TD n° 6.

Positional control of a DC servo-motor.

1) DC servo-motor modelling.

$$G(s) = \frac{\Theta(s)}{U(s)} = \frac{K}{s(1+Ts)} = \frac{\overset{\text{rad}}{\Theta(s)}}{\underset{\text{rad/s}}{\Omega(s)}} \times \frac{\overset{\text{rad/s}}{\Omega(s)}}{\underset{V}{U(s)}} = \frac{1}{s} \times \frac{\overset{\text{rad/(V.s)}}{K}}{1+Ts}$$

$\theta(t) = \mathcal{L}^{-1}(\Theta(s))$ is the angular position of the motor shaft (in rad)

$u(t) = \mathcal{L}^{-1}(U(s))$ is the applied motor voltage (in V)

K : motor gain in rad/(V.s)

T : motor time-constant in second

1-a) DC servo-motor input : $u(t)$ applied motor voltage (in V)

DC servo-motor output : $\theta(t)$ angular position of the motor shaft (in rad).

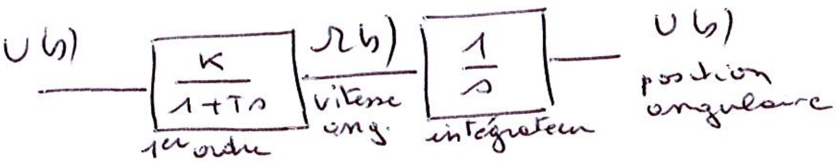
1.b) $\omega(t) = \dfrac{d\theta(t)}{dt}$

$\mathcal{L}$

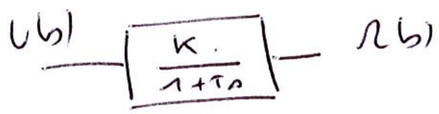
$\Omega(s) = s\,\Theta(s)$ or $\Theta(s) = \dfrac{1}{s}\Omega(s) \Leftrightarrow \dfrac{\Theta(s)}{\Omega(s)} = \dfrac{1}{s}$.

$$\frac{\Theta(s)}{U(s)} = \frac{\Theta(s)}{\Omega(s)} \times \frac{\Omega(s)}{U(s)} = \frac{1}{s} \times \frac{K}{1+Ts}.$$

$$\Theta(s) = \frac{K}{1+Ts} \times \frac{1}{s}\, U(s)$$



1-c)



From the given step response, we can compute

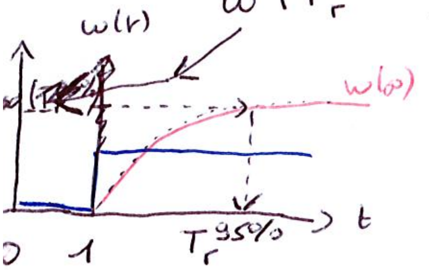
$$K = \frac{\Delta\omega(t)}{\Delta u(t)} = \frac{46-0}{2-0} = 23 \text{ rad/(V.s)}$$

$$\cancel{\omega(T_r^{63\%}) = 63\% \times 46 = 29 \text{ rad/s}}.$$

$$\cancel{T = T_r^{63\%} = 0{,}15}$$

$$\omega(T_r^{95\%}) = 95\% \times \omega(\infty)$$

$$\omega(T_r^{95\%}) = 0,95 \times 46 \simeq 43,7$$



$$T_r^{95\%} = 1,6 - 1 = 0,6\ s.$$

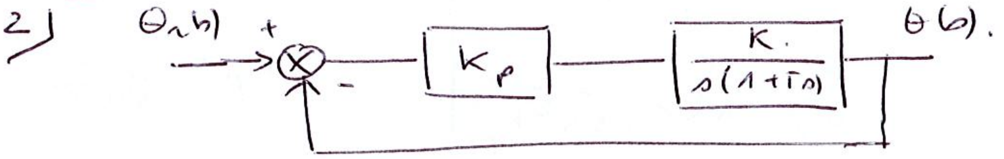
$\hat{L}$ instant when the step is sent to the motor

We know that for a first-order model

settling-time at 5%: $T_r^{95\%} \simeq 3T$ and so $\boxed{T = \dfrac{T_r^{95\%}}{3} = 0,2\ s}$

$\boxed{\text{P- Control.}}$

2)



2-a) $F_{OL}(s) = \dfrac{KK_p}{s(1+Ts)}$.

2-b) $F_{CL}(s) = \dfrac{KK_p}{s(1+Ts) + KK_p} = \dfrac{KK_p}{Ts^2 + s + KK_p}$

2-c) Use of Routh-Hurwitz criterion is easy here.

a) $K>0$ and $T>0$ $\Rightarrow K_p > 0$ (Necessary condition).

b) Routh table

s^2	T	KK_p.
s^1	1	0
s^0	KK_p	

The closed-loop P control is stable if and only if $K_p > 0$.

2-d) $\lim\limits_{t\to+\infty} \varepsilon(t) = \lim\limits_{s\to 0} s\,\mathcal{E}(s)$

$= \lim\limits_{s\to 0} s\,\theta_r(s)\left[1 - F_{CL}(s)\right]$ (unity feedback)

$= \lim\limits_{s\to 0} s \times \dfrac{A}{s}\left[1 - \dfrac{KK_p}{Ts^2+s+KK_p}\right]$

for $K_p>0$ $\lim\limits_{t\to+\infty} \varepsilon(t) = 0$ when $\theta_r(t) = A\,\Gamma(t)$. The closed-loop P control is accurate (or precise) to a step setpoint

(~~tournez svp~~)

(3)

2-e) Let us determine K_p such that $\zeta = \frac{\sqrt{2}}{2}$ for $F_{CL}(s)$

$$F_{CL}(s) = \frac{1}{\frac{s^2}{\omega_0^2} + \frac{2\zeta}{\omega_0}s + 1} = \frac{1}{\frac{T}{KK_p}s^2 + \frac{1}{KK_p}s + 1}$$

By direct identification of the denominator coefficients

$$\left.\begin{array}{l} \zeta = \frac{\sqrt{2}}{2} \\ \frac{2\zeta}{\omega_0} = \frac{1}{KK_p} \\ \frac{1}{\omega_0^2} = \frac{T}{KK_p} \end{array}\right\} \quad \left.\begin{array}{l} \zeta = \frac{\sqrt{2}}{2} \\ \frac{1}{\omega_0} = \frac{1}{\sqrt{2}KK_p} \\ \frac{1}{\omega_0^2} = \frac{T}{\cancel{K}K_p} = \frac{1}{2K^{\cancel{2}}K_p^{\cancel{2}}} \end{array}\right\} \quad \left.\begin{array}{l} \zeta = \frac{\sqrt{2}}{2} \\ \omega_0 = \sqrt{2}KK_p \\ K_p = \frac{1}{2KT} \end{array}\right\}$$

$$\left\{\begin{array}{l} \zeta = \frac{\sqrt{2}}{2} \\ K_p = \frac{1}{2 \times 23 \times 0,2} = 0,1087 \\ \omega_0 = \sqrt{2} \times 23 \times 0,1087 = 3,5 \text{ rad/s.} \end{array}\right.$$

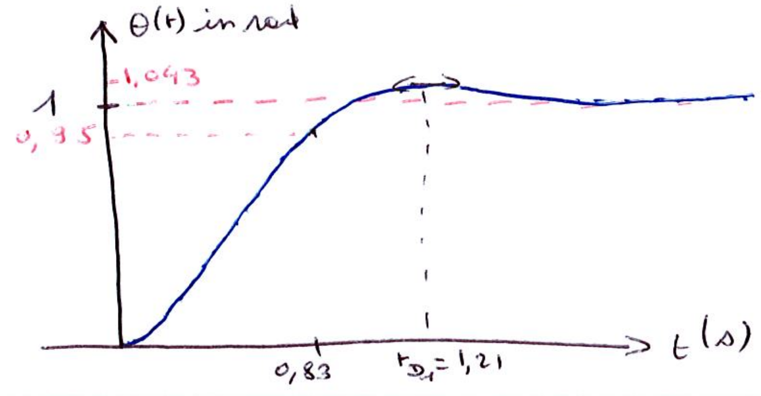
2.f) $D_{1\%} = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100 = 4,3\ \%$

$t_{D_1} = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}} = 1,27\ s.$

To determine the settling-time at 5% we need to use the abacus given in the appendix which plots $\omega_0 \times T_r^{5\%} = f(\zeta)$

From $\zeta = 0,707$, we read on the y-axis $\omega_0 T_r^{5\%} = 3$

and so $T_r^{5\%} = \frac{3}{\omega_0} = \frac{3}{3,5} = 0,83\ s.$



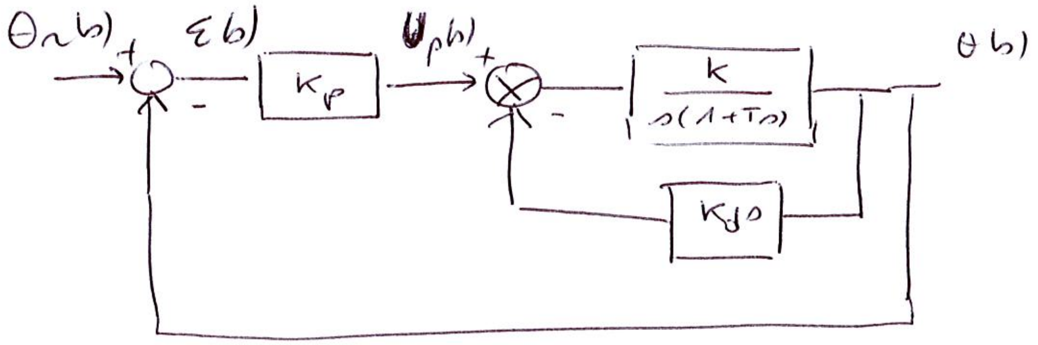
Performance requirements are satisfied

$D_{1\%} = 4,3\% < 5\%$

$T_r^{5\%} = 0,83\ s$ quite small

Best we can do with this P-control.

3) Servo-motor control using PD.

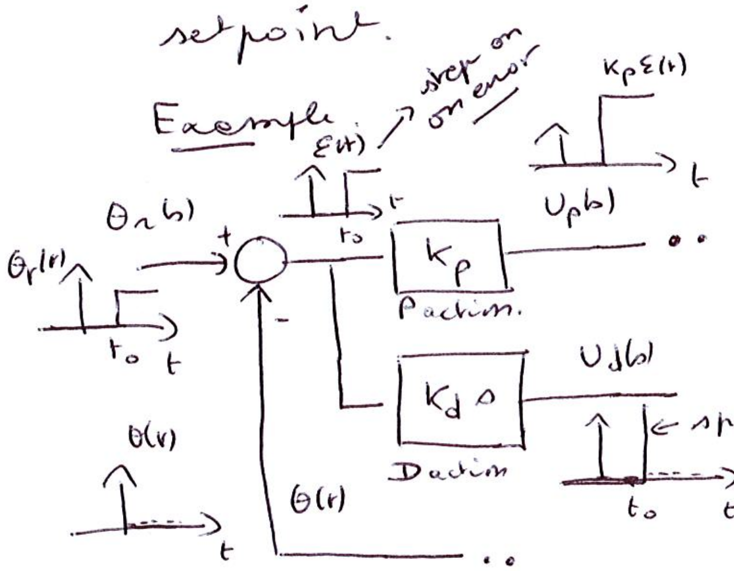


3. a) See lecture slides ~~PID~~ G-Auto-Effects of PID actions.

slide n° 23.

The goal / advantage of having the derivative action/term on the output is to avoid the spike effect after a step change on the setpoint.

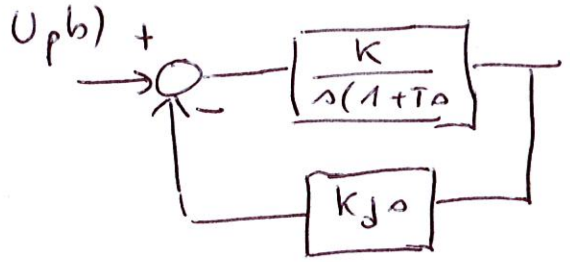
Example



← spike effect since the time-derivative of ~~the~~ a step is a Dirac impulse

There is a risk to get a control signal that total becomes saturated. (or to destroyed the actuator)

3 b)

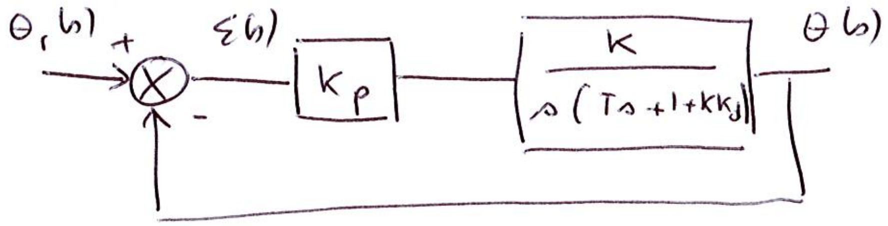


$$F_i(s) = \frac{\theta(s)}{U_p(s)} = \frac{C(s)\,G(s)}{1 + C(s)G(s)H(s)} \quad \text{with} \quad C(s) = 1 \quad G(s) = \frac{K}{s(1+Ts)}$$

$$H(s) = K_d s.$$

$$F_i(s) = \frac{\frac{K}{s(1+Ts)}}{1 + \frac{KK_d s}{s(1+Ts)}} = \frac{K}{s(Ts + 1 + KK_d)}$$

3. c)



3. d)

$$F_{c2}(s) = \frac{K K_p}{Ts^2 + (1+KK_d)s + KK_p}.$$

3. e) Use of Routh - Hurwitz criterion.

a) Necessary condition

$T > 0$

$1 + KK_d > 0 \qquad K_d > -\frac{1}{K}$

$KK_p > 0 \qquad K_p > 0$

b) Routh table.

s^2	T	KK_p
s^1	$1+KK_d$	0
s^0	KK_p.	

$$\Rightarrow \boxed{\begin{array}{c} K_d > -\frac{1}{K} \\ K_p > 0 \end{array}}$$

to make the feedback PD control system stable.

3. f) $\lim_{t \to +\infty} \varepsilon(t) = \lim_{s \to 0} s\, \varepsilon(s)$

$= \lim_{s \to 0} s\, \Theta_r(s) \left[1 - F_{CL}(s)\right]$ (unitary feedback here)

$$= \lim_{s \to 0} s \times \frac{A}{s} \left[1 - \frac{KK_p}{Ts^2 + (1+KK_d)s + KK_p}\right]$$

when $K_d > -\frac{1}{K}$, $K_p > 0$ | $\lim_{t \to +\infty} \varepsilon(t) = 0$ for $\Theta_r(t) = A\,\Gamma(t)$

The closed-loop PD control will be accurate to a step setpoint.

3. g) Let us determine K_p and K_d to get a $F_{CL}(s)$ step response with $D_{1\%} = 4{,}3\%$ and $T_r^{5\%} = 0{,}05\,s$.

From $D_{1\%}$ and $T_r^{5\%}$ we can calculate z and ω_0

$$z = \sqrt{\frac{(\ln D_1)^2}{(\ln D_1)^2 + \pi^2}} = 0{,}707 = \frac{\sqrt{2}}{2}$$ (with $D_1 = 0{,}043$)

ω_0 can be determined by using the abacus given in the appendix. For $z \simeq 0{,}707$, we read a product $\omega_0 \times T_r^{5\%} = 3$

$$\omega_0 = \frac{3}{T_r^{5\%}} = \frac{3}{0{,}05} = 60\ \text{rad/s}$$

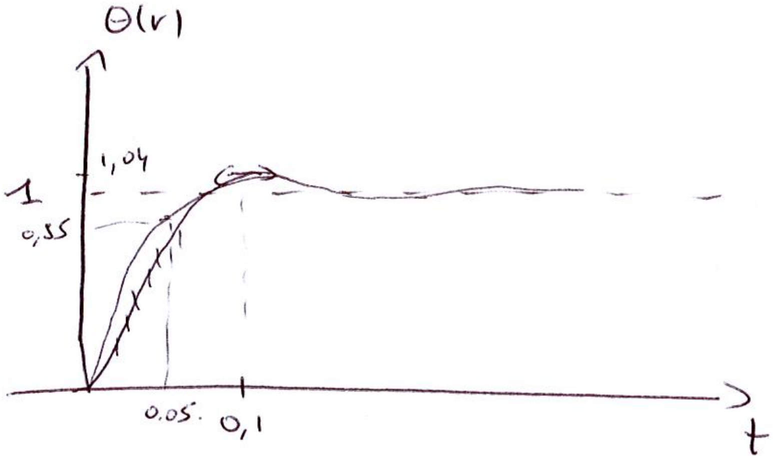
We have

$$F_{CL}(s) = \frac{1}{\frac{1}{\omega_0^2}s^2 + \frac{2z}{\omega_0}s + 1} = \frac{1}{\frac{T}{KK_p}s^2 + \frac{1+KK_d}{KK_p}s + 1}$$

By direct identification of the denominator coefficients

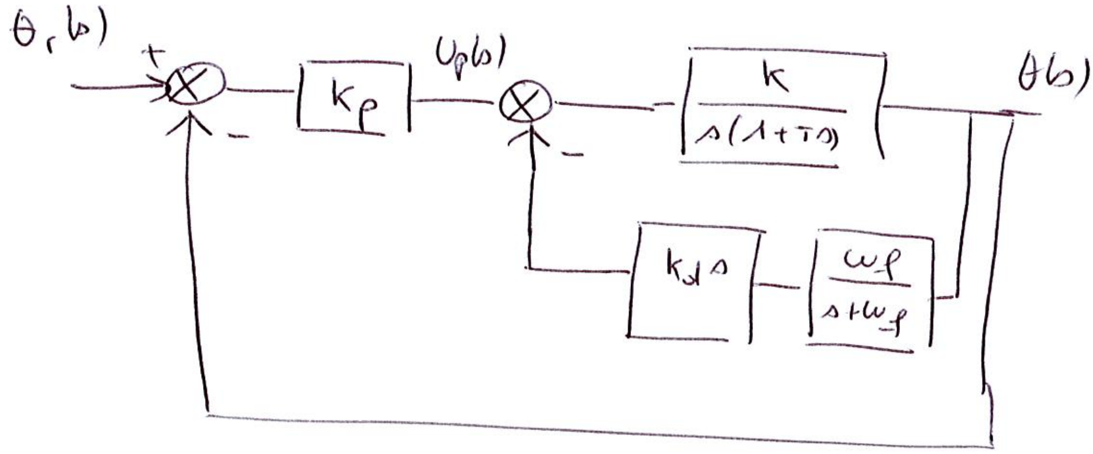
$$\left.\begin{aligned} \frac{1}{\omega_0^2} &= \frac{T}{KK_p} \\ \frac{2z}{\omega_0} &= \frac{1+KK_d}{KK_p} \end{aligned}\right\} \quad \begin{aligned} K_p &= \frac{\omega_0^2 T}{K} = \frac{60^2 \times 0{,}2}{23} = 31{,}3 \\ K_d &= \frac{1}{K}\left(\frac{2zKK_p}{\omega_0} - 1\right) = \frac{2zK_p}{\omega_0} - \frac{1}{K} = 0{,}69 \end{aligned}$$

3. h) The performance required are satisfied
- no steady-state error for a stepsetpoint
- $D_{1\%} < 5\%$
- $T_r^{5\%}$ much smaller than with the P control



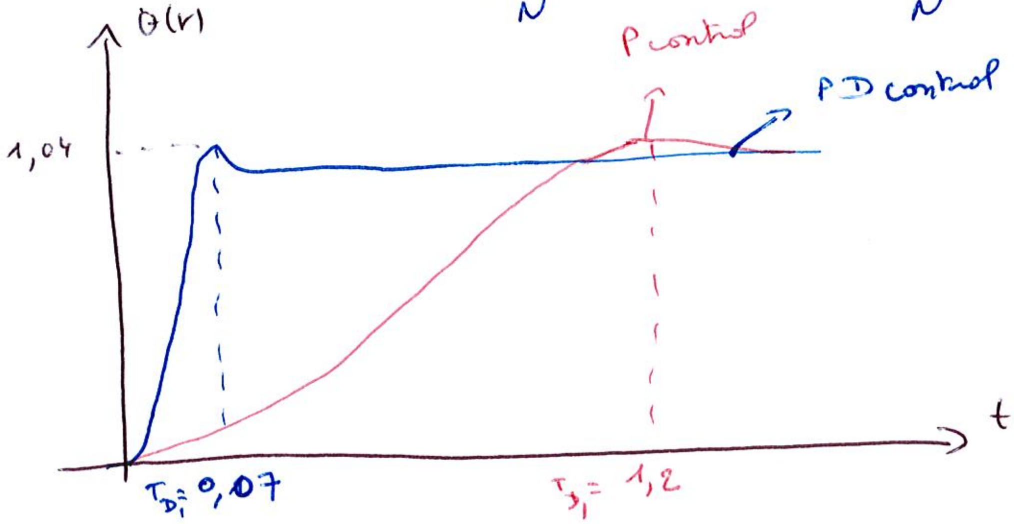
$$T_{D_1} = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}} \simeq \frac{\pi}{60\sqrt{1-\frac{1}{2}}} = 0{,}0738.$$

3. i)



4) ~~Effect of load on servomotor control~~.

$$K_d s \simeq \frac{K_d s}{1+\frac{T_d}{N}s} = T_d \cdot \frac{s}{1+\frac{1}{N}s}.$$



PD control much faster