4A I2S & M3

Digital Signal Processing

Final exam 2019/2020 - 1h45

Name & firstname: ....

Diploma (I2S/M3): ....

Instructions:

1. Do not forget to write your name above and include all these pages with your copy.
2. You can answer in French or in English.
3. The only material you can consult is your personal A4 recto-verso sheet.
4. You may use a hand calculator but with no communication capabilities.
5. The exercises must be solved in the given order.
6. Good luck !

Multiple choice questions *(there is one correct answer from the choices only)*

Why is the DFT (or its fast FFT version) used extensively in the field of signal processing?

☐ The DFT is the only Fourier transform that can be evaluated exactly with a computer

☐ The DFT represents continuous-time periodic signals

☐ The DFT has more terms than the DTFT

☐ Three letter acronyms are always preferable

If an N point DFT is taken of a discrete-time signal $x(k)$, how many values are computed in the frequency domain representation $X(m)$?

☐ $N/2$

☐ $N \log_2(N)$

☐ N

☐ 1024

What filter would you recommend to cancel out abnormal spikes or outliers in a data set?

☐ an IIR filter

☐ a median filter

☐ a FIR filter

☐ a Butterworth filter

1 Difference equation of an elliptic digital filter

An elliptic filter, also known as the Chebyshev-Cauer filter, allows a sharper cut-off in the transition-band by allowing ripples in both the pass-band and in the stop-band. A digital elliptic low-pass digital filter has been designed in MATLAB using the ellip() function, giving the transfer function

$$H(z) = \frac{0.02z^2 + 0.01z + 0.02}{z^2 - 1.70z + 0.78}$$

1.1 Write the recursive difference equation you would use to implement the filter.

1.2 Is this filter causal. Justify your answer.

2 Bilinear transform of an analog integrator

An analog integrator has a transfer function $H(s) = \frac{1}{s}$.

2.1 Use the bilinear transform to find the discrete time transfer function $H(z)$.

2.2 Give the difference equation of the digital version of the analog integrator.

3 Analysis of a digital all-pass filter

A linear time-invariant filter is used to process sampled data (with sampling frequency f_s). The filter is described by the constant coefficient difference equation:

$$y(k) = -0.5y(k-1) + 0.5x(k) + x(k-1)$$

3.1 State whether the filter is FIR or IIR. Justify your answer.

3.2 Show that the transfer function $H(z)$ of the filter is given by

$$H(z) = \frac{0.5 + z^{-1}}{1 + 0.5z^{-1}}$$

3.3 Determine and plot the poles and zeros of $H(z)$ in the z-plane.

3.4 Is the filter stable ? Justify your answer.

3.5 From $H(z)$, give the filter frequency response function $H(f)$.

3.6 By using

$$e^{-j2\pi fT_s} = \cos(2\pi fT_s) - j\sin(2\pi fT_s)$$

Show that the filter is a so-called "all-pass" filter, that is

$$|H(f)| = 1 \quad \text{for all } f$$

4 DTFT and DFT

Consider a discrete-time signal $x(k)$ whose z-transform is:

$$X(z) = 1 + z^{-1} + z^{-3} + z^{-4}$$

4.1 Determine and plot $x(k)$. Express it as a sum of delayed impulses of Kronecker.

4.2 Compute the DTFT $X(f)$ of $x(k)$.

4.3 Consider now the finite-length signal $y(k)$ defined as

$$y(k) = x(k), k = 0, 1, 2, 3$$

Compute its four DFT coefficients $Y(m)$, for $m = 0, 1, 2, 3$.

5 Sampling frequency choice and perfect reconstruction

Consider a pure co-sine signal

$$s(t) = A\cos(2\pi f_0 t)$$

where $A = 2,\ f_0 = 200$ Hz. The signal is sampled at a sampling frequency f_s. We study the effect of two different choices for f_s on the continuous-time reconstructed signal in the case of the perfect interpolation.

5.1 What continuous-time signal $s_1(t)$ would be obtained by assuming a perfect reconstruction *(that is when an ideal low-pass filter is used with a cut-off frequency of* $\frac{f_s}{2}$*)* if $f_{s1} = 500$ Hz. Use a graphical spectral analysis to justify your answer.

5.2 What continuous-time signal $s_2(t)$ would be obtained by assuming a perfect reconstruction *(that is when an ideal low-pass filter is used with a cut-off frequency of* $\frac{f_s}{2}$*)* if $f_{s2} = 300$ Hz. Use a graphical spectral analysis to justify your answer.

5.3 Based on your analysis, which of the two sampling frequencies would you recommend ?

6 Stochastic process

Consider the discrete-time stationary stochastic process $\mathbf{y}(k)$ that is generated from:

$$\mathbf{y}(k) = e(k) + c_1\mathbf{e}(k-1) + c_2\mathbf{e}(k-2) \tag{1}$$

where $\mathbf{e}(k)$ is a white noise of variance $\sigma_{\mathbf{e}}^2$ while c_1 and c_2 are real scalars.

6.1 Determine $E\big(\mathbf{y}(k)\big)$.

6.2 Determine $R_{\mathbf{y}}(\tau)$ for every value of τ.

6.3 Determine the power spectrum $\Phi_{\mathbf{y}}(f)$ of $\mathbf{y}(k)$ obtained at the output of the filter.

7 AR model estimation

Consider the discrete-time stationary stochastic process $\mathbf{y}(k)$ that is generated from:

$$\mathbf{y}(k) - 0.6\mathbf{y}(k-1) = 3\mathbf{v}(k) - 1.6\mathbf{v}(k-1)$$

where $\mathbf{v}(k)$ is a white noise of unit variance.

7.1 Write the stochastic process in polynomial form as

$$\mathbf{y}(k) = \frac{C(q^{-1})}{D(q^{-1})}\mathbf{v}(k)$$

where q^{-1} is the shift operator. Give the usual name for this stochastic process (AR, MA or ARMA), its order and its block-diagram representation.

7.2 Based on N measurements $y(1), y(2), \dots, y(N)$ generated by the model above, we want to find the best AR(1) model, i.e., a model on the form

$$y(k) + ay(k-1) = e(k),$$

where $e(k)$ is a white noise. The cost function $V(a, Z^N)$ to be minimized to estimate the model parameter a is chosen here to be the sum of the squares of the prediction error. Recall the expression of the cost function and the condition to determine its minimum.

7.3 Show that the analytical solution of the least squares (LS) estimates for a is given by

$$\hat{a} = -\Big(\sum_{k=1}^{N} y^2(k-1)\Big)^{-1}\sum_{k=1}^{N}\big(y(k-1)y(k)\big)$$

Appendices

Continuous-time Fourier transform (CTFT)

The CTFT is defined as:

$$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt$$

The inverse CTFT is defined as:

$$x(t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}df$$

Some useful properties of the Fourier transform

$$\begin{aligned}
F\left(ax(t)+by(t)\right) =& aX(f)+bY(f)\\
F\left(x(t-t_0)\right) =& e^{-j2\pi ft_0}X(f)\\
F\left(e^{j2\pi f_0 t}x(t)\right) =& X(f-f_0)\\
F\left(x(at)\right) =& \frac{1}{|a|}X\left(\frac{f}{a}\right)\\
F\left(x(t)*y(t)\right) =& X(f)\times Y(f)\\
F\left(x(t)\times y(t)\right) =& X(f)*Y(f)
\end{aligned}$$

Discrete-time Fourier transform (DTFT)

The DTFT is defined as:

$$X(f) = \sum_{k=-\infty}^{+\infty} x(k)e^{-j2\pi fkT_s}$$

The inverse DTFT is defined as:

$$x(k) = \int_{-f_s/2}^{+f_s/2} X(f)e^{j2\pi fkT_s}df$$

Discrete Fourier Transform (DFT)

The DFT is defined as (*set $N = K_0$ if the discrete-time signal is periodic of period K_0*):

$$X(m) = \sum_{k=0}^{N-1} x(k)e^{-j2\pi\frac{km}{N}} \text{ for } m = 0, 1, \ldots, N-1.$$

The inverse DFT is defined as:

$$x(k) = \frac{1}{N}\sum_{m=0}^{N-1} X(m)e^{j2\pi\frac{km}{N}} \text{ for } k = 0, 1, \ldots, N-1.$$

Table of common unilateral z-transforms

The table below provides a number of z-transforms and their region of convergence (ROC).

Signal	z-transform	ROC
$\delta(k)$	1	All z
$\delta(k-i)$	z^{-i}	$z \neq 0$
$u(k)$	$\frac{z}{z-1}$	$\|z\| > 1$
$ku(k)$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
$k^2u(k)$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
$a^ku(k)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
$ka^ku(k)$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
$k^2a^ku(k)$	$\frac{az(z+a)}{(z-a)^3}$	$\|z\| > \|a\|$
$\cos(\omega_0 k)u(k)$	$\frac{z(z-\cos(\omega_0))}{z^2-2\cos(\omega_0)z+1}$	$\|z\| > \|1\|$
$\sin(\omega_0 k)u(k)$	$\frac{\sin(\omega_0)z}{z^2-2\cos(\omega_0)z+1}$	$\|z\| > \|1\|$

Useful properties of the unilateral z-transform

Some useful properties which have found practical use in signal processing are summarized below.

Property	signal	z-transform
linearity	$ax(k) + by(k)$	$aX(z) + bY(z)$
delays (or shifts)	$x(k-1)$	$z^{-1}X(z)$
	$x(k-i)$	$z^{-i}X(z)$
convolution	$y(k) = h(k) * u(k)$	$Y(z) = H(z)U(z)$
	$y(k) = \sum_{i=-\infty}^{+\infty} h(i)u(k-i)$	
differentiation	$kx(k)$	$-z\frac{dX(z)}{dz}$
accumulation	$\sum_{k=0}^{n} x(k)$	$\frac{1}{1-z^{-1}}X(z)$
initial value theorem	if $x(k) = 0$ for $k < 0$	$x(0) = \lim_{z\to+\infty} X(z)$
final value theorem	$\lim_{k\to+\infty} x(k) = \lim_{z\to 1}(z-1)X(z)$	if the limit exists