can see in Figure 5(b) that the zeros obtained with the fixed GHF (26) are *close* to the origin for $\Delta > 0.1$. However, when the sampling period is reduced further the unmodelled pole at $s = -10^2$ in (30) becomes significant and the zeros clearly depart from the desired values.

In Figure 6, we can see even more clearly the *bandwidth of validity* for the nominal model when



(a) Zero Order Hold (b) Generalised Hold

Fig. 7. Zero magnitudes for Example 3: fast pole at $s = -10$.

the unmodelled fast pole is assumed to be at $s = -10^3$. On the other hand, the plots in Figure 7 correspond to an unmodelled pole at $s = -10$, where we can see that the GHF (26) is not able to assign the zeros near the origin, but because the model (24) used for design poorly describes the true system. □

The example confirms the heuristic notion that the *system relative degree* and our GHF design procedure, should be considered in terms of a *bandwidth of validity* for the nominal model.

VII. Conclusions

In this paper we have shown that a generalised hold can be designed to shift the *sampling zeros* of a sampled-data system to the origin. Moreover the required hold is a function **only** on the plant relative degree. Robustness to both finite sample periods and unmodelled dynamics has also been investigated.

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APPENDIX
PROOF OF LEMMA 3

We first need the following result:

Lemma 4: Using the polynomials defined in (2)–(3), and defining $B_0(z) = z^{-1}$, we have:

$$\sum_{k=1}^{\infty} k^p z^{-k} = \frac{zB_p(z)}{(z-1)^{p+1}} \quad ; \forall p \geq 0 \tag{31}$$

Proof: For $p = 0$, the result follows straightforward, and for $p = 1$, we have that:

$$\sum_{k=1}^{\infty} kz^{-k} = \mathcal{Z}\left\{k\right\} = -z\tfrac{d}{dz}\mathcal{Z}\left\{1\right\} = \tfrac{zB_1(z)}{(z-1)^2} \tag{32}$$

Now, assuming that (31) holds for p, we prove the result for $p+1$:

$$\begin{aligned}\sum_{k=1}^{\infty} k^{p+1} z^{-k} &= \mathcal{Z}\left\{k^{p+1}\right\} = -z\tfrac{d}{dz}\mathcal{Z}\left\{k^p\right\} \\ &= \tfrac{z\left[z(1-z)B_p{}'(z)+(pz+1)B_p(z)\right]}{(z-1)^{p+2}}\end{aligned} \tag{33}$$

Now, the result follows from the recursion satisfied by the polynomials $B_p(z)$ [18]:

$$z(1-z)B_p{}'(z) + (pz+1)B_p(z) = B_{p+1}(z) \tag{34}$$

for all $p \geq 0$. □

Proof of Lemma 3: The proof follows from (17), where we obtain the $\mathcal{Z}$-transform of the sampled output of the n-th order integrator, when the input is equal to the impulse response of the GHF (11). The latter is a piecewise constant function defined in n subintervals. The corresponding impulse response and its Laplace transform are given by:

$$f_n(t) = \sum_{\ell=1}^{n} g_\ell \left[\mu\left(t - \tfrac{(\ell-1)\Delta}{n}\right) - \mu\left(t - \tfrac{\ell\Delta}{n}\right)\right] \tag{35}$$

$$F_n(s) = \sum_{\ell=1}^{n} g_\ell F_{n\ell}(s) \tag{36}$$

where $\mu(\cdot)$ is the unitary step function, and:

$$F_{n\ell}(s) = \frac{1}{s}\left(e^{-s\frac{(\ell-1)\Delta}{n}} - e^{-s\frac{\ell\Delta}{n}}\right) \tag{37}$$

We are interested in the impulse response of the combined continuous-time model:

$$H(s) = G(s)F_n(s) = \sum_{\ell=1}^{n} g_\ell H_\ell(s) \tag{38}$$

where we can compute the inverse Laplace transform of each element in the sum:

$$H_\ell(s) = G(s)F_{n\ell}(s) = \frac{e^{-s\frac{(\ell-1)\Delta}{n}} - e^{-s\frac{\ell\Delta}{n}}}{s^{n+1}} \tag{39}$$

$$\begin{aligned}h_\ell(t) = &\tfrac{(t-\frac{\ell-1}{n}\Delta)^n}{n!}\mu\left(t - \tfrac{(\ell-1)\Delta}{n}\right) \\ &- \tfrac{(t-\frac{\ell}{n}\Delta)^n}{n!}\mu\left(t - \tfrac{\ell\Delta}{n}\right)\end{aligned} \tag{40}$$

If we consider this signal at the sampling instants $h_\ell[k] = h_\ell(k\Delta)$, we have that $h_\ell[0] = 0$, and for $k \geq 1$ we can use the binomial theorem to obtain:

$$\begin{aligned}h_\ell[k] = \frac{\Delta^n}{n!}\sum_{p=0}^{n-1} k^p \tbinom{n}{p}\left(\tfrac{-1}{n}\right)^{n-p} \\ \times\left[(\ell-1)^{n-p} - \ell^{n-p}\right]\end{aligned} \tag{41}$$

The $\mathcal{Z}$-transform of this signal is then:

$$\begin{aligned}H_\ell(z) = \frac{\Delta^n}{n!}\sum_{p=0}^{n-1}\Bigg[\tbinom{n}{p}\left(\tfrac{-1}{n}\right)^{n-p} \\ \times\left[(\ell-1)^{n-p} - \ell^{n-p}\right]\sum_{k=1}^{\infty} k^p z^{-k}\Bigg]\end{aligned} \tag{42}$$

where, applying the result in Lemma 4, we have:

$$\begin{aligned}H_\ell(z) = \frac{\Delta^n}{n!}\sum_{p=0}^{n-1}\Bigg[\tbinom{n}{p}\left(\tfrac{-1}{n}\right)^{n-p} \\ \times\left[(\ell-1)^{n-p} - \ell^{n-p}\right]\tfrac{z\,B_p(z)}{(z-1)^{p+1}}\Bigg]\end{aligned} \tag{43}$$

Finally, the result is obtained by replacing (43) into the linear combination in (38). □