3A IA2R

Digital control

Final exam 2023/2024 - 1h30

Name & firstname: ....

Instructions:

1. Do not forget to write your name above and include all these pages with your copy.
2. You can answer in French or in English but do not mix both languages.
3. The only material you can consult is your personal A4 recto-verso piece of paper.
4. You may use a hand calculator but with no communication capabilities.
5. The exercises must be solved on your copy in the given order.
6. Good luck !

Multiple choice questions *(there is one correct answer from the choices only. Wrong answers will not be penalized.)*

A sampled system is stable if all the poles of the transfer function lie INSIDE the unit circle of the z-plane ?

☐ *True*

☐ *False*

What is the role of a Zero-Order Hold in a digital closed-loop block diagram ?

☐ it interpolates the discrete-time sequence generated by the digital controller

☐ it samples the continuous-time signals

☐ it avoids the aliasing effect

☐ it guarantees the appropriate choice of the sampling period

What is the name of the criterion that can be used to state if a discrete-time transfer function is stable or not

☐ Ragazzani's stability criterion

☐ Shannon's stability criterion

☐ Nyquist's stability criterion

☐ Jury's stability criterion

Exercise 1
Consider the second-order difference equation of a digital system

$$y(k+2) + 0.4y(k+1) + 0.2y(k) = u(k), \quad \text{with } y(0) = y(1) = 0.$$

Determine its z-transfer function.

Exercise 2
Consider the following transfer function

$$G(z) = \frac{1}{z^2 + 0.4z + 0.2}$$

State whether the system is stable or not. Explain your answer.

Exercise 3
Give the difference equation that implements the control for the digital controller

$$C(z) = \frac{U(z)}{\varepsilon(z)} = \frac{0.3 - 0.4z^{-1} + 0.2z^{-2}}{1 - z^{-1}}$$

Exercise 4
Consider the following sampled transfer function

$$G_{zoh}(z) = \frac{Y(z)}{U(z)} = \frac{0.5z^{-1}}{1 - 0.8z^{-1}}$$

1. Find its order, its steady-state gain and its pole.
2. Determine the system response to a unit step, i.e. when $u(k) = \Gamma(k)$.
3. Determine the corresponding continuous-time transfer function $G(s)$ when the sampling period is $T_s = 0.1$s.

Problem 1 - Direct design method of the digital controller
One possibility for determining a controller directly in discrete-time is to start from a desired closed-loop transfer function $G_{ref}(z)$ and to solve the closed-loop transfer function $G_{cl}(z)$ for the controller transfer function according to

$$G_{cl}(z) = G_{ref}(z) \Rightarrow C(z) = \frac{1}{G_{zoh}(z)} \times \frac{G_{ref}(z)}{1 - G_{ref}(z)} \tag{1}$$

where $G_{zoh}(z)$ is the sampled transfer function of the open-loop plant.
Given the sampled transfer function

$$G_{zoh}(z) = \frac{1}{(z-1)(z-0.5)}$$

and the desired closed-loop transfer function as

$$G_{ref}(z) = \frac{1}{z}$$

1. Show that the digital controller $C(z)$ takes the following form:

$$C(z) = \frac{U(z)}{\varepsilon(z)} = z - 0.5$$

2. Determine the corresponding difference equation.
3. Is-it possible to implement this control law in practice ? If not, clearly state why.
4. Suggest a solution so that the control can be implemented.

Problem 2 - Altitude control for a mini-drone via the approximation of a PD analog controller

The goal is to design an altitude control for the Tello mini-drone as shown in Figure 1.

Figure 1: Tello mini-drone from DJI

The input and output of the mini-drone altitude control system are:

- $u(t)$: percentage of maximum velocity on the Z axis (related in some way to the thrust generated by the 4 rotors);
- $y(t)$: altitude or position along the Z axis of the mini-drone in cm.

The maximum velocity percentage-to-altitude transfer function takes the form of a first-order plus pure integrator model

$$\frac{Y(s)}{U(s)} = \frac{K}{s(1+Ts)} \tag{2}$$

where $Y(s) = \mathcal{L}[y(t)]$ and $U(s) = \mathcal{L}[u(t)]$.

K is the gain, T is the model time-constant. We will assume

- $K = 0.8$;
- $T = 0.3$s.

The performance requirements for the mini-drone altitude control are described in Table 1.

Requirement	Assessment criteria	Level
Control	Step reference tracking	No steady-state error
the mini-drone altitude	Peak overshoot	$D_{1\%} = 4.3\%$
	Settling time at 5 %	$t_s^{5\%} = 0.5$ s

Table 1: Performance requirements for the mini-drone altitude control

The proposed strategy is to use a variation of the standard PD control, where unlike the standard PD where the derivative term is usually applied to the error, it is applied to the output, as shown in Figure 2.

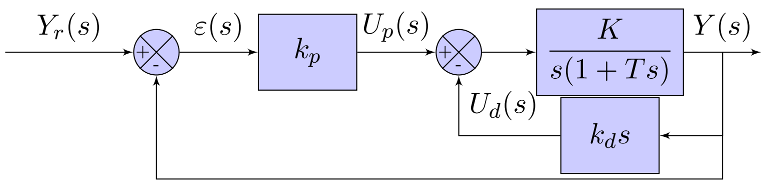

Figure 2: Block-diagram of the PD feedback configuration of the altitude control system

1. Determine the internal closed-loop transfer function $F_i(s) = \dfrac{Y(s)}{U_p(s)}$ in terms of k_d, K and T.

2. Plot the simplified closed-loop block-diagram.

3. Determine the closed-loop transfer function $G_{cl}(s) = \dfrac{Y(s)}{Y_r(s)}$ in terms of k_p, k_d, K and T.

4. Determine the range of values for k_p and k_d that ensure the stability of the closed loop system.

5. Calculate the steady-state tracking error in response to a step $y_r(t) = A\Gamma(t)$, *i.e.* :

$$\lim_{t\to+\infty} \varepsilon(t) = \lim_{t\to+\infty} (y_r(t) - y(t))$$

6. Determine the value for the damping coefficient ζ and undamped natural frequency ω_0 that make the desired closed-loop transfer function step response to have $D_{1\%} = 4.3$ % and $t_s^{5\%} = 0.5$ s. It is recalled that ζ and ω_0 are function of the step response performance indices (settling time $t_s^{5\%}$ and peak overshoot D_1)

$$\zeta = \sqrt{\frac{(\ln(D_1))^2}{\pi^2 + (\ln(D_1))^2}} \qquad \omega_0 = \frac{3}{\zeta t_s^{5\%}}$$

7. Give the transfer function of the desired closed-loop system $G_{ref}(s)$.

8. A method for determining the PD controller $C(s)$ is to consider the desired continuous-time second-order closed-loop transfer function $G_{ref}(s)$ and to solve the gains k_p and k_d of the PD controller. Show that this method leads to

$$G_{cl}(s) = G_{ref}(s) \Rightarrow \begin{cases} k_p = \dfrac{T\omega_0^2}{K} \\ k_d = \dfrac{2\zeta k_p}{\omega_0} - \dfrac{1}{K} \end{cases}$$

9. Determine the approximated digital version $C_d(z)$ that results from the use of the Tustin rule

$$s = \frac{2}{T_s}\frac{1 - z^{-1}}{1 + z^{-1}}$$

for the derivative part of the PD controller:

$$C_d(s) = k_d s$$

10. Show that the difference equation of the digital controller output $u(k)$ can be written as:

$$u(k) = u_p(k) - u_d(k)$$

11. Express $u_p(k)$ as a function of k_p and $\varepsilon(k)$ and $u_d(k)$ as a function of k_d, T_s, $u_d(k-1)$, $y(k)$ and $y(k-1)$.

Appendices

Table of common unilateral Laplace and z-transforms

$f(t)$	$F(s)$	$f(k)$	$F(z)$	ROC
$\delta(t)$	1	$\delta(k)$	1	All z
$\delta(t-t_0)$	e^{-st_0}	$\delta(k-i)$	z^{-i}	$z \neq 0$
$\Gamma(t)$	$\frac{1}{s}$	$\Gamma(k)$	$\frac{z}{z-1}$	$\|z\| > 1$
$t\Gamma(t)$	$\frac{1}{s^2}$	$kT_s\Gamma(k)$	$\frac{zT_s}{(z-1)^2}$	$\|z\| > 1$
$t^2\Gamma(t)$	$\frac{2}{s^3}$	$(kT_s)^2\Gamma(k)$	$\frac{z(z+1)T_s^2}{(z-1)^3}$	$\|z\| > 1$
$e^{-at}\Gamma(t)$	$\frac{1}{s+a}$	$a^k\Gamma(k)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$

Useful properties of the unilateral z-transform

Property	signal	z-transform
linearity	$ax(k) + by(k)$	$aX(z) + bY(z)$
delays (or shifts)	$x(k-1)$	$z^{-1}X(z)$
	$x(k-i)$	$z^{-i}X(z)$
time-advance	$x(k+1)$	$zX(z) - zx(0)$
	$x(k+2)$	$z^2X(z) - z^2x(0) - zx(1)$
final value theorem	$\lim\limits_{k\to+\infty} x(k) = \lim\limits_{z\to 1}(z-1)X(z)$	if the limit exists

Sampled transfer function of continuous-time first-order systems
Consider a continuous-time first-order system given by its Laplace transfer function:

$$G(s) = \frac{b}{s+a}$$

Its equivalent ZOH (zero-order hold) sampled transfer function is given by:

$$G_{ZOH}(z) = Z\Big(B_o(s)G(s)\Big) = \left(1 - z^{-1}\right) Z\left(\frac{b}{s(s+a)}\right) = \frac{b_1 z^{-1}}{1 + a_1 z^{-1}}$$

where

$$\begin{cases} a_1 = -e^{-aT_s} \\ b_1 = \dfrac{b}{a}(1 + a_1) \end{cases}$$

and T_s denotes the sampling period.