Tutorials
Digital control
n°1

H. Garnier
F. Collin

The only dumb question is the one that remains unasked. Albert Einstein.

The z-transform and its application for solving linear time-invariant difference equations

A knowledge of the z-transform is very important in digital control but also in digital signal processing. It is a valuable tool for representing, analyzing and designing discrete-time systems or digital filters. The unilateral z-transform of a discrete-time signal is defined as:

$$X(z) = \sum_{k=0}^{+\infty} x(k)z^{-k}$$

The z-transform of common discrete-time signals have been determined in closed form. They are usually given in the form of table (see Appendix). Such a table is particularly useful in finding the inverse z-transform.

Just as linear continuous-time systems are described by differential equations, linear discrete-time systems are described by difference equations. The Laplace transform can be used for solving linear time-invariant differential equations. Similarly, the z-transform is an operational method for solving linear time-invariant difference equations.

Exercise 1.1
Plot each of the following discrete-time signals. By using the properties and the table of z-transforms (see Appendix), find their z-transform :

$$\begin{aligned}
&(a) \quad x_1(k) = \delta(k-1) + 3\delta(k-2) + \delta(k-3)\\
&(b) \quad x_2(k) = \Gamma(k)\\
&(c) \quad x_3(k) = a^k\Gamma(k)\\
&(d) \quad x_4(k) = \big(-1 + 2^k\big)\Gamma(k)
\end{aligned}$$

where $\delta(k)$ and $\Gamma(k)$ denote the Kronecker impulse and discrete-time step respectively.

Exercise 1.2
Given the following z-transform of a discrete-time sequence,

$$X(z) = 4z^{-2} + 2 + 3z^{-1}$$

determine the inverse z-transform by direct inspection.
Express $x(k)$ in as a sum of weighted discrete-time delayed Kronecker impulses and plot the discrete-time sequence.

Exercise 1.3
Given the following z-transforms, determine each original discrete-time sequence by using the partial fraction expansion method.

$$X_1(z) = \frac{z^2}{(z-1)(z-2)}; \qquad X_2(z) = \frac{z}{z^2 + 6z + 8}$$

Give, when it is possible, the first four terms of each sequence and the final value $x(+\infty)$. Use the initial value theorem to check the correctness of your inverse z-transform calculation.

Exercise 1.4
Using the z-transform, solve the following difference equation:

$$s(k) - 3s(k-1) = e(k)$$

when $e(k) = 4\Gamma(k)$

Exercise 1.5
Show that the difference equation:

$$s(k) - 5s(k-1) + 6s(k-2) = \Gamma(k)$$

has the following solution

$$s(k) = \left(\frac{1}{2} - (2)^{k+2} + \frac{1}{2}(3)^{k+2}\right)\Gamma(k)$$

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Sampled transfer functions

Exercise 2.1

1. Recall the digital closed-loop control block-diagram of a continuous-time systems described by its transfer function $G(s)$.

2. Plot the output of the zero-order hold (ZOH) block when the controller output takes the form of the discrete sequence displayed in Figure 2.1.

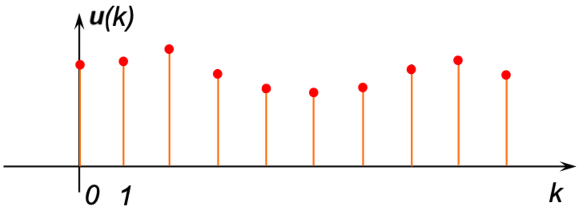

Figure 2.1: Sampled controller output

3. Recall the plot of the impulse response of the ZOH. Deduce from the plot its mathematical expression $h_0(t)$ and derive its transfer function $H_0(s)$ from it.

4. Recall the formula of the so-called *sampled* transfer function $G_{ZOH}(z)$ (which is also denoted as the diskretized version of the plant $G(s)$ by using the zero-order hold method).

5. Determine the sampled transfer function $G_{ZOH}(z)$ when the continuous-time plant $G(s)$ is a first-order model:
$$G(s) = \frac{b}{s+a}$$
Note the dependency of the sampled transfer function parameters to the sampling period T_s.

Exercise 2.2

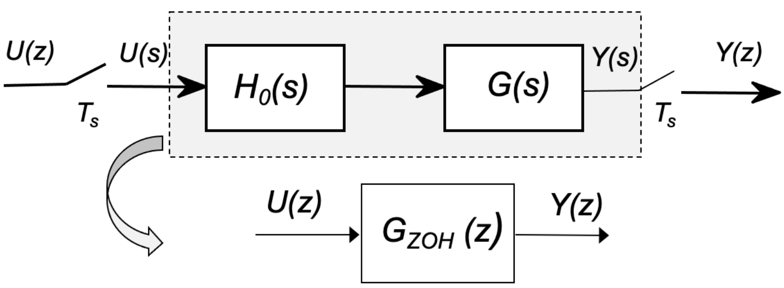

Figure 2.2: Sampled transfer function

Consider a sampled system constituted by the cascade of a zero-order hold (ZOH) $H_0(s)$, a continuous-time transfer function $G(s)$ and a sampler as shown in Figure 2.2 where:

$$G(s) = \frac{2}{1+2s}$$

1. Give the steady-state gain, the time-constant, the pole and zero of $G(s)$.
2. Determine the sampled transfer function $G_{ZOH}(z)$ when $T_s = 1$s.
3. Check your results with Matlab by typing the following commands:
```
s = tf('s');
G=2/(1+2*s)
Gzoh=c2d(G,1,'zoh')
```
4. Calculate the pole and zero of $G_{ZOH}(z)$.
5. Recall the relationship between the poles of $G(s)$ and $G_{ZOH}(z)$.
6. Give the difference equation of the sampled system.
7. Calculate the response $y(k)$ to a unit discrete-time step $u(k) = \Gamma(k)$ for $k = 0, 1, 2, 3, 4$ and compare it with the step response of the continuous-time system $G(s)$ recalled below :

$$y_c(t) = K(1 - e^{-t/T})\Gamma(t)$$

computed at the appropriate sampled time-instants $t = kT_s$ for $k = 0, 1, 2, 3, 4$.

Exercise 2.3

Consider a sampled system constituted by the cascade of a zero-order hold (ZOH) $H_0(s)$, a continuous-time transfer function $G(s)$ and a sampler where

$$G(s) = \frac{s+4}{(s+1)^3}$$

1. Calculate the poles and zero of $G(s)$.
2. Use Matlab to plot zero-pole diagram and the step response of the continuous-time transfer function. Note that the continuous-time process exhibits an aperiodic step response which is due to its zero in the left half of the s-plane
3. Given that the unit of time is minutes, determine the sampled transfer function $G_{ZOH}(z)$ by using Matlab and investigate the location of its zeros with respect to the unit circle when the sampling period T_s is 2 and 0.5 min.
4. For each case, plot the step response of the sampled transfer function and check whether the sampled transfer function exhibits an non-inverse response.

Exercise 2.4

Consider a sampled system constituted by the cascade of a zero-order hold (ZOH) $H_0(s)$, a continuous-time transfer function $G(s)$ and a sampler where

$$G(s) = \frac{-3s+1}{(2s+1)(5s+1)}$$

1. Use Matlab to plot the step response of the continuous-time transfer function. Note that the continuous-time second-order process exhibits an inverse step response which is due to its zero in the right half of the s-plane.
2. Given that the unit of time is minutes, determine the sampled transfer function $G_{ZOH}(z)$ by using Matlab and investigate the location of its zero with respect to the unit circle when the sampling period is 1, 2, 6, 7, 8, and 10 min.
3. For each case, plot the step response of the sampled transfer function and check whether the sampled transfer function exhibits also an inverse response.

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Analysis of discrete-time systems

Exercise 3.1
Consider the sampled system given the following discrete-time transfer function

$$G_{ZOH}(z) = \frac{2z^{-5}}{(z-1)^2}\frac{0.2}{z-0.8}$$

1. Give the system order, its zero and poles.
2. Determine the gain, the number of integrators and number of samples of the time-delay.

Exercise 3.2
Consider the sampled system given by its transfer function

$$G_{ZOH}(z) = \frac{Y(z)}{U(z)} = \frac{0.4z^{-1}}{1-0.8z^{-1}}$$

1. Give the system order and determine its steady-state gain, its zero and pole.
2. Plot the zero-pole diagram. Conclude about the stability of the sampled system.
3. Check your answers by executing the following Matlab code:
```
Gd=tf([0 0.4],[1 -0.8],pi)
dcgain(Gd)
zero(Gd)
pole(Gd)
pzplot(Gd)
```
4. Determine the difference equation of the discrete-time system.
5. Compute its response to a discrete-time unit step input.
6. Use the final value theorem to determine the step response value when $k \to \infty$.

Exercise 3.3
State whether the discrete-time transfer functions given below are stable, marginally stable or unstable. Justify your answer.

$$G_1(z) = \frac{5z}{(z+0.2)(z-0.8)} \qquad G_2(z) = \frac{5z}{(z+1.2)(z-0.8)}$$
$$G_3(z) = \frac{5(z+1)}{z(z-1)(z-0.8)} \qquad G_4(z) = \frac{5(z+1.2)}{z^2(z-1)^2(z+0.1)}$$

Exercise 3.4
The characteristic equation of a digital transfer function is given as

$$D(z) = z^3 + z^2 + 0.5z + 0.25 = 0$$

State by using Jury's stability criterion whether the system is stable or not.
Exercise 3.5

The characteristic equation of a linear digital feedback control system is given as

$$D(z) = z^2 + z + K_p$$

where K_p is a real positive constant.

Find the range of values for K_p so that the digital feedback control system is stable.

Exercise 3.6
Consider a sampled system constituted by the cascade of a zero-order hold (ZOH) $H_0(s)$, a continuous-time transfer function $G(s)$ and a sampler

$$G(s) = \frac{2}{(s+2)(s+1)}$$

1. Give the block-diagram representation of the sampled system.
2. Determine the sampled transfer function $G_{ZOH}(z)$ when $T_s = 0.25$s.
3. A unity feedback proportional digital control is implemented for the continuous-time system. Represent the block-diagram of the closed-loop digital control.
4. Represent the equivalent digital control block-diagram with the sampled transfer function $G_{ZOH}(z)$.
5. Determine the range of proportional gain K_p which guarantees the digital closed-loop system to be stable.
6. Determine the final value response of the closed-loop output when the setpoint is a discrete-time unit step.

Exercise 3.7
A continuous-time system is modeled by the following first-order plus delay transfer function

$$G(s) = \frac{1}{1+8s}e^{-2s}$$

1. Find the PI controller that results by applying the Ziegler-Nichols tuning rules to the process (see Appendix).
2. Determine the transfer function of the digital controller $C(z)$ by using the backward difference approximation for the integral part when T_s =0.1s.
3. Give the difference equation of the digital controller output.
4. Give the Matlab code that implements the closed-loop control for the digital PID controller.

POLYTECH NANCY

Tutorials
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Model-based design of digital controllers

Exercise 4 - Rotary speed control for Qube-Servo 2

Control of speed is a common problem encountered by many control engineers, perhaps the most common well-known situation being the cruise control fitted to many automobiles. Here it will be assumed that the speed to be controlled is rotary. We investigate the model-based rotary speed control for the Qube-Servo 2 platform equipped with the inertia disk, as shown in Figure 4.1, that was used during one of the labs in the previous semester.

Figure 4.1: Qube-Servo 2 platform equipped with the inertia disk

The input of the system to be controlled is the voltage of the motor $u(t)$ in V while the output is the angular velocity or rotary speed $\omega(t)$ in rad/s.
The servomotor voltage-to-rotary speed transfer function takes the well-known first-order model form whose key parameters have been estimated from a step response:

$$G(s) = \frac{\Omega(s)}{U(s)} = \frac{K}{1+Ts} = \frac{23}{1+0.2s} \tag{1}$$

where $U(s) = \mathcal{L}[u(t)]$ and $\Omega(s) = \mathcal{L}[\omega(t)]$ are the Laplace transform of the motor voltage and disk rotary speed respectively.
K in rad/(V-s) is the model steady-state gain, T in s is the model time-constant.
The performance requirements for the rotary speed control are described in Table 1.

Requirement	Assessment criteria	Level
Control the rotary speed of the inertia disk	Step reference tracking Peak overshoot Settling time at 5 %	No steady-state error $D_{1\%} = 4.3\%$ $t_s^{5\%} = 50$ ms

Table 1: Performance requirements for the rotary speed control

To design a digital control, we seek for a z-domain transfer function (or difference equation) model of the controller that meets the given performance requirements. Two main approaches can be used:

- the first starts from a continuous-time model of the plant which is used to design in the s-domain a continuous-time controller that meets the specification. The digital controller results then from the approximation (through discretization) of the designed continuous-time controller;
- the second approach starts from a discrete-time model (the sampled version of the continuous-time model) of the plant. The digital controller is then designed in the z-domain by using pole placement, model reference or the (so-called) direct design methods.

The two approaches are investigated here.

It should be noted from the design specifications that the desired transient step response of the closed-loop control system is known. When only a pair of dominant complex poles is retained, the desired transfer function model that serves as the model reference for the closed-loop system takes the well-known form of a second order:

$$G_{ref}(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \tag{2}$$

with damping factor ζ and undamped natural frequency ω_0.

The poles are function of ζ and ω_0 and are given, when $\zeta < 1$, by:

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0 \Rightarrow s_{1,2} = -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2}$$

It is also recalled that ζ and ω_0 are function of the step response performance indices (settling time $t_s^{5\%}$ and peak overshoot D_1)

$$\zeta = \sqrt{\frac{(\ln(D_1))^2}{\pi^2 + (\ln(D_1))^2}} \qquad \omega_0 = \frac{3}{\zeta t_s^{5\%}}$$

Therefore, shaping the desired transient step response of the closed-loop system is similar to placing the dominant poles of the closed-loop transfer function as shown in Figure 4.2.

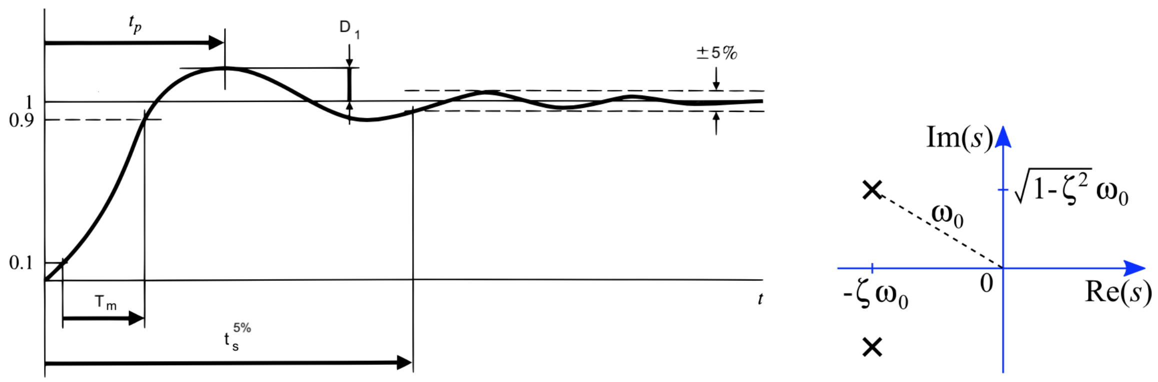

Figure 4.2: Step response with design performance indices (left) and s-plane poles (right)

4.1. Digital controller design via the approximation of a continuous-time PI controller

PI controllers are commonly used in speed control. The proposed strategy is to use a variation of the classical PI control as shown in Figure 4.3, where unlike the standard PI where the proportional term is usually applied to the error, it is applied to the output. This PI variation presents the advantage of obtaining a closed-loop transfer function (for a first-order plant model) without any zero as the desired $G_{ref}(s)$.

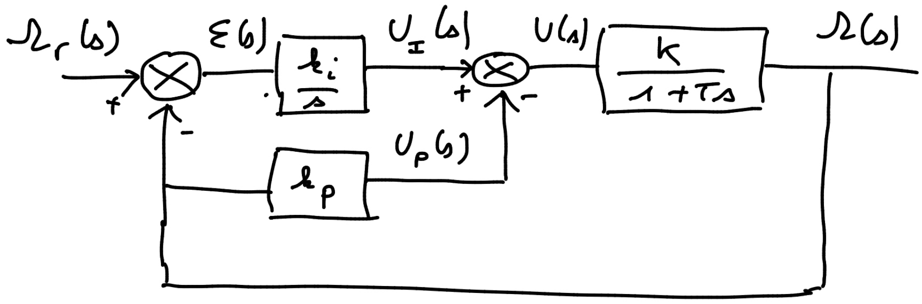

Figure 4.3: Block-diagram of the PI feedback configuration for the rotary speed servo system with proportional effect on the output.

1. Determine the closed-loop transfer function $G_{cl}(s) = \dfrac{\Omega(s)}{\Omega_r(s)}$ in terms of k_p, k_i, K and T.

2. Determine the range of values for k_p and k_i that ensure the stability of the closed loop.

3. Determine the steady-state tracking error in response to a step $\omega_r(t) = 10\Gamma(t)$ by using the final value theorem:
$$\lim_{t\to+\infty} \varepsilon(t) = \lim_{s\to 0} s\varepsilon(s)$$

4. Determine the value for ζ and ω_0 that make the desired closed-loop transfer function step response to have $D_{1\%} = 4.3$ % and $T_r^{5\%} = 50$ ms.

5. Give the transfer function of the desired closed-loop system $G_{ref}(s)$ and calculate the s-plane poles of the desired closed-loop transfer function.

6. A classical method for determining the PI controller $C(s)$ in the continuous-time domain is to consider the desired continuous-time second-order closed-loop transfer function $G_{ref}(s)$ and to solve the gains k_p and k_i of the PI controller. Show that this method leads to
$$G_{cl}(s) = G_{ref}(s) \Rightarrow \begin{cases} k_i = \dfrac{T\omega_0^2}{K} \\ k_p = \dfrac{2\zeta k_i}{\omega_0} - \dfrac{1}{K} \end{cases}$$

7. Are the performance requirements satisfied with this PI structure and its design from the continuous-time plant model?

8. For a given continuous-time controller, several approximated digital controller forms can be obtained based on the chosen discretisation methods: backward Euler, forward Euler, pole-zero matched, Tustin, etc. The Tustin method gives the nearest form of the continuous-time controller. It is therefore chosen here where the Laplace variable is replaced by:
$$s = \frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}}$$
when the sampling interval is set to T_s.
Determine the discretised version $C_i(z)$ that results from the use of the Tustin rule for the integral part of the PI controller:
$$C_i(s) = \frac{k_i}{s}$$

9. Show that the difference equation of the digital controller output $u(k)$ can be written as:
$$u(k) = u_i(k) - u_p(k)$$

10. Express $u_i(k)$ and $u_p(k)$ as a function of k_p, k_i, T_s, $u_i(k-1)$, $\varepsilon(k)$, $\varepsilon(k-1)$ and $\omega(k)$.

11. Represent the block-diagram of the closed-loop digital control.

12. It is recalled that the choice of the sampling period T_s for implementing a digital control should be done according to the desired closed-loop response. A rule of thumb often used to make the choice of the sampling period is:
$$\frac{t_s^{1\%}}{100} < T_s < \frac{t_s^{1\%}}{5}$$
where $t_s^{1\%}$ is the settling time at 1%. What sampling period T_s value would you recommend to determine and implement a digital controller via the approximation of the continuous-time PI controller ?

4.2. Direct design method of the digital controller from the sampled plant model

We now consider the design in the z-domain of a digital controller $C(z)$ so that the closed-loop step response respects the performance requirements.

1. Show that the sampled transfer function $G_{zoh}(z)$ of $G(s)$ for a given sampling period T_s has the form:

$$G_{zoh}(z) = \frac{b_1}{z + a_1}$$

2. Express a_1 and b_1 in terms of K, T and T_s.
3. Represent the block-diagram of the closed-loop digital control system with a controller $C(z)$ and the sampled transfer function $G_{zoh}(z)$.
4. Derive the discrete-time closed-loop transfer function and show it is given by:

$$G_{cl}(z) = \frac{C(z)G_{zoh}(z)}{1 + C(z)G_{zoh}(z)}$$

5. A possibility for determining the controller $C(z)$ directly in discrete-time is to start from a desired discrete-time closed-loop transfer function $G_{ref}(z)$ and to solve the closed-loop transfer function for the controller transfer function. Show that this so-called direct design method leads to

$$G_{cl}(z) = G_{ref}(z) \Rightarrow C(z) = \frac{1}{G_{zoh}(z)} \times \frac{G_{ref}(z)}{1 - G_{ref}(z)} \tag{3}$$

   It should be noted that the sampled plant model $G_{zoh}(z)$ appears in $C(z)$ through its inverse; this feature is one of the characteristics of the direct design method. This could induce a number of problems as the controller poles will be the sampled model plant zeros and vice versa. It will be necessary to check whether this controller is stable and physically realizable. This design method should be carefully used for plant having unstable poles or zeros (which is not the case here).

6. To compute the digital controller from (3), we need to determine a discrete-time version $G_{ref}(z)$ of the desired continuous-time closed-loop transfer function $G_{ref}(s)$. As the setpoint is piecewise constant, the ZOH discretization method is chosen so that the desired performance requirements are retained. Assuming that the sampled (ZOH) transfer function $G_{ref}(z)$ of $G_{ref}(s)$ for a given sampling period T_s takes the form:

$$G_{ref}(z) = \frac{\beta_1 z + \beta_0}{z^2 + \alpha_1 z + \alpha_0}$$

   Show that the digital controller $C(z)$ takes the following form:

$$C(z) = \frac{[\beta_1 z^2 + (\beta_0 + a_1\beta_1)z + a_1\beta_0]/b_1}{z^2 + (\alpha_1 - \beta_1)z + (\alpha_0 - \beta_0)}$$

7. Determine the difference equation of the digital controller output $u(k)$ as a function of α_0, α_1, β_0, β_1, a_1, b_1, $u(k-1)$, $u(k-2)$, $\varepsilon(k)$, $\varepsilon(k-1)$ and $\varepsilon(k-2)$ that results from this direct design method.
8. What sampling period T_s value would you recommend to determine and implement the digital controller ? Explain your choice.
9. For the chosen T_s, by using the mapping

$$z_{1,2} = e^{s_{1,2}T_s}$$

   Calculate the z-plane poles of the discrete-time closed-loop transfer function. Is the digital closed-loop system stable ?

10. Use Matlab to compute the digital controller parameters and verify that the controller is stable.

English to French translation

bandwidth	:	bande passante
closed-loop system	:	système bouclé
cut-off frequency	:	fréquence (ou pulsation) de coupure
damped frequency	:	pulsation amortie
damping ratio	:	coefficient d'amortissement
feedback	:	contre-réaction
feedback system	:	système à contre-réaction
impulse response	:	réponse impulsionnelle
integral wind-up	:	emballement du terme intégral
impulse response	:	réponse impulsionnelle
input	:	entrée
gain	:	gain
linear time-invariant (LTI)	:	linéaire invariant dans le temps
output	:	sortie
overshoot	:	dépassement
rise time	:	time de montée
root locus	:	lieu des racines
sampled	:	échantillonné
sampled systems	:	systèmes échantillonnés
setpoint	:	consigne
settling time	:	temps de réponse
steady-state gain	:	gain statique
steady-state response	:	réponse en régime permanent
step response	:	réponse indicielle
time-invariant	:	invariant dans le temps
transient response	:	réponse transitoire
undamped natural frequency	:	pulsation propre non amortie
zero-order holder	:	bloqueur d'ordre zéro

Appendix

Table of common unilateral Laplace and z-transforms

The table below provides a number of Laplace and z-transform pairs and their region of convergence (ROC).
Remarks:

$f(t)$	$F(s)$	$f(k)$	$F(z)$	ROC
$\delta(t)$	1	$\delta(k)$	1	All z
$\delta(t-t_0)$	e^{-st_0}	$\delta(k-i)$	z^{-i}	$z \neq 0$
$\Gamma(t)$	$\frac{1}{s}$	$\Gamma(k)$	$\frac{z}{z-1}$	$\|z\| > 1$
$t\Gamma(t)$	$\frac{1}{s^2}$	$kT_s\Gamma(k)$	$\frac{zT_s}{(z-1)^2}$	$\|z\| > 1$
		$a^k\Gamma(k)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
$e^{-at}\Gamma(t)$	$\frac{1}{s+a}$	$e^{-akT_s}\Gamma(k)$	$\frac{z}{z-e^{-aT_s}}$	$\|z\| > \|a\|$
$te^{-at}\Gamma(t)$	$\frac{1}{(s+a)^2}$	$kT_se^{-akT_s}\Gamma(k)$	$\frac{T_se^{-aT_s}z}{(z-e^{-aT_s})^2}$	$\|z\| > \|a\|$
$\cos(\omega_0 t)\Gamma(t)$	$\frac{s}{s^2+\omega_0^2}$	$\cos(\omega_0 kT_s)\Gamma(k)$	$\frac{z(z-\cos(\omega_0 T_s))}{z^2-2\cos(\omega_0 T_s)z+1}$	$\|z\| > \|1\|$
$\sin(\omega_0 t)\Gamma(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\sin(\omega_0 kT_s)\Gamma(k)$	$\frac{\sin(\omega_0 T_s)z}{z^2-2\cos(\omega_0 T_s)z+1}$	$\|z\| > \|1\|$

• The notation for z found in the table above may differ from that found in other tables. For example, the basic z-transform of a unit discrete-time step $\Gamma(k)$ can be written as either of the following two expressions, which are equivalent:

$$Z(\Gamma(k)) = \Gamma(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

• The ROC for a given sequence $x(k)$, is defined as the range of z for which the z-transform converges. Since the z-transform is a power series, it converges when $x(k)z^{-k}$ is absolutely summable. Stated differently,

$$\sum_{k=0}^{+\infty} |x(k)z^{-k}| < \infty \quad \text{must be satisfied for convergence.}$$

Property 1. if $x(k)$ is of finite duration, then the ROC is the entire z-plane, except possibly $z = 0$ and/or $z = \infty$.

Property 2. The ROC does not contain any poles.

Useful properties of geometric series

finite sum of geometric series	$\sum_{k=0}^{N} q^k = N+1$	if $q = 1$
	$\sum_{k=0}^{N} q^k = \frac{1-q^{N+1}}{1-q}$	if $q \neq 1$
infinite sum of geometric series	$\lim_{N\to+\infty} \sum_{k=0}^{N} q^k = \frac{1}{1-q}$	if $\|q\| < 1$

Useful properties of the unilateral z-transform

Some useful properties which have found practical use are summarized below.

Property	signal	z-transform
linearity	$ax(k) + by(k)$	$aX(z) + bY(z)$
time-delay	$x(k-1)$	$z^{-1}X(z)$ if $x(k)$ is causal[1]
	$x(k-i)$	$z^{-i}X(z)$ if $x(k)$ is causal
	$x(k-1)$	$z^{-1}X(z) + z^{-2}x(-1)$ if $x(k)$ is non-causal
	$x(k-i)$	$z^{-i}X(z) + \sum_{j=-i}^{-1} z^{j-i}x(j)$ if $x(k)$ is non-causal
time-advance	$x(k+1)$	$zX(z) - zx(0)$
	$x(k+2)$	$z^2X(z) - z^2x(0) - zx(1)$
	$x(k+i)$	$z^iX(z) - \sum_{j=0}^{i-1} z^{i-j}x(j)$
convolution	$y(k) = h(k) * u(k)$	$Y(z) = H(z)U(z)$
	$y(k) = \sum_{i=-\infty}^{+\infty} h(i)u(k-i)$	
differentiation	$kx(k)$	$-z\frac{dX(z)}{dz}$
accumulation	$\sum_{i=0}^{k} x(i)$	$\frac{1}{1-z^{-1}}X(z)$
initial value theorem	if $x(k) = 0$ for $k < 0$	$x(0) = \lim_{z\to+\infty} X(z)$
final value theorem	$\lim_{k\to+\infty} x(k) = \lim_{z\to 1}(z-1)X(z)$	if the limit exists

[1] A signal is causal if $x(k) = 0$, for all $k < 0$

Sampled transfer functions of continuous-time systems

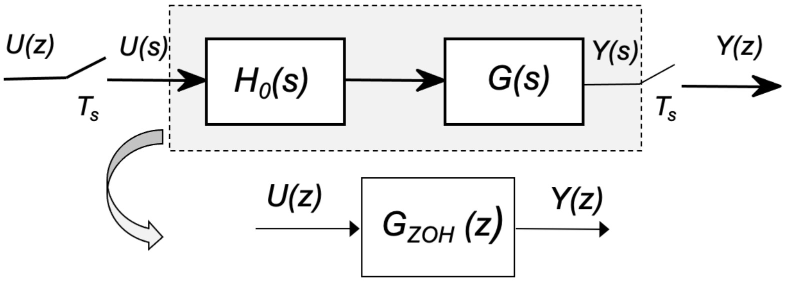

Figure A.1: Sampled system

Consider a Zero-Order Hold (ZOH) $H_0(s)$ cascaded with a continuous-time transfer function $G(s)$ and a sampler as shown in Figure A.1, the corresponding sampled transfer function is given by:

$$G_{ZOH}(z) = \frac{Y(z)}{U(z)} = Z\Big(H_0(s)G(s)\Big) = \left(1 - z^{-1}\right) Z\left(\frac{G(s)}{s}\right) = \left(\frac{z-1}{z}\right) Z\left(\frac{G(s)}{s}\right)$$

First-order systems
Consider a continuous-time first order system given by its Laplace transfer function:

$$G(s) = \frac{b}{s+a}$$

Its equivalent ZOH sampled transfer function is given by:

$$G_{ZOH}(z) = Z\Big(H_0(s)G(s)\Big) = \left(1 - z^{-1}\right) Z\left(\frac{b}{s(s+a)}\right) = \frac{b_1 z^{-1}}{1 + a_1 z^{-1}}$$

where

$$\begin{cases} a_1 = -e^{-aT_s} \\ b_1 = \dfrac{b}{a}(1 + a_1) \end{cases}$$

and T_s denotes the sampling period.

Second-order systems
Consider a continuous-time second order system given by its Laplace transfer function:

$$G(s) = \frac{a \times b}{(s+a)(s+b)}$$

Its equivalent ZOH sampled transfer function is given by:

$$G_{ZOH}(z) = Z\Big(H_0(s)G(s)\Big) = \left(1 - z^{-1}\right) Z\left(\frac{a \times b}{s(s+a)(s+b)}\right)$$

$$G_{ZOH}(z) = \frac{b_1 z^{-1} + b_2 z^{-2}}{(1 + \alpha z^{-1})(1 + \beta z^{-1})}$$

where

$$\begin{cases} \alpha = -e^{-aT_s} \\ \beta = -e^{-bT_s} \\ b_1 = \dfrac{b \times \alpha - a \times \beta}{b - a} + 1 \\ b_2 = \dfrac{b \times \beta - a \times \alpha}{b - a} + \alpha \times \beta \end{cases}$$

and T_s denotes the sampling period.

PID tuning by using empirical rules

A PID controller defined in its so-called *ideal* form is defined as:

$$C(s) = K_p \left(1 + \frac{1}{T_i s} + \frac{T_d s}{1 + \frac{T_d}{N} s} \right) = K_p + K_i \frac{1}{s} + K_d \frac{s}{1 + \frac{T_d}{N} s} \tag{1}$$

The individual effects of these three K_p, K_i and K_d parameters appearing in (1) on the closed-loop performance of stable plants are recalled in Table 1 below.

TABLE 1 Effects of independent P, I, and D tuning on closed-loop response. For example, while K_I and K_D are fixed, increasing K_P alone can decrease rise time, increase overshoot, slightly increase settling time, decrease the steady-state error, and decrease stability margins.

	Rise Time	Overshoot	Settling Time	Steady-State Error	Stability
Increasing K_P	Decrease	Increase	Small Increase	Decrease	Degrade
Increasing K_I	Small Decrease	Increase	Increase	Large Decrease	Degrade
Increasing K_D	Small Decrease	Decrease	Decrease	Minor Change	Improve

Tuning of the PID controller by using the Ziegler-Nichols rules

Assume a continuous-time process is reasonably well modeled by the first-order plus time-delay transfer function model:

$$G(s) = \frac{Ke^{-\tau s}}{1 + Ts}$$

Several tuning rules are available for determining the PID controller parameter values. The proposed rules address different design specifications such as load disturbance rejection or setpoint tracking.
The most popular tuning rules are those attributed to Ziegler-Nichols. Their aim is to provide *satisfactory load disturbance rejection.*
Table 2 shows the Ziegler-Nichols rules for P, PI, or PID controllers defined in its so-called *ideal* form (see (1)).

Controller type	K_p	T_i	T_d
P	$\frac{T}{K\tau}$		
PI	$0.9\frac{T}{K\tau}$	3τ	
PID	$1.2\frac{T}{K\tau}$	2τ	0.5τ

Table 2: Ziegler-Nichols disturbance rejection tuning rules for a first-order-plus delay model determined from an open-loop step response