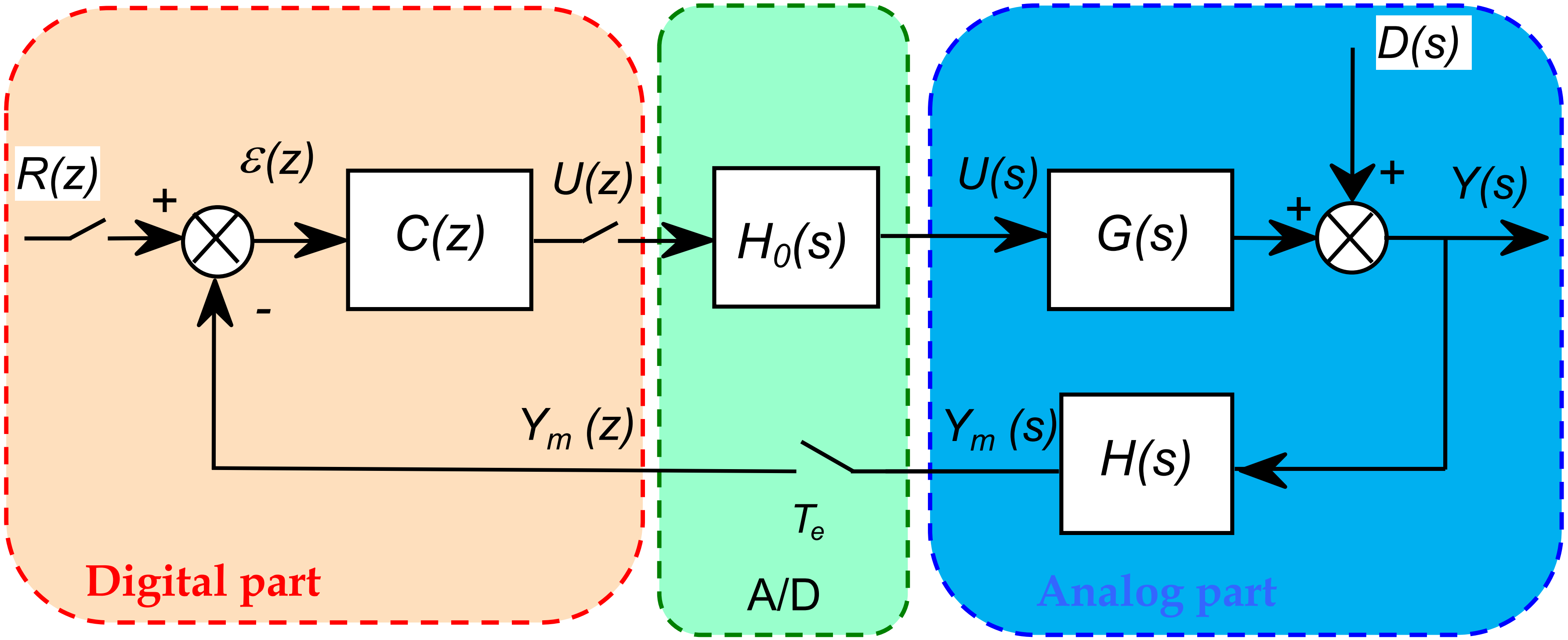
Discrete-time systems

Hugues GARNIER

hugues.garnier@univ-lorraine.fr

The video of the day

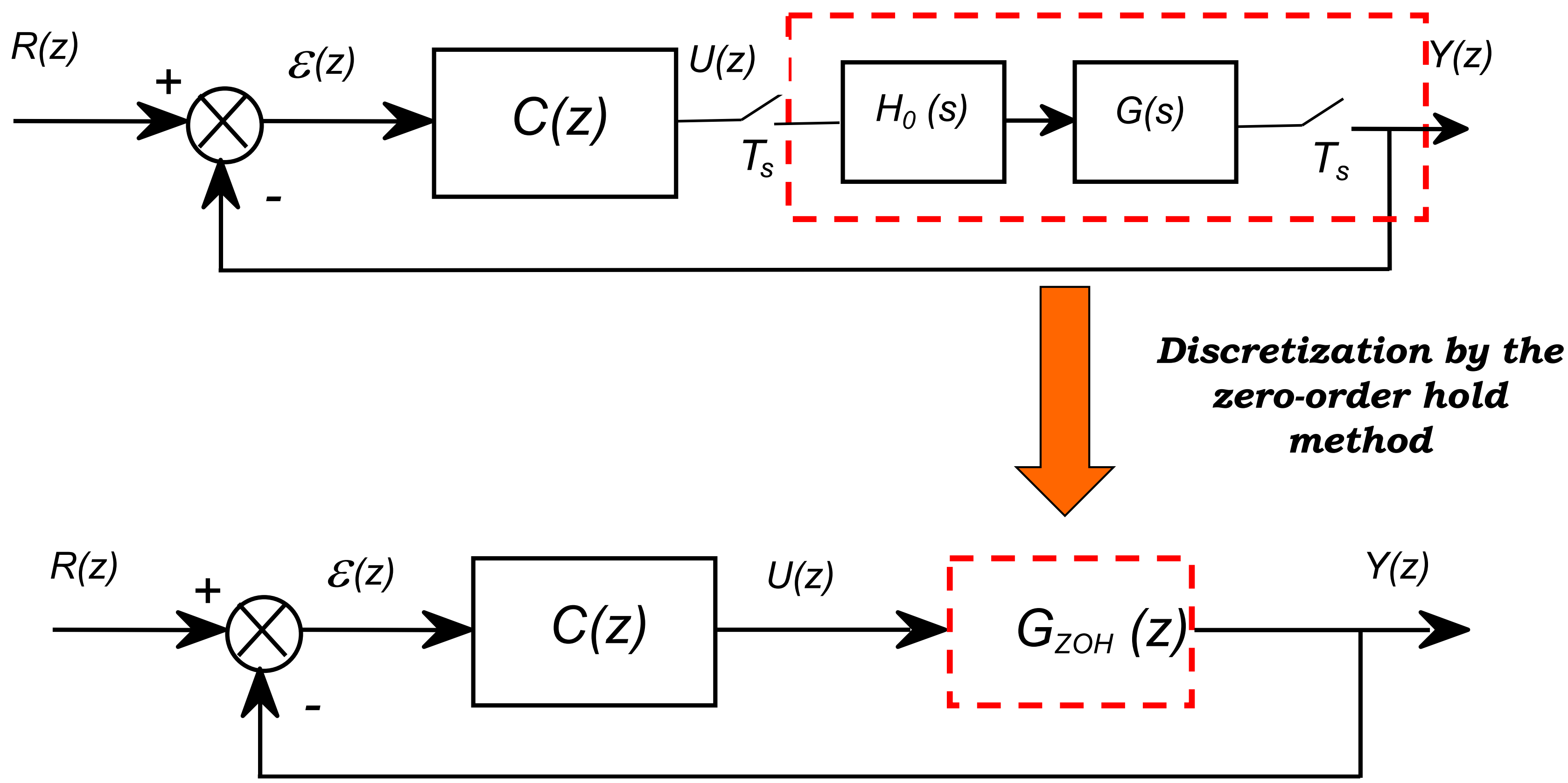
Digital control block-diagram



- Need for blocks to make analog and digital parts interact:

D/A & A/D

Digital control block-diagram



Discrete-time signals and systems
Why study them for digital control?

- Digital control
  - Actuator - Process – Sensor = continuous-time system
    - Control and output signals: continuous-time signals
  - Controller = discrete-time system
    - Input and output signals: discrete-time signals
- Tools required
  - Tools for modeling discrete-time signals and systems: sampler, holds, digital controller, sampled systems...
  - Discretization tools/methods for
    - switching from a continuous-time to a sampled model
    - digital simulation of the system response and/or the discrete-time controller design requires prior discretization of the continuous-time system
    - if an analog controller has already been designed, its digital implementation requires discretization

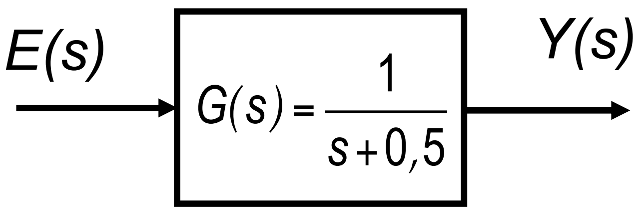
Reminders – The various tools for analyzing continuous-time signals and linear systems

- Continuous-time signal $y(t)$
- Fourier Transform (TFtc) $Y(f)=\int_{-\infty}^{+\infty} y(t)e^{-j2\pi ft}dt$
- Laplace transform $Y(s)=\int_{0}^{+\infty} y(t)e^{-st}dt$
- Continuous system response

$$y(t)=g(t)*e(t)$$
$$Y(f)=G(f)\times E(f)$$
$$Y(s)=G(s)\times E(s)$$

- Example

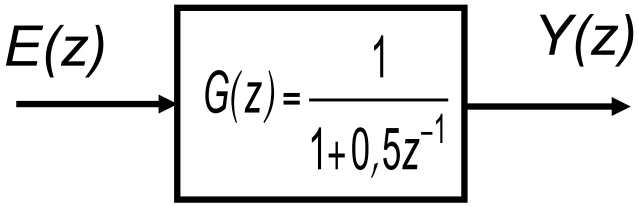
$$\dot{y}(t)+0,5y(t)=e(t)$$
$$\left(s+0,5\right)Y(s)=E(s)$$



The various tools for analyzing of *discrete-time* signals and linear systems

- Discrete-time signal $y(k)$
- Fourier Transform (TFtd) $Y(f)=\sum_{k=-\infty}^{+\infty} y(k)e^{-j2\pi fkT_e}$
- *z-transform* $Y(z)=\sum_{k=0}^{+\infty} y(k)z^{-k}$
- Digital system response $y(k)=g(k)*e(k)$
  $Y(f)=G(f)\times E(f)$
  $Y(z)=G(z)\times E(z)$
- Example

$$y(k)+0,5y(k-1)=e(k)$$
$$\left(1+0,5z^{-1}\right)Y(z)=E(z)$$



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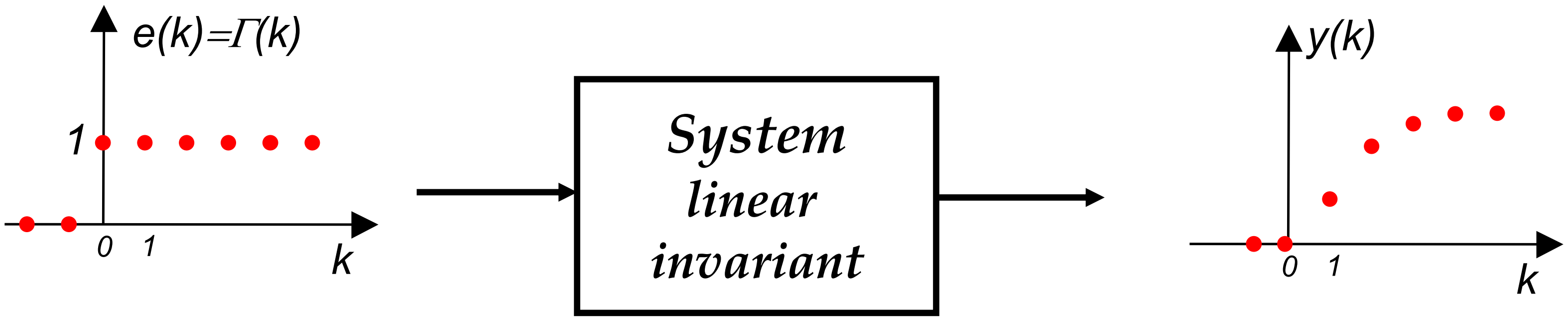
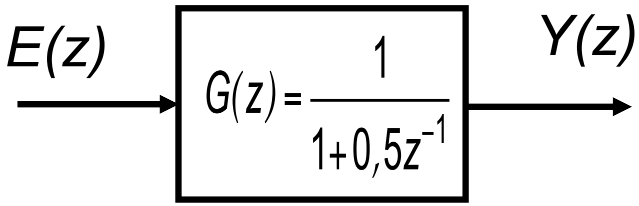
Discrete-time model of a linear invariant system

- An excitation (e.g. $\Gamma(k)$) is sent to the input of a system. We measure the sampled output $y(k)$



What is the form of the discrete-time model?

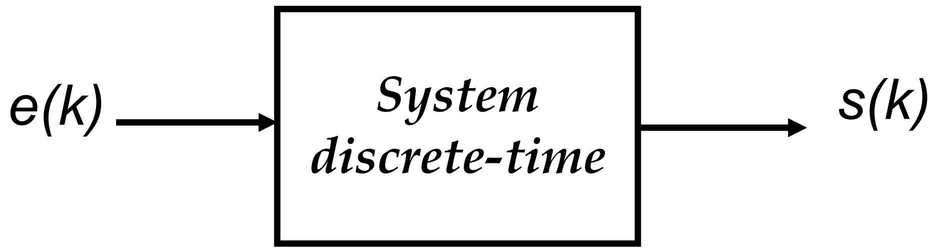


$$y(k)+0{,}5y(k-1)=e(k)$$

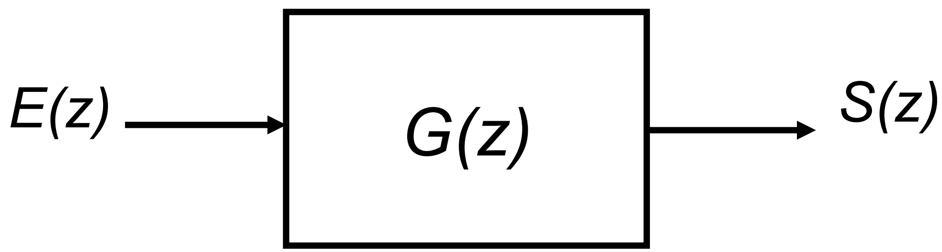
$$\left(1+0{,}5z^{-1}\right)Y(z)=E(z)$$

Discrete-time system

- A discrete-time system is defined as an operator between ***two discrete-time signals***

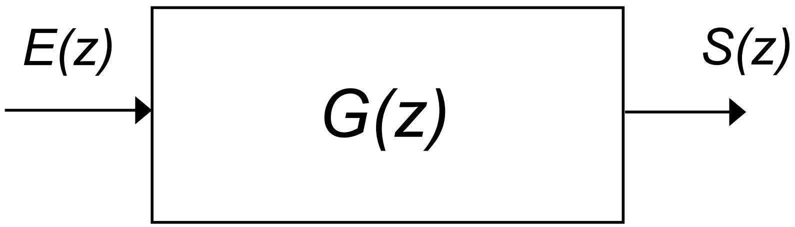


- The mathematical tool used to facilitate its analysis is the ***z-transform***



Block diagram

*In **digital control**, a system is represented by a block diagram that links the z-transform of the input E(z) to the z-transform of the output S(z) via its transfer function G(z).*



From the block-diagram, we can deduce the following relationships

$$S(z) = G(z)E(z)$$

ou

$$G(z) = \frac{S(z)}{E(z)}$$

Description of a discrete-time linear system

- A discrete-time linear system can be described by :
  - a convolution product
  - a difference equation
  - its transfer function in z

- ***The mathematical tool*** used to facilitate the analysis of discrete-time linear systems is the ***z-transform***

Convolution product

- The impulse response *g(k) is* used to calculate the filter output *s(k)* for any input *e(k)* via the ***discrete-time convolution product***

$$s(k) = g(k) * e(k) = \sum_{i=-\infty}^{+\infty} g(i)\, e(k-i) = \sum_{i=-\infty}^{+\infty} g(k-i)\, e(i)$$

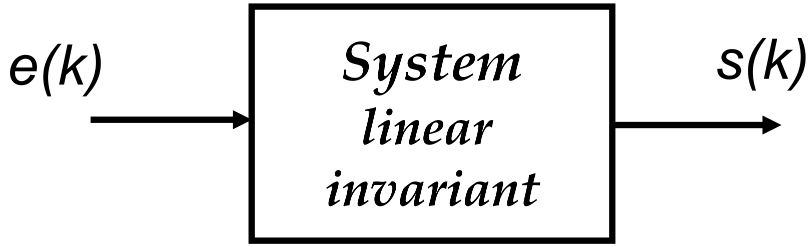
- If the filter is causal: *g(k)=0* for all *k < 0*

$$s(k) = g(k) * e(k) = \sum_{i=0}^{+\infty} g(i)\, e(k-i)$$

$$Z(s(k)) = Z(g(k) * e(k))$$
$$S(z) = G(z) \times E(z)$$

Difference equation

- A linear time-invariant ***discrete-time*** system having an input *e(k)* and an output *s(k)* is described by a ***difference equation*** with constant coefficients:



$$a_0 s(k) + a_1 s(k-1) + \ldots + a_{n_a} s(k-n_a) = b_0 e(k) + b_1 e(k-1) + \ldots + b_{n_b} e(k-n_b)$$

Transfer function in z

Let the difference equation express the relationship between the input signal *e(k)* and the output signal *s(k)* of a discrete-time system:

$$a_0 s(k) + a_1 s(k-1) + \dots + a_{n_a} s(k-n_a) = b_0 e(k) + b_1 e(k-1) + \dots + b_{n_b} e(k-n_b)$$

By applying the z-transform to the 2 members of the equation and using :

$$Z\big(x(k-i)\big) = z^{-i} X(z)$$

$$\left(a_0 + a_1 z^{-1} + \dots + a_{n_a} z^{-n_a}\right) S(z) = \left(b_0 + b_1 z^{-1} + \dots + b_{n_b} z^{-n_b}\right) E(z)$$

$$G(z) = \frac{S(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_{n_b} z^{-n_b}}{a_0 + a_1 z^{-1} + \dots + a_{n_a} z^{-n_a}} = \frac{b_0 z^{n_a} + b_1 z^{n_a-1} + \dots + b_{n_b} z^{n_a-n_b}}{a_0 z^{n_a} + a_1 z^{n_a-1} + \dots + a_{n_a}}$$

n_a : filter order

Steady-state gain of a discrete-time system

- Consider a linear system described by the transfer function :

$$G(z)=\frac{S(z)}{E(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\ldots+b_{n_b}z^{-n_b}}{a_0+a_1z^{-1}+a_2z^{-2}+\ldots+a_{n_a}z^{-n_a}}$$

- Definition
  - *If G(z) is known, **the steady-state gain** of a discrete-time system is the value of G(z) for z=1*

$$K=\lim_{z\to 1}G(z)$$

  - If we have recorded the step response of a *stable system*, we also have:

$$K=\frac{\lim_{k\to+\infty} s(k)-s(0)}{\lim_{k\to+\infty} e(k)-e(0)}$$

z-transfer function - Example

$$s(k)-0,8s(k-1)=0,2e(k)$$

$$G(z)=? \qquad K=? \qquad n_a=?$$

$$\left(1-0,8z^{-1}\right)S(z)=0,2\ E(z)$$

$$G(z)=\frac{S(z)}{E(z)}=\frac{0,2}{1-0,8z^{-1}}=\frac{0,2z}{z-0,8}$$

system order: $n_a=1$

$$K=\lim_{z\to 1}G(z)=\frac{0,2}{1-0,8}=1$$

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Determining the difference equation from the knowledge of the z-transfer function

- Consider the z-transfer function *G(z)* of a digital system

$$G(z)=\frac{0{,}2z}{z-0{,}8}$$ How can we determine its difference equation?

G(z) is expressed **in negative power** of *z*

$$G(z)=\frac{0{,}2z}{z-0{,}8}\times\frac{z^{-1}}{z^{-1}}=\frac{0{,}2}{1-0{,}8z^{-1}}$$

$$G(z)=\frac{S(z)}{E(z)}$$

$$\frac{S(z)}{E(z)}=\frac{0{,}2}{1-0{,}8z^{-1}}$$

$$(1-0{,}8z^{-1})S(z)=0{,}2E(z)$$

$$S(z)-0{,}8z^{-1}S(z)=0{,}2E(z)$$

$$s(k)-0{,}8s(k-1)=0{,}2e(k)$$

car $Z^{-1}\left(z^{-i}\,S(z)\right)=s(k-i)$ ici $i=1$

General form of a discrete transfer function

$$G(z) = \frac{K}{(1-z)^m} z^{-r} \frac{N(z)}{D(z)}$$

$$K = \lim_{z \to 1} (1-z)^m G(z)$$

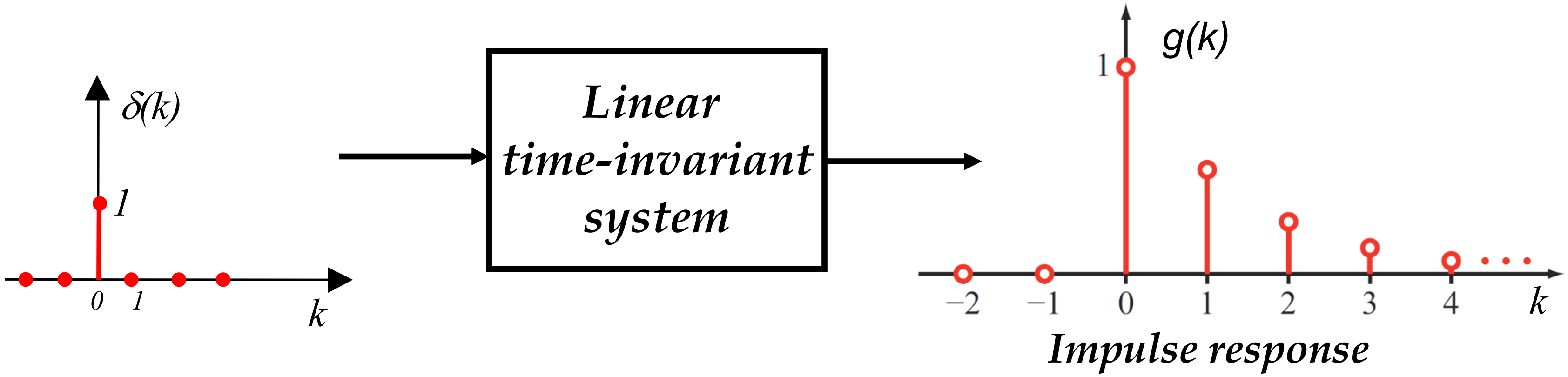
- The general form of a discrete transfer function makes it easy to visualize :
  - K gain (= steady-state gain if *m=0, no pure integrator)*
  - the presence of pure integrators (pole at $z = 1$) of order m
  - number of samples for pure delay (pole $z = 0$ of order r)

Analysis of a discrete-time linear system

- The characteristics of a linear discrete-time system are classically analyzed via:
  - its impulse response
  - its step response
  - its frequency response
  - its diagram of poles and zeros

Impulse response

- It corresponds to the response *g(k)* obtained when a Kronecker pulse *δ(k) is* sent to the input.



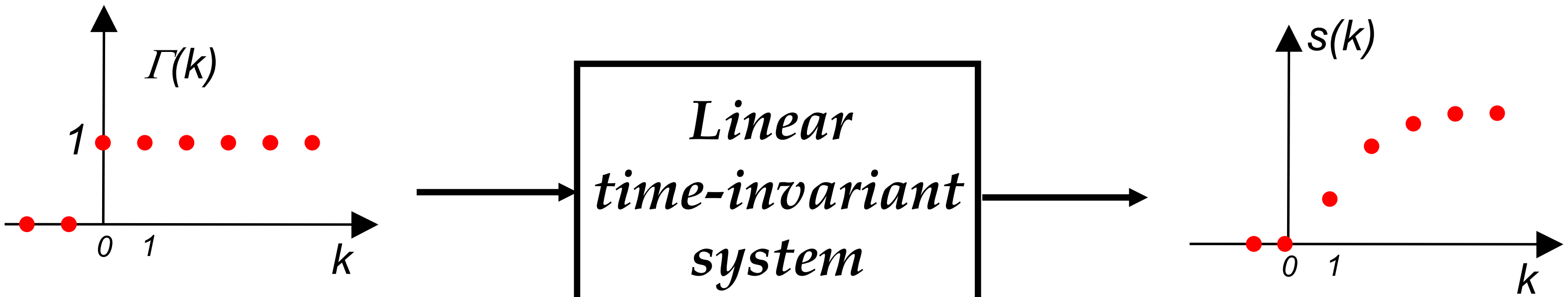
- If *g(k)=0* for *k<0,* the system is ***causal***

$$\mathcal{Z}(s(k)) = \mathcal{Z}(g(k) * \delta(k))$$
$$S(z) = G(z) \times 1$$
$$S(z) = G(z)$$
$$s(k) = g(k)$$

$$Z(\delta(k)) = 1$$

Step response

- It corresponds to the response obtained when a step $\Gamma(k)$ is sent to the system



Step

Step response

$$Z\left(\Gamma(k)\right) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$Z\left(s(k)\right) = Z\left(g(k) * \Gamma(k)\right)$$

$$S(z) = G(z) \times \frac{z}{z-1}$$

Frequency response

- If we know *G(z)*

$$\left.\begin{array}{l} z = e^{sT_e} \\ s = j\omega = j2\pi f \end{array}\right\} \Rightarrow \quad z = e^{j2\pi f T_e}$$

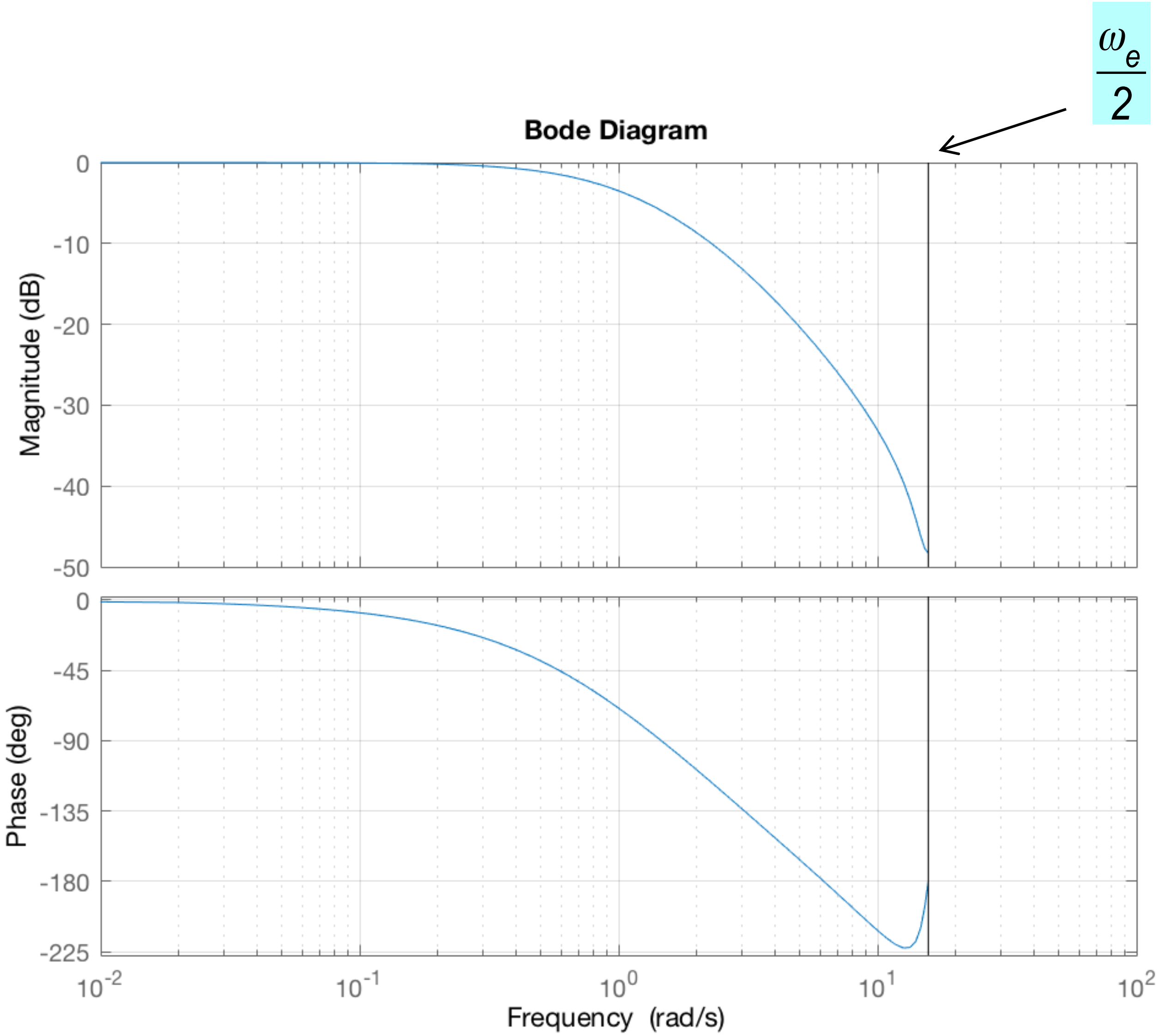
$$G(f) = G(z)\Big|_{z=e^{j2\pi f T_e}} = \left|G(f)\right| e^{j\varphi(f)}$$

The frequency response can be plotted in amplitude and phase from

$$\left|G(f)\right| = \left|\frac{Y(f)}{E(f)}\right| \qquad \varphi(f) = Arg\left(G(f)\right)$$

- Frequency response characteristics of discrete-time models
  - ***They are periodic with "period" f_s***
    - ***The analysis and plot are limited to the frequency range [0 ; f_s /2].***
  - ***No particular slopes in the Bode diagram***

Bode diagram of a discrete-time model
Example



Causality of a discrete-time system

Important concept for real-time implementation

- A system is causal if its output at any instant depends only on the values of the input at present and past instants.

- Examples
  - Causal system $u(k) = 0.7\, u(k-1) + 0.3\, \varepsilon(k)$
  - Non-causal system $u(k) = 0.5u(k-1) + 0.2\, \varepsilon(k) + 0.1\, \varepsilon(k+1)$

- The response of a causal system varies only after the input appears ($k \geq 0$) and is zero for $k<0$.

In particular, the impulse response $g(k) = 0$ for all $k < 0$

- Convolution product for a causal system

$$y(k) = \sum_{i=-\infty}^{+\infty} g(i)u(k-i) = \sum_{i=0}^{+\infty} g(i)u(k-i)$$

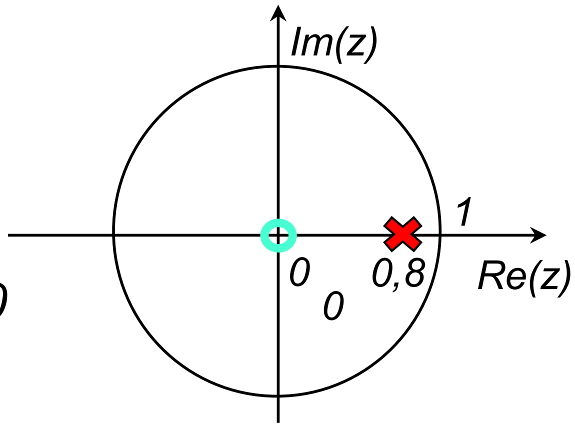
Pole-zero diagram

- Given a transfer function

$$G(z)=\frac{B(z)}{A(z)}=C\frac{\prod_{j=1}^{M}(z-z_j)}{\prod_{i=1}^{N}(z-p_i)}$$

- Definitions

  - ○ ***Zeros*** $\mathbf{z_j}$ are the roots of the numerator $B(z)=0$
  - ✖ ***Poles*** $\mathbf{p_i}$ are the roots of the denominator $A(z)=0$



Pole-zero diagram

- Example

$$G(z)=\frac{B(z)}{A(z)}=\frac{0,2}{1-0,8z^{-1}}=\frac{0,2z}{z-0,8}$$

Always write G(z) in positive power of z to determine poles and zeros

Stability of a discrete-time system

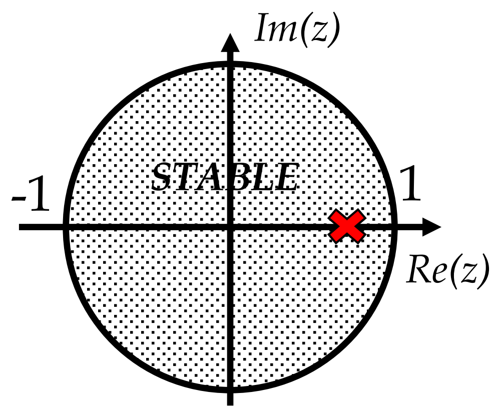
- If we know the z-transfer function of the discrete-time system

$$G(z) = C\frac{\prod_{j=1}^{M}(z-z_j)}{\prod_{i=1}^{N}(z-p_i)}$$

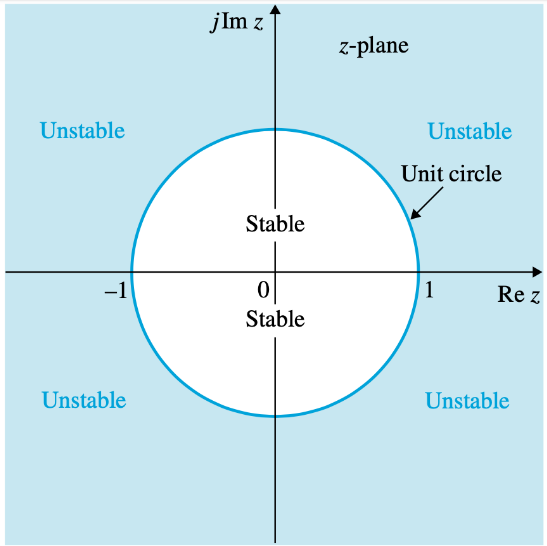
The discrete-time system is stable if all of its poles p_i have a complex modulus less than 1, i.e. if they ***lie inside the unit circle***

$$|p_i| < 1$$

Example $G(z) = \frac{0,2z}{z-0,8}$



Stability of discrete-time systems
Examples



$$M(z) = \frac{5z}{(z-0.2)(z-0.8)}$$ Stable system

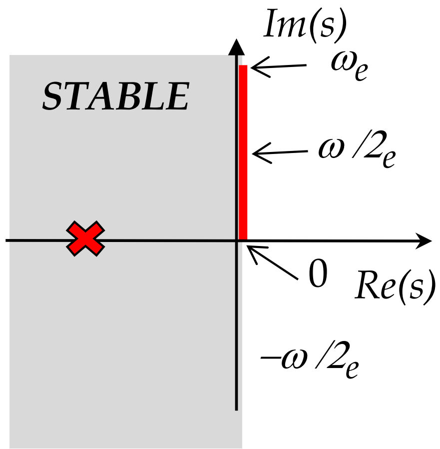
$$M(z) = \frac{5z}{(z+1.2)(z-0.8)}$$ Unstable system due to the pole at $z = -1.2$

$$M(z) = \frac{5(z+1)}{z(z-1)(z-0.8)}$$ Marginally stable due to $z = 1$

$$M(z) = \frac{5(z+1.2)}{z^2(z+1)^2(z+0.1)}$$ Unstable due to second-order pole at $z = -1$

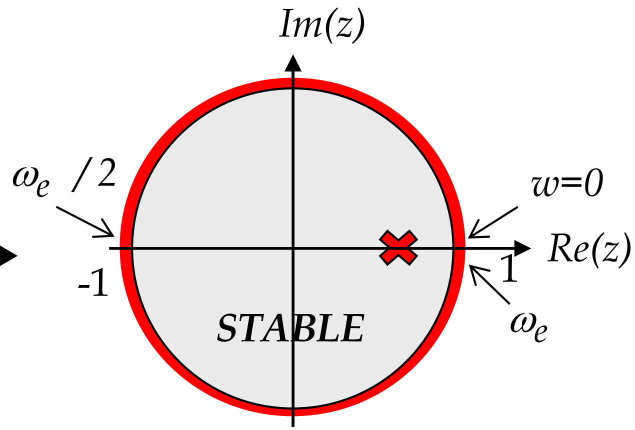
Stability conditions
continuous-time systems/ discrete-time systems

Analog systems



$$z = e^{sT_e}$$

Digital systems



$$Re\ (\text{of all poles}) < 0$$

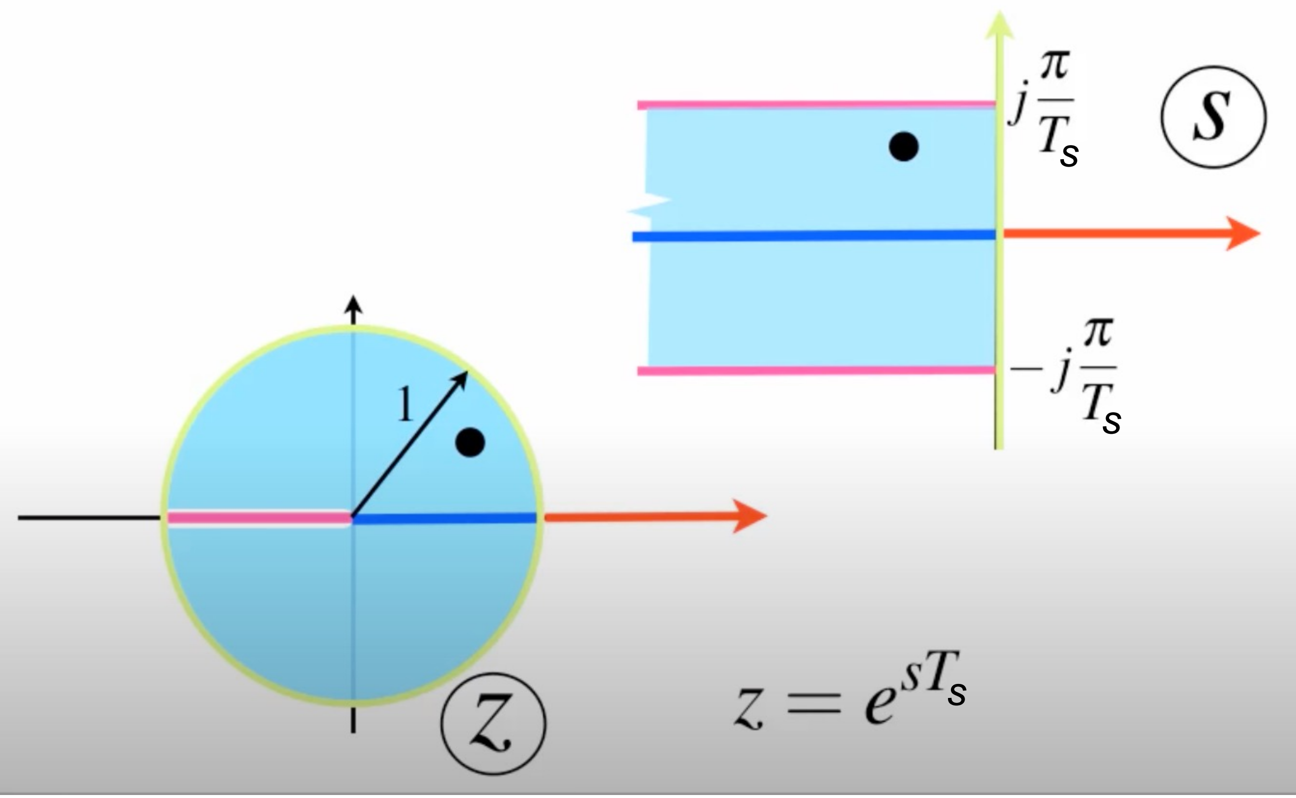
Real part

$$\left| \text{of all poles} \right| < 1$$

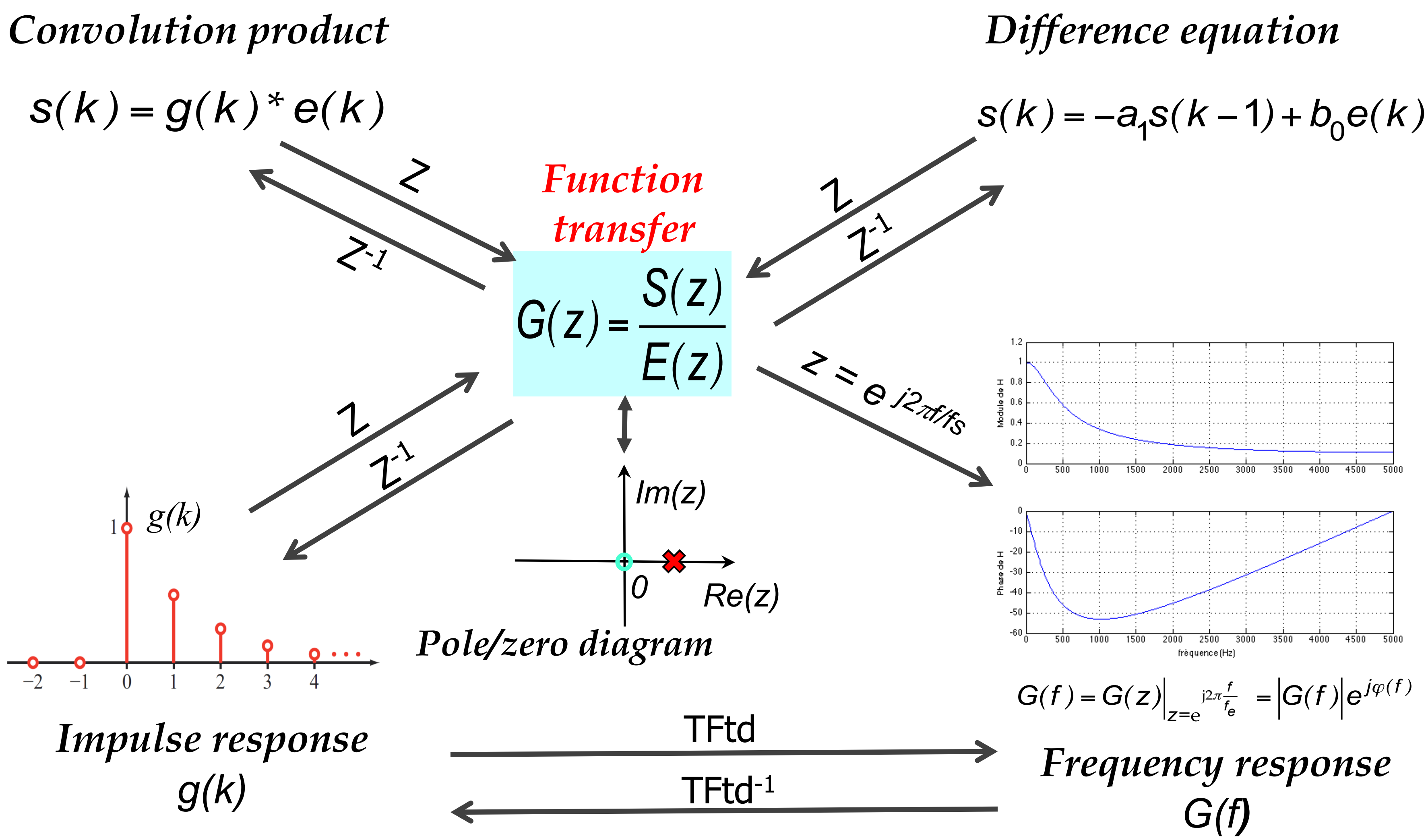
Complex module

Relationship between s-domain poles and z-domain poles

The z-plane

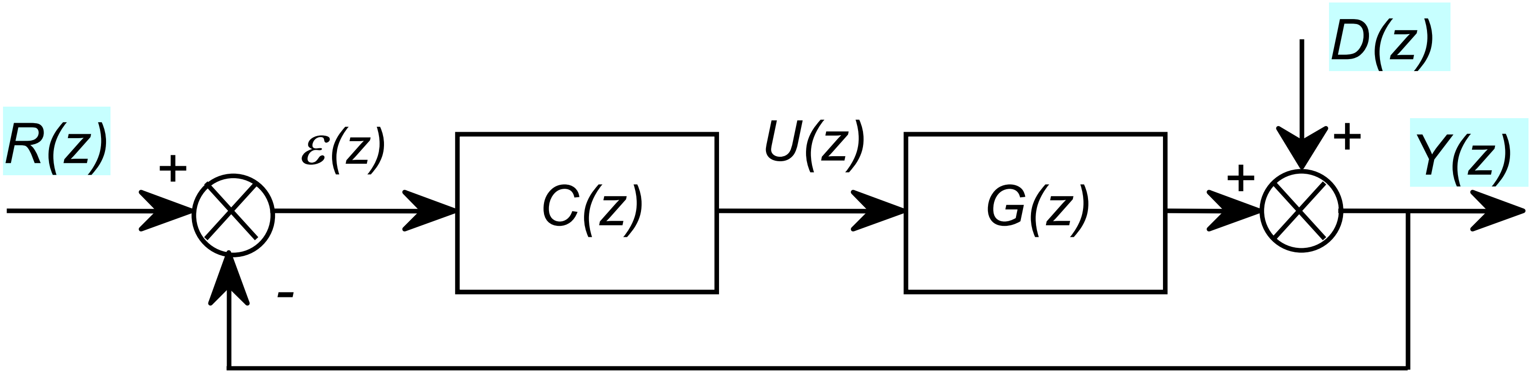


Analysis of a discrete-time system
In a nutshell!



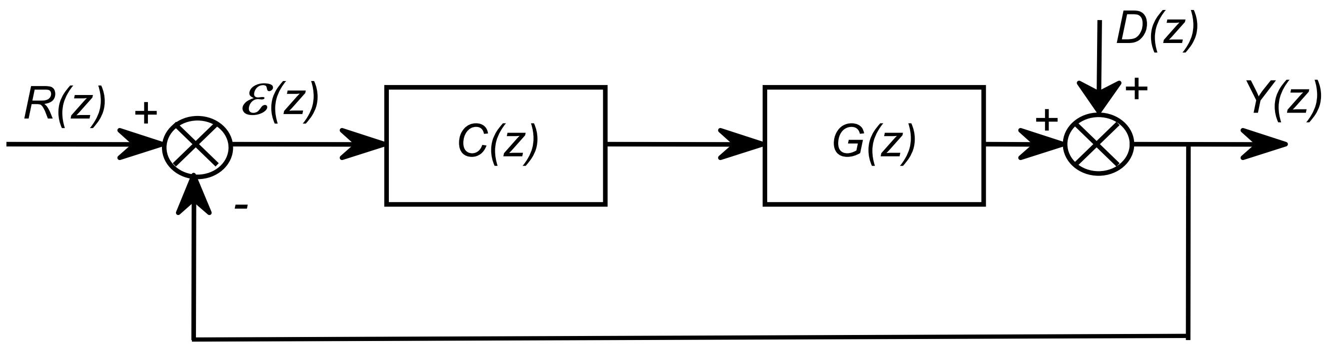
Digital control block-diagram

- The design of a control (and therefore for $C(z)$) using a fully discrete-time approach is based on:
  - a model $G(z)$ of the zero-order hold + actuator + system + sensor + sampler elements
  - the type of external signals: reference $R(z)$, disturbance $D(z)$



$$Y(z) = \frac{C(z)G(z)}{1+C(z)G(z)} R(z) + \frac{1}{1+C(z)G(z)} D(z)$$

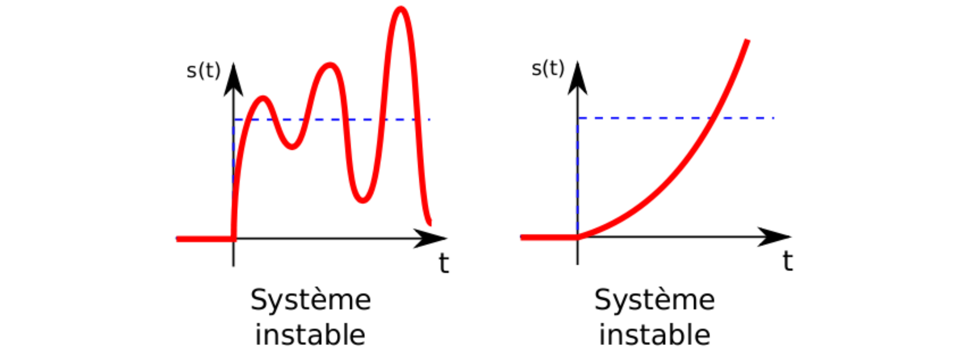
Tools for evaluating the performance of a digital control system



- ***Tools*** for evaluating the closed-loop system performance
  - ***its stability***
  - ***its accuracy***

Stability of a closed-loop system

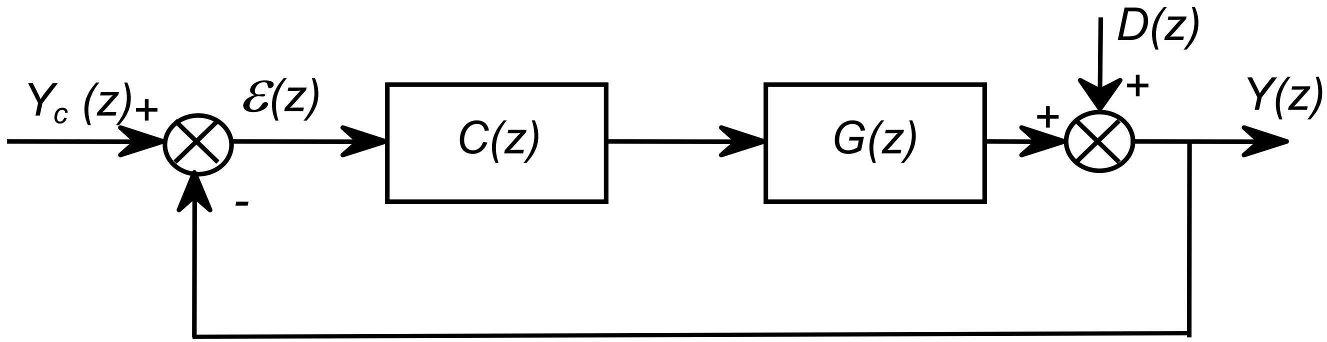
- Poor control design can lead to an unstable closed-loop system!



- *Before examining other performances, we must must guarantee the stability of the closed-loop system with the chosen controller C(z)*

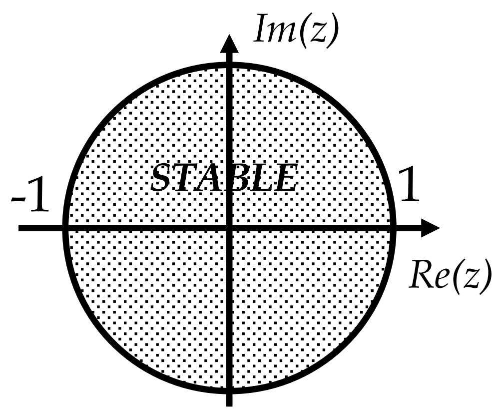
How can you predict the loop stability before closing it?

Tools for analyzing the stability of a closed-loop system



$$Y(z) = F_{BF}(z)Y_c(z) + F_D(z)D(z)$$

- The closed-loop system is stable if all poles p_i of $F_{BF}(z)$ are inside the unit circle



Tools for analyzing the stability of a closed-loop system
The Jury criterion

The Jury criterion is used to determine stability on the basis of the knowledge of the characteristic polynomial without calculating its roots :

Eliahu **Jury**
1923-2020

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

A discrete-time linear system is asymptotically stable if and only if the coefficients of its characteristic polynomial satisfy the following relations. The conditions depend on the order of the system.

We give them only for **n=2** and **n=3**. Higher orders can be generated without difficulty, but are computationally tedious.

Assume that $a_n > 0$. If this is not the case, simply multiply by -1.

$$\mathbf{n=2:} \quad \begin{cases} a_0 + a_1 + a_2 & > 0 \\ a_0 - a_1 + a_2 & > 0 \\ a_2 - a_0 & > 0 \end{cases}$$

$$\mathbf{n=3:} \quad \begin{cases} a_0 + a_1 + a_2 + a_3 & > 0 \\ -a_0 + a_1 - a_2 + a_3 & > 0 \\ a_3 - |a_0| & > 0 \\ a_0 a_2 - a_1 a_3 - a_0^2 + a_3^2 & > 0 \end{cases}$$

Jury criterion - Example

Exemple 3.5 Soit le système dont le polynôme caractéristique s'écrit :

$$P(z) = z^3 + (K - 0.75)z - 0.25$$

L'application du critère de Jury conduit à l'ensemble d'équations :

$$\left\{ \begin{array}{rcll} a_0 + a_1 + a_2 + a_3 & = & K & > 0 \\ -a_0 + a_1 - a_2 + a_3 & = & K + 0.5 & > 0 \\ a_3 - |a_0| & = & 1 - 0.25 & > 0 \\ a_0 a_2 - a_1 a_3 - a_0^2 + a_3^2 & = & -K + 1.6875 & > 0 \end{array} \right.$$

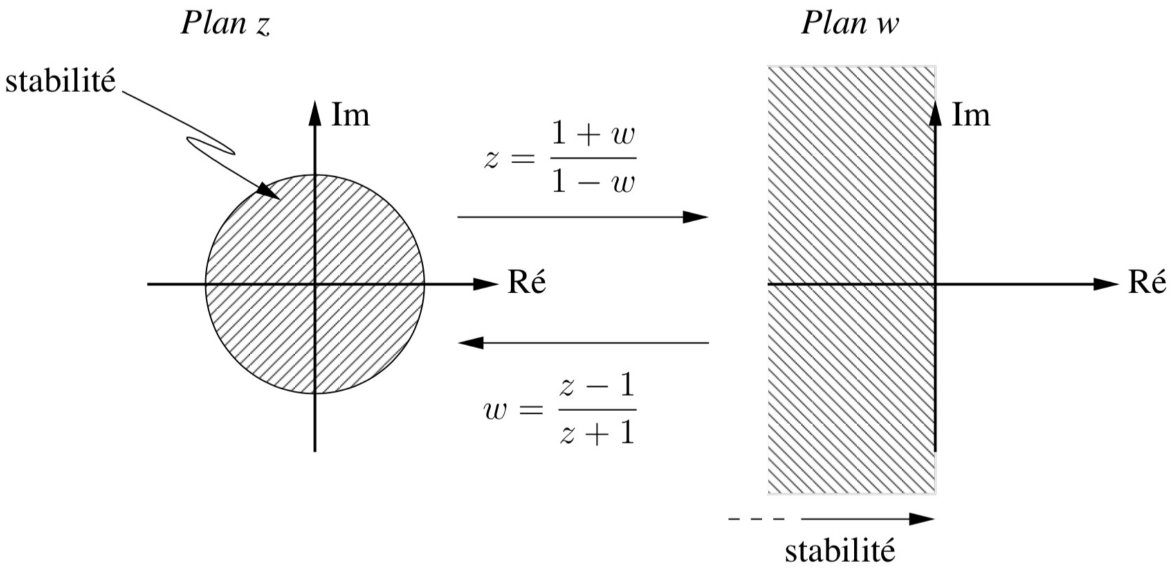
dont l'intersection donne $0 < K < 1.6875$ comme condition de stabilité. $\triangleleft$

Tools for analyzing the stability of a closed-loop system
The adapted Routh-Hurwitz criterion

The ***Routh-Hurwitz*** criterion ***(see S5 Control course)*** is used to assess that the roots of a polynomial belong to the left half-plane. It is therefore not directly applicable to discrete-time systems

However, we can use the algebraic Routh-Hurwitz criterion applied to the polynomial $P(w)$ obtained by applying the bilinear transformation

$$z = \frac{1+w}{1-w}$$



2. Critère algébrique de Routh-Hurwitz (Reminder)

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} \cdots + a_1 s + a_0}$$

P(s)

1. Vérifier que $\forall\ a_i \neq 0$ et $\forall\ a_i$ ont le même signe puis construire le tableau
2. Recopier les coefficients du dénominateur dans les deux 1ères lignes
3. Compléter le tableau selon la règle : $b_{i,j} = \frac{b_{i-1,1} b_{i-2,j+1} - b_{i-1,j+1} b_{i-2,1}}{b_{i-1,1}}$
4. $G(s)$ stable $\Leftrightarrow$ tous les termes de la 1ère colonne sont de même signe.

Le nombre de pôles instables correspond au nombre de changement de signe dans la 1ère colonne.

s^n	a_n	a_{n-2}	a_{n-4}	$\cdots$	a_0
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	0
s^{n-2}	$b_{3,1}$	$b_{3,2}$	$\cdots$		
s^{n-3}	$b_{4,1}$	$b_{4,2}$	$\cdots$		
$\vdots$	$\vdots$				
s^0	a_0				

$$b_{3,1} = \frac{a_{n-1} a_{n-2} - a_{n-3} a_n}{a_{n-1}}, \quad b_{3,2} = \frac{a_{n-1} a_{n-4} - a_{n-5} a_n}{a_{n-1}}, \ldots, b_{4,1} = \frac{b_{3,1} a_{n-3} - b_{3,2} a_{n-1}}{b_{3,1}}, \ldots$$

Routh-Hurwitz criterion - Example

Exemple 3.6 Reprenons l'exemple précédent dont le polynôme caractéristique s'écrit :

$$P(z) = z^3 + (K - 0.75)\,z - 0.25 \qquad z = \frac{1+w}{1-w}$$

Par transformation bilinéaire, il vient le polynôme :

$$w^3(K+0.5) + w^2(3-K) + w(4.5-K) + K$$

La table de Routh correspondante s'écrit :

$$\begin{vmatrix} K+0.5 & 4.5-K & 0 \\ 3-K & K & 0 \\ \frac{-8K+13.5}{3-K} & 0 & \\ K & 0 & \end{vmatrix}$$

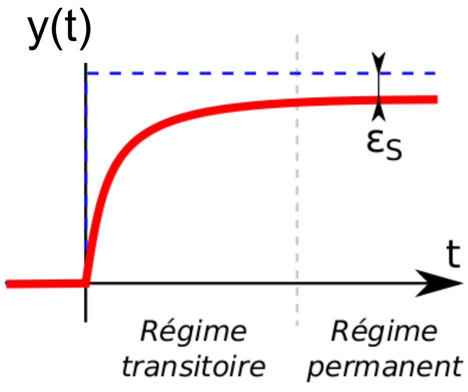
Le polynôme en w aura toutes ses racines à partie réelle négative (et par conséquent, le polynôme $P(z)$ aura ses racines de module inférieur à 1) si tous ses coefficients sont de même signe et si les éléments de la première colonne de la table de Routh sont de même signe. Ceci conduit à la satisfaction simultanée de l'ensemble de conditions suivantes :

$$\begin{cases} K+0.5 & > 0 \\ 3-K & > 0 \\ 4.5-K & > 0 \\ K & > 0 \\ -8K+13.5 & > 0 \end{cases}$$

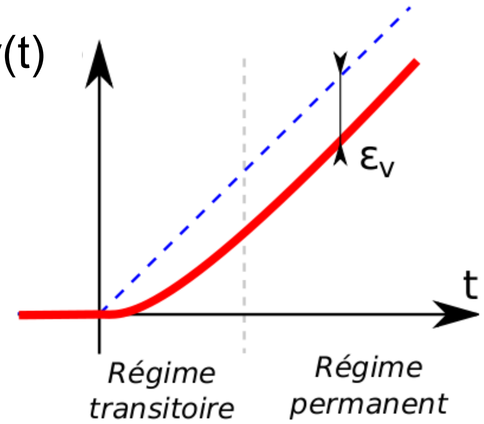
soit la condition de stabilité $0 < K < 1.6875$. ◁

Accuracy of a closed-loop system – Reminders

- A *stable* loop system is ***accurate if the*** steady-state error between setpoint and output is zero when the setpoint is changed.



ε_S : static error
or position



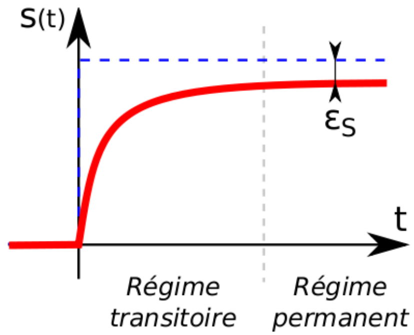
ε_V : drag error
or speed

- Accuracy is defined as a function of the ***type of setpoint y_c (k)*** :
  - ***step*** setpoint: static accuracy or position error
  - ***ramp*** setpoint: speed accuracy or drag error
  - ***parabola*** setpoint: acceleration precision

Tool for analyzing the accuracy of a closed-loop system

- Analyzing steady-state accuracy means studying the error between setpoint and steady-state output ($k \rightarrow +\infty$).

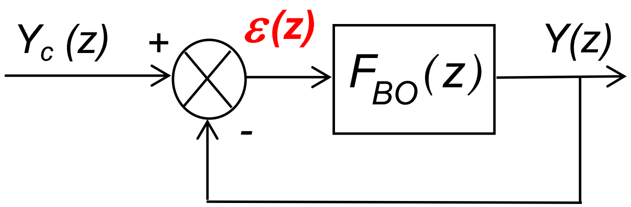
$$\varepsilon_S = \lim_{k \to +\infty} \left(y_c(k) - y(k) \right) = \lim_{k \to +\infty} \varepsilon(k)$$



- The closed-loop system is said:
  - ***accurate*** if this limit is zero
  - ***not accurate*** if this limit is not zero
    - *Nevertheless, the smaller the error, the more accurate the system.*

- We exploit the final value theorem

$$\lim_{k \to +\infty} \varepsilon(k) = \lim_{z \to 1} (z-1)\varepsilon(z)$$



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Tool for analyzing the accuracy of a closed-loop system

- We exploit the final value theorem $\qquad H(z)=1$

$$\lim_{k\to\infty}\varepsilon(k)=\lim_{z\to 1}(z-1)\varepsilon(z)=\lim_{z\to 1}(z-1)\left(Y_c(z)-Y(z)\right)=\lim_{z\to 1}(z-1)Y_c(z)\left(1-\frac{Y(z)}{Y_c(z)}\right)$$

$$=\lim_{z\to 1}(z-1)Y_c(z)\left(1-F_{BF}(z)\right)=\lim_{z\to 1}(z-1)Y_c(z)\left(1-\frac{C(z)G(z)}{1+C(z)G(z)}\right)$$

$$=\lim_{z\to 1}(z-1)Y_c(z)\left(1-\frac{F_{BO}(z)}{1+F_{BO}(z)}\right)$$

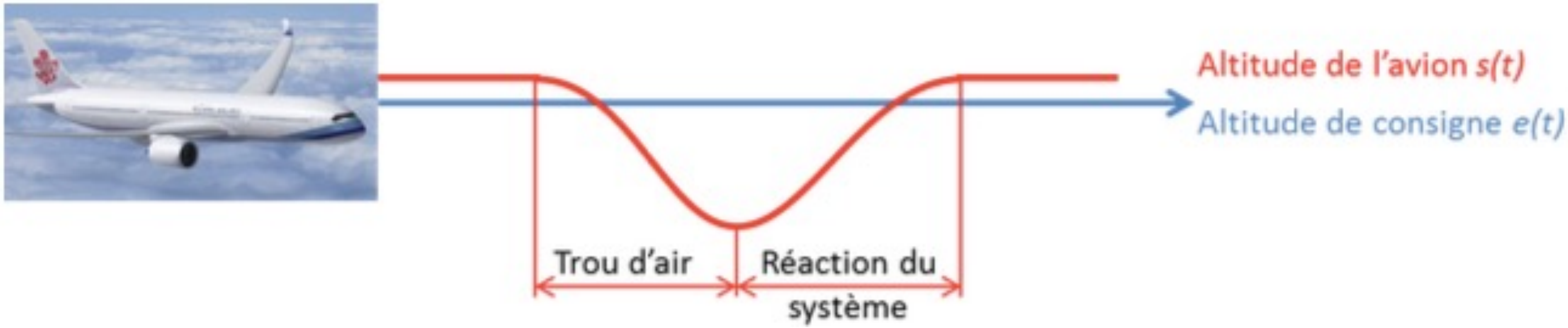
$$\lim_{k\to\infty}\varepsilon(k)=\lim_{z\to 1}(z-1)Y_c(z)\left(\frac{1}{1+F_{BO}(z)}\right)$$

➔ The accuracy of the closed-loop system depends on

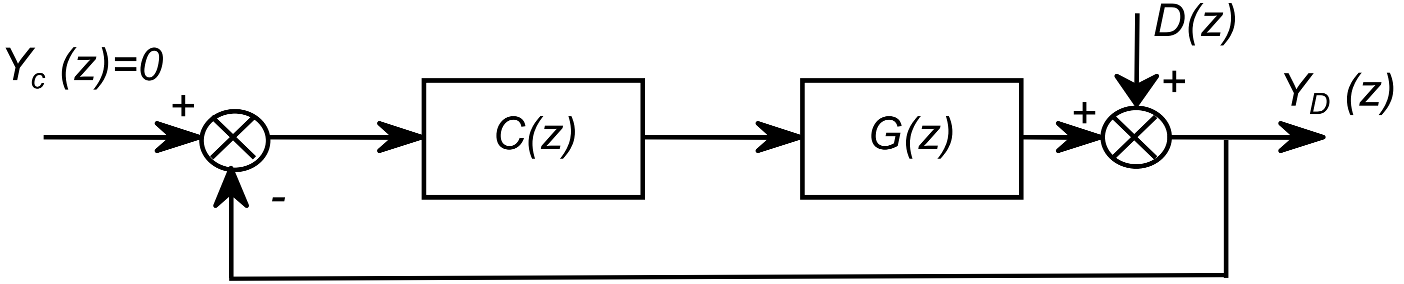
- of $Y_c(z)$ and therefore of the type of setpoint: *step, ramp, parabola,* etc.
- of the open-loop transfer function $F_{BO}(z)=C(z)G(z)$

Influence of disturbances on the accuracy of a looped system

- Reminder
  - Disturbance: external input that interferes with system operation



Method for analyzing the influence of a *step* disturbance on the accuracy of a closed-loop system



- When $y_c(k)=0$, we want to characterize $y_D(k)$ for $\mathbf{d(k)=\Gamma(k)}$

$$\lim_{k\to+\infty} y_D(k) = \lim_{z\to1}(z-1)Y_D(z) = \lim_{z\to1}(z-1)F_D(z)D(z)$$

$$F_D(z) = \frac{1}{1+F_{BO}(z)}; \quad Y_D(z) = \frac{z}{z-1}$$

$$\lim_{k\to+\infty} y_D(k) = \overset{?}{\rightarrow} 0$$

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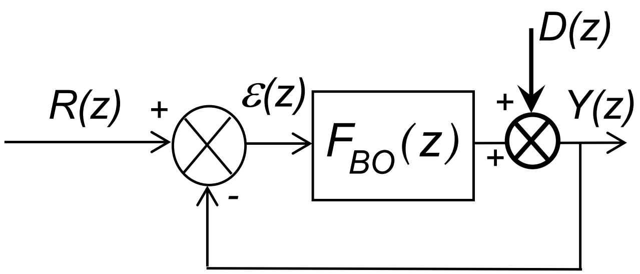
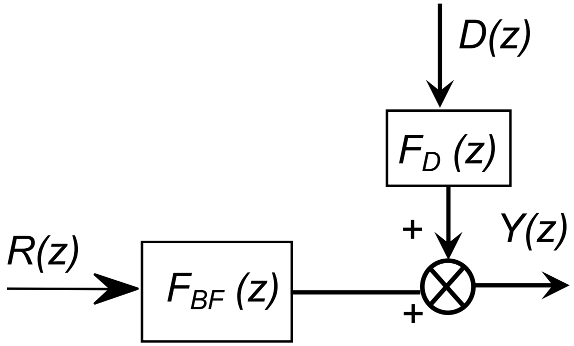
Stability and performance of digital control loop structures
Summary of analysis tools

① From the loop system block diagram





① Calculate F_{BO} (z). *Deduce* F_{BF} (z) *and* F_D (z)

② ***The stability of*** the closed-loop system is analyzed via

- Calculating the poles of F_{BF} (z)
- Jury criteria
- the Routh-Hurwitz criterion via the bilinear transformation

③ Performance criteria are evaluated. *Do they meet the specifications*?

- ***Accuracy***
  - Calculation of accuracy error
- ***Rapidity*** and ***damping***
  - Calculation of response time at $n\%$ and D_{max}

H. Garnier