

*The only dumb question is the one that remains unasked.* Albert Einstein.

## The $z$ -transform and its application for solving linear time-invariant difference equations

A knowledge of the  $z$ -transform is very important in digital control but also in digital signal processing. It is a valuable tool for representing, analyzing and designing discrete-time systems or digital filters. The unilateral  $z$ -transform of a discrete-time signal is defined as:

$$X(z) = \sum_{k=0}^{+\infty} x(k)z^{-k}$$

The  $z$ -transform of common discrete-time signals have been determined in closed form. They are usually given in the form of table (see Appendix). Such a table is particularly useful in finding the inverse  $z$ -transform.

Just as linear continuous-time systems are described by differential equations, linear discrete-time systems are described by difference equations. The Laplace transform can be used for solving linear time-invariant differential equations. Similarly, the  $z$ -transform is an operational method for solving linear time-invariant difference equations.

### Exercise 1.1

Plot each of the following discrete-time signals. By using the properties and the table of  $z$ -transforms (see Appendix), find their  $z$ -transform :

- (a)  $x_1(k) = \delta(k-1) + 3\delta(k-2) + \delta(k-3)$
- (b)  $x_2(k) = \Gamma(k)$
- (c)  $x_3(k) = a^k \Gamma(k)$
- (d)  $x_4(k) = (-1 + 2^k) \Gamma(k)$

where  $\delta(k)$  and  $\Gamma(k)$  denote the Kronecker impulse and discrete-time step respectively.

### Exercise 1.2

Given the following  $z$ -transform of a discrete-time sequence,

$$X(z) = 4z^{-2} + 2 + 3z^{-1}$$

determine the inverse  $z$ -transform by direct inspection.

Express  $x(k)$  in as a sum of weighted discrete-time delayed Kronecker impulses and plot the discrete-time sequence.

### Exercise 1.3

Given the following  $z$ -transforms, determine each original discrete-time sequence by using the partial fraction expansion method.

$$X_1(z) = \frac{z^2}{(z-1)(z-2)}; \quad X_2(z) = \frac{z}{z^2 + 6z + 8}$$

Give, when it is possible, the first four terms of each sequence and the final value  $x(+\infty)$ . Use the initial value theorem to check the correctness of your inverse  $z$ -transform calculation.

**Exercise 1.4**

Using the  $z$ -transform, solve the following difference equation:

$$s(k) - 3s(k-1) = e(k)$$

when  $e(k) = 4\Gamma(k)$

**Exercise 1.5**

Show that the difference equation:

$$s(k) - 5s(k-1) + 6s(k-2) = \Gamma(k)$$

has the following solution

$$s(k) = \left( \frac{1}{2} - (2)^{k+2} + \frac{1}{2}(3)^{k+2} \right) \Gamma(k)$$

*The only dumb question is the one that remains unasked. Albert Einstein.*

## Sampled transfer functions

### Exercise 2.1

1. Recall the digital closed-loop control block-diagram of a continuous-time systems described by its transfer function  $G(s)$ .
2. Plot the output of the zero-order hold (ZOH) block when the controller output takes the form of the discrete sequence displayed in Figure 2.1.

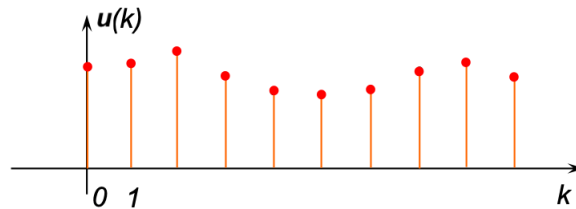


Figure 2.1: Sampled controller output

3. Recall the plot of the impulse response of the ZOH. Deduce from the plot its mathematical expression  $h_0(t)$  and derive its transfer function  $H_0(s)$  from it.
4. Recall the formula of the so-called *sampled* transfer function  $G_{ZOH}(z)$  (which is also denoted as the diskretized version of the plant  $G(s)$  by using the zero-order hold method).
5. Determine the sampled transfer function  $G_{ZOH}(z)$  when the continuous-time plant  $G(s)$  is a first-order model:

$$G(s) = \frac{b}{s+a}$$

Note the dependency of the sampled transfer function parameters to the sampling period  $T_s$ .

### Exercise 2.2

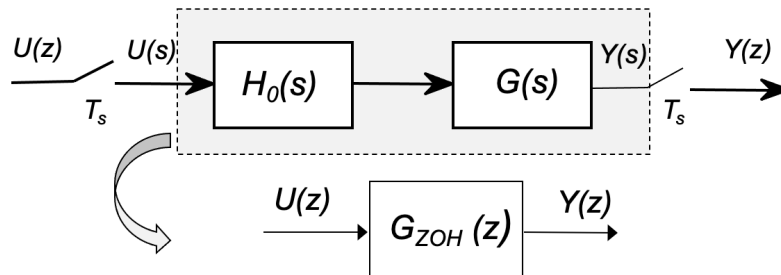


Figure 2.2: Sampled transfer function

Consider a sampled system constituted by the cascade of a zero-order hold (ZOH)  $H_0(s)$ , a continuous-time transfer function  $G(s)$  and a sampler as shown in Figure 2.2 where:

$$G(s) = \frac{2}{1 + 2s}$$

1. Give the steady-state gain, the time-constant, the pole and zero of  $G(s)$ .
2. Determine the sampled transfer function  $G_{ZOH}(z)$  when  $T_s = 1s$ .
3. Check your results with Matlab by typing the following commands:  

```
s = tf('s');
G=2/(1+2*s)
Gzoh=c2d(G,1,'zoh')
```
4. Calculate the pole and zero of  $G_{ZOH}(z)$ .
5. Recall the relationship between the poles of  $G(s)$  and  $G_{ZOH}(z)$ .
6. Give the difference equation of the sampled system.
7. Calculate the response  $y(k)$  to a unit discrete-time step  $u(k) = \Gamma(k)$  for  $k = 0, 1, 2, 3, 4$  and compare it with the step response of the continuous-time system  $G(s)$  recalled below :

$$y_c(t) = K(1 - e^{-t/T})\Gamma(t)$$

computed at the appropriate sampled time-instants  $t = kT_s$  for  $k = 0, 1, 2, 3, 4$ .

### Exercise 2.3

Consider a sampled system constituted by the cascade of a zero-order hold (ZOH)  $H_0(s)$ , a continuous-time transfer function  $G(s)$  and a sampler where

$$G(s) = \frac{s + 4}{(s + 1)^3}$$

1. Calculate the poles and zero of  $G(s)$ .
2. Use Matlab to plot zero-pole diagram and the step response of the continuous-time transfer function. Note that the continuous-time process exhibits an aperiodic step response which is due to its zero in the left half of the  $s$ -plane
3. Given that the unit of time is minutes, determine the sampled transfer function  $G_{ZOH}(z)$  by using Matlab and investigate the location of its zeros with respect to the unit circle when the sampling period  $T_s$  is 2 and 0.5 min.
4. For each case, plot the step response of the sampled transfer function and check whether the sampled transfer function exhibits an non-inverse response.

### Exercise 2.4

Consider a sampled system constituted by the cascade of a zero-order hold (ZOH)  $H_0(s)$ , a continuous-time transfer function  $G(s)$  and a sampler where

$$G(s) = \frac{-3s + 1}{(2s + 1)(5s + 1)}$$

1. Use Matlab to plot the step response of the continuous-time transfer function. Note that the continuous-time second-order process exhibits an inverse step response which is due to its zero in the right half of the  $s$ -plane.
2. Given that the unit of time is minutes, determine the sampled transfer function  $G_{ZOH}(z)$  by using Matlab and investigate the location of its zero with respect to the unit circle when the sampling period is 1, 2, 6, 7, 8, and 10 min.
3. For each case, plot the step response of the sampled transfer function and check whether the sampled transfer function exhibits also an inverse response.

*The only dumb question is the one that remains unasked.* Albert Einstein.

## Analysis of discrete-time systems

### Exercise 3.1

Consider the sampled system given the following discrete-time transfer function

$$G_{ZOH}(z) = \frac{2z^{-5}}{(z-1)^2} \frac{0.2}{z-0.8}$$

1. Give the system order, its zero and poles.
2. Determine the gain, the number of integrators and number of samples of the time-delay.

### Exercise 3.2

Consider the sampled system given by its transfer function

$$G_{ZOH}(z) = \frac{Y(z)}{U(z)} = \frac{0.4z^{-1}}{1-0.8z^{-1}}$$

1. Give the system order and determine its steady-state gain, its zero and pole.
2. Plot the zero-pole diagram. Conclude about the stability of the sampled system.
3. Check your answers by executing the following Matlab code:  

```
Gd=tf([0 0.4],[1 -0.8],pi)
dcgain(Gd)
zero(Gd)
pole(Gd)
pzplot(Gd)
```
4. Determine the difference equation of the discrete-time system.
5. Compute its response to a discrete-time unit step input.
6. Use the final value theorem to determine the step response value when  $k \rightarrow \infty$ .

### Exercise 3.3

State whether the discrete-time transfer functions given below are stable, marginally stable or unstable. Justify your answer.

$$G_1(z) = \frac{5z}{(z+0.2)(z-0.8)} \quad G_2(z) = \frac{5z}{(z+1.2)(z-0.8)}$$
$$G_3(z) = \frac{5(z+1)}{z(z-1)(z-0.8)} \quad G_4(z) = \frac{5(z+1.2)}{z^2(z-1)^2(z+0.1)}$$

### Exercise 3.4

The characteristic equation of a digital transfer function is given as

$$D(z) = z^3 + z^2 + 0.5z + 0.25 = 0$$

State by using Jury's stability criterion whether the system is stable or not.

### Exercise 3.5

The characteristic equation of a linear digital feedback control system is given as

$$D(z) = z^2 + z + K_p$$

where  $K_p$  is a real positive constant.

Find the range of values for  $K_p$  so that the digital feedback control system is stable.

**Exercise 3.6**

Consider a sampled system constituted by the cascade of a zero-order hold (ZOH)  $H_0(s)$ , a continuous-time transfer function  $G(s)$  and a sampler

$$G(s) = \frac{2}{(s+2)(s+1)}$$

1. Give the block-diagram representation of the sampled system.
2. Determine the sampled transfer function  $G_{ZOH}(z)$  when  $T_s = 0.25s$ .
3. A unity feedback proportional digital control is implemented for the continuous-time system. Represent the block-diagram of the closed-loop digital control.
4. Represent the equivalent digital control block-diagram with the sampled transfer function  $G_{ZOH}(z)$ .
5. Determine the range of proportional gain  $K_p$  which guarantees the digital closed-loop system to be stable.
6. Determine the final value response of the closed-loop output when the setpoint is a discrete-time unit step.

**Exercise 3.7**

A continuous-time system is modeled by the following first-order plus delay transfer function

$$G(s) = \frac{1}{1+8s}e^{-2s}$$

1. Find the PI controller that results by applying the Ziegler-Nichols tuning rules to the process (see Appendix).
2. Determine the transfer function of the digital controller  $C(z)$  by using the backward difference approximation for the integral part when  $T_s = 0.1s$ .
3. Give the difference equation of the digital controller output.
4. Give the Matlab code that implements the closed-loop control for the digital PID controller.

*The only dumb question is the one that remains unasked.* Albert Einstein.

## Model-based design of digital controllers

### Exercise 4 - Altitude control for a mini-drone via the approximation of a PD analog controller

The goal is to design an altitude control for the Tello mini-drone as shown in Figure 4.1.



Figure 4.1: Tello mini-drone from DJI

The input and output of the mini-drone altitude control system are:

- $u(t)$ : motor speed in % (related in some way to the thrust generated by the 4 rotors);
- $y(t)$ : altitude or position along the  $Z$  axis of the mini-drone in cm.

The maximum velocity percentage-to-altitude transfer function takes the form of a first-order plus pure integrator model

$$\frac{Y(s)}{U(s)} = \frac{K}{s(1 + Ts)} \quad (1)$$

where  $Y(s) = \mathcal{L}[y(t)]$  and  $U(s) = \mathcal{L}[u(t)]$ .

$K$  is the gain,  $T$  is the model time-constant. We will assume

- $K = 0.8$ ;
- $T = 0.3\text{s}$ .

The performance requirements for the mini-drone altitude control are described in Table 1.

| Requirement                     | Assessment criteria   | Level   |
|---------------------------------|---|---|
| Control the mini-drone altitude | Step reference tracking<br>Peak overshoot<br>Settling time at 5 % | No steady-state error<br>$D_{1\%} = 4.3\%$<br>$t_s^{5\%} = 1 \text{ s}$ |

Table 1: Performance requirements for the mini-drone altitude control

The proposed strategy is to use a variation of the standard PD control, where unlike the standard PD where the derivative term is usually applied to the error, it is applied to the output, as shown in Figure 4.2.

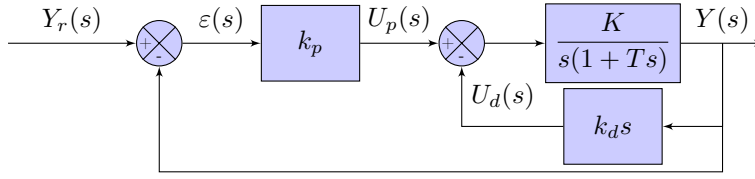


Figure 4.2: Block-diagram of the PD feedback configuration of the altitude control system

1. Determine the internal closed-loop transfer function  $F_i(s) = \frac{Y(s)}{U_p(s)}$  in terms of  $k_d$ ,  $K$  and  $T$ .
2. Plot the simplified closed-loop block-diagram.
3. Determine the closed-loop transfer function  $G_{cl}(s) = \frac{Y(s)}{Y_r(s)}$  in terms of  $k_p$ ,  $k_d$ ,  $K$  and  $T$ .
4. Determine the range of values for  $k_p$  and  $k_d$  that ensure the stability of the closed loop system.
5. Calculate the steady-state tracking error in response to a step  $y_r(t) = A\Gamma(t)$ , *i.e.* :

$$\lim_{t \rightarrow +\infty} \varepsilon(t) = \lim_{t \rightarrow +\infty} (y_r(t) - y(t))$$

6. Determine the value for the damping coefficient  $\zeta$  and undamped natural frequency  $\omega_0$  that make the desired closed-loop transfer function step response to have  $D_{1\%} = 4.3\%$  and  $t_s^{5\%} = 1$  s. It is recalled that  $\zeta$  and  $\omega_0$  are function of the step response performance indices (settling time  $t_s^{5\%}$  and peak overshoot  $D_1$ )

$$\zeta = \sqrt{\frac{(\ln(D_1))^2}{\pi^2 + (\ln(D_1))^2}} \quad \omega_0 = \frac{3}{t_s^{5\%}} \text{ when } \zeta \approx 0.707$$

7. Give the transfer function of the desired closed-loop system  $G_{ref}(s)$ .
8. A method for determining the PD controller  $C(s)$  is to consider the desired continuous-time second-order closed-loop transfer function  $G_{ref}(s)$  and to solve the gains  $k_p$  and  $k_d$  of the PD controller. Show that this method leads to

$$G_{cl}(s) = G_{ref}(s) \Rightarrow \begin{cases} k_p = \frac{T\omega_0^2}{K} \\ k_d = \frac{2\zeta k_p}{\omega_0} - \frac{1}{K} \end{cases}$$

9. Determine the approximated digital version  $C_d(z)$  that results from the use of the backward Euler rule

$$s = \frac{1 - z^{-1}}{T_s}$$

for the derivative part of the PD controller:

$$C_d(s) = k_d s$$

10. Show from the block-diagram 4.2 that the difference equation of the digital controller output  $u(k)$  can be written as:

$$u(k) = u_p(k) - u_d(k)$$

11. Express  $u_p(k)$  as a function of  $k_p$  and  $\varepsilon(k)$  and  $u_d(k)$  as a function of  $k_d$ ,  $T_s$ ,  $y(k)$  and  $y(k-1)$ .



## English to French translation

|                             |   |                                     |
|-----------------------------|---|-------------------------------------|
| bandwidth                   | : | bande passante                      |
| closed-loop system          | : | système bouclé                      |
| cut-off frequency           | : | fréquence (ou pulsation) de coupure |
| damped frequency            | : | pulsation amortie                   |
| damping ratio               | : | coefficient d'amortissement         |
| feedback                    | : | contre-réaction                     |
| feedback system             | : | système à contre-réaction           |
| impulse response            | : | réponse impulsionnelle              |
| integral wind-up            | : | emballement du terme intégral       |
| impulse response            | : | réponse impulsionnelle              |
| input                       | : | entrée                              |
| gain                        | : | gain                                |
| linear time-invariant (LTI) | : | linéaire invariant dans le temps    |
| output                      | : | sortie                              |
| overshoot                   | : | dépassement                         |
| rise time                   | : | time de montée                      |
| root locus                  | : | lieu des racines                    |
| sampled                     | : | échantillonné                       |
| sampled systems             | : | systèmes échantillonnés             |
| setpoint                    | : | consigne                            |
| settling time               | : | temps de réponse                    |
| steady-state gain           | : | gain statique                       |
| steady-state response       | : | réponse en régime permanent         |
| step response               | : | réponse indicielle                  |
| time-invariant              | : | invariant dans le temps             |
| transient response          | : | réponse transitoire                 |
| undamped natural frequency  | : | pulsation propre non amortie        |
| zero-order holder           | : | bloqueur d'ordre zéro               |

## Appendix

### Table of common unilateral Laplace and $z$ -transforms

The table below provides a number of Laplace and  $z$ -transform pairs and their region of convergence (ROC).  
Remarks:

| $f(t)$                      | $F(s)$                              | $f(k)$                         | $F(z)$   | ROC         |
|-----------------------------|-------------------------------------|--------------------------------|--|-------------|
| $\delta(t)$                 | 1                                   | $\delta(k)$                    | 1  | All $z$     |
| $\delta(t - t_0)$           | $e^{-st_0}$                         | $\delta(k - i)$                | $z^{-i}$   | $z \neq 0$  |
| $\Gamma(t)$                 | $\frac{1}{s}$                       | $\Gamma(k)$                    | $\frac{z}{z-1}$  | $ z  > 1$   |
| $t\Gamma(t)$                | $\frac{1}{s^2}$                     | $kT_s\Gamma(k)$                | $\frac{zT_s}{(z-1)^2}$   | $ z  > 1$   |
|                             |                                     | $a^k\Gamma(k)$                 | $\frac{z}{z-a}$  | $ z  >  a $ |
| $e^{-at}\Gamma(t)$          | $\frac{1}{s+a}$                     | $e^{-akT_s}\Gamma(k)$          | $\frac{z}{z-e^{-aT_s}}$  | $ z  >  a $ |
| $te^{-at}\Gamma(t)$         | $\frac{1}{(s+a)^2}$                 | $kT_s e^{-akT_s}\Gamma(k)$     | $\frac{T_s e^{-aT_s} z}{(z-e^{-aT_s})^2}$                          | $ z  >  a $ |
| $\cos(\omega_0 t)\Gamma(t)$ | $\frac{s}{s^2 + \omega_0^2}$        | $\cos(\omega_0 kT_s)\Gamma(k)$ | $\frac{z(z - \cos(\omega_0 T_s))}{z^2 - 2\cos(\omega_0 T_s)z + 1}$ | $ z  >  1 $ |
| $\sin(\omega_0 t)\Gamma(t)$ | $\frac{\omega_0}{s^2 + \omega_0^2}$ | $\sin(\omega_0 kT_s)\Gamma(k)$ | $\frac{\sin(\omega_0 T_s)z}{z^2 - 2\cos(\omega_0 T_s)z + 1}$       | $ z  >  1 $ |

• The notation for  $z$  found in the table above may differ from that found in other tables. For example, the basic  $z$ -transform of a unit discrete-time step  $\Gamma(k)$  can be written as either of the following two expressions, which are equivalent:

$$Z(\Gamma(k)) = \Gamma(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

• The ROC for a given sequence  $x(k)$ , is defined as the range of  $z$  for which the  $z$ -transform converges. Since the  $z$ -transform is a power series, it converges when  $x(k)z^{-k}$  is absolutely summable. Stated differently,

$$\sum_{k=0}^{+\infty} |x(k)z^{-k}| < \infty \quad \text{must be satisfied for convergence.}$$

*Property 1.* if  $x(k)$  is of finite duration, then the ROC is the entire  $z$ -plane, except possibly  $z = 0$  and/or  $z = \infty$ .

*Property 2.* The ROC does not contain any poles.

### Useful properties of geometric series

|                                  |   |               |
|----------------------------------|---|---------------|
| finite sum of geometric series   | $\sum_{k=0}^N q^k = N + 1$  | if $q = 1$    |
|                                  | $\sum_{k=0}^N q^k = \frac{1 - q^{N+1}}{1 - q}$                    | if $q \neq 1$ |
| infinite sum of geometric series | $\lim_{N \rightarrow +\infty} \sum_{k=0}^N q^k = \frac{1}{1 - q}$ | if $ q  < 1$  |

---

### Useful properties of the unilateral $z$ -transform

Some useful properties which have found practical use are summarized below.

| Property              | signal   | $z$ -transform  |
|-----------------------|--|---|
| linearity             | $ax(k) + by(k)$  | $aX(z) + bY(z)$   |
| time-delay            | $x(k-1)$   | $z^{-1}X(z)$ if $x(k)$ is causal <sup>1</sup>                       |
|                       | $x(k-i)$   | $z^{-i}X(z)$ if $x(k)$ is causal                                    |
|                       | $x(k-1)$   | $z^{-1}X(z) + z^{-2}x(-1)$ if $x(k)$ is non-causal                  |
|                       | $x(k-i)$   | $z^{-i}X(z) + \sum_{j=-i}^{-1} z^{j-i}x(j)$ if $x(k)$ is non-causal |
| time-advance          | $x(k+1)$   | $zX(z) - zx(0)$   |
|                       | $x(k+2)$   | $z^2X(z) - z^2x(0) - zx(1)$   |
|                       | $x(k+i)$   | $z^iX(z) - \sum_{j=0}^{i-1} z^{i-j}x(j)$                            |
| convolution           | $y(k) = h(k) * u(k)$<br>$y(k) = \sum_{i=-\infty}^{+\infty} h(i)u(k-i)$ | $Y(z) = H(z)U(z)$   |
| differentiation       | $kx(k)$  | $-z \frac{dX(z)}{dz}$   |
| accumulation          | $\sum_{i=0}^k x(i)$  | $\frac{1}{1-z^{-1}}X(z)$  |
| initial value theorem | if $x(k) = 0$ for $k < 0$  | $x(0) = \lim_{z \rightarrow +\infty} X(z)$                          |
| final value theorem   | $\lim_{k \rightarrow +\infty} x(k) = \lim_{z \rightarrow 1} (z-1)X(z)$ | if the limit exists   |

---

<sup>1</sup> A signal is causal if  $x(k) = 0$ , for all  $k < 0$

---

**Sampled transfer functions of continuous-time systems**

**Sampled transfer functions of continuous-time systems**

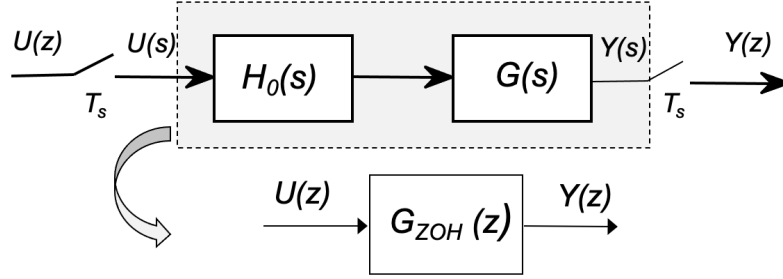


Figure A.1: Sampled system

Consider a Zero-Order Hold (ZOH)  $H_0(s)$  cascaded with a continuous-time transfer function  $G(s)$  and a sampler as shown in Figure A.1, the corresponding sampled transfer function is given by:

$$G_{ZOH}(z) = \frac{Y(z)}{U(z)} = Z\left(H_0(s)G(s)\right) = (1 - z^{-1}) Z\left(\frac{G(s)}{s}\right) = \left(\frac{z-1}{z}\right) Z\left(\frac{G(s)}{s}\right)$$

**First-order systems**

Consider a continuous-time first order system given by its Laplace transfer function:

$$G(s) = \frac{b}{s+a}$$

Its equivalent ZOH sampled transfer function is given by:

$$G_{ZOH}(z) = Z\left(H_0(s)G(s)\right) = (1 - z^{-1}) Z\left(\frac{b}{s(s+a)}\right) = \frac{b_1 z^{-1}}{1 + a_1 z^{-1}}$$

where

$$\begin{cases} a_1 = -e^{-aT_s} \\ b_1 = \frac{b}{a}(1 + a_1) \end{cases}$$

and  $T_s$  denotes the sampling period.

**Second-order systems**

Consider a continuous-time second order system given by its Laplace transfer function:

$$G(s) = \frac{a \times b}{(s+a)(s+b)}$$

Its equivalent ZOH sampled transfer function is given by:

$$G_{ZOH}(z) = Z\left(H_0(s)G(s)\right) = (1 - z^{-1}) Z\left(\frac{a \times b}{s(s+a)(s+b)}\right)$$

$$G_{ZOH}(z) = \frac{b_1 z^{-1} + b_2 z^{-2}}{(1 + \alpha z^{-1})(1 + \beta z^{-1})}$$

where

$$\begin{cases} \alpha = -e^{-aT_s} \\ \beta = -e^{-bT_s} \\ b_1 = \frac{b \times \alpha - a \times \beta}{b - a} + 1 \\ b_2 = \frac{b \times \beta - a \times \alpha}{b - a} + \alpha \times \beta \end{cases}$$

and  $T_s$  denotes the sampling period.

## PID tuning by using empirical rules

A PID controller defined in its so-called *ideal* form is defined as:

$$C(s) = K_p \left( 1 + \frac{1}{T_i s} + \frac{T_d s}{1 + \frac{T_d}{N} s} \right) = K_p + K_i \frac{1}{s} + K_d \frac{s}{1 + \frac{T_d}{N} s} \quad (1)$$

The individual effects of these three  $K_p$ ,  $K_i$  and  $K_d$  parameters appearing in (1) on the closed-loop performance of stable plants are recalled in Table 1 below.

**TABLE 1** Effects of independent P, I, and D tuning on closed-loop response.

For example, while  $K_I$  and  $K_D$  are fixed, increasing  $K_P$  alone can decrease rise time, increase overshoot, slightly increase settling time, decrease the steady-state error, and decrease stability margins.

|                  | Rise Time      | Overshoot | Settling Time  | Steady-State Error | Stability |
|------------------|----------------|-----------|----------------|--------------------|-----------|
| Increasing $K_P$ | Decrease       | Increase  | Small Increase | Decrease           | Degrade   |
| Increasing $K_I$ | Small Decrease | Increase  | Increase       | Large Decrease     | Degrade   |
| Increasing $K_D$ | Small Decrease | Decrease  | Decrease       | Minor Change       | Improve   |

### Tuning of the PID controller by using the Ziegler-Nichols rules

Assume a continuous-time process is reasonably well modeled by the first-order plus time-delay transfer function model:

$$G(s) = \frac{K e^{-\tau s}}{1 + T s}$$

Several tuning rules are available for determining the PID controller parameter values. The proposed rules address different design specifications such as load disturbance rejection or setpoint tracking.

The most popular tuning rules are those attributed to Ziegler-Nichols. Their aim is to provide *satisfactory load disturbance rejection*.

Table 2 shows the Ziegler-Nichols rules for P, PI, or PID controllers defined in its so-called *ideal* form (see (1)).

| Controller type | $K_p$                 | $T_i$   | $T_d$     |
|-----------------|-----------------------|---------|-----------|
| P               | $\frac{T}{K\tau}$     |         |           |
| PI              | $0.9 \frac{T}{K\tau}$ | $3\tau$ |           |
| PID             | $1.2 \frac{T}{K\tau}$ | $2\tau$ | $0.5\tau$ |

Table 2: Ziegler-Nichols disturbance rejection tuning rules for a first-order-plus delay model determined from an open-loop step response