Digital control systems

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Homework Set - Digital Control Systems

Due date: Tuesday 26th April 2022, at 2:00pm.

This homework set is the final exam from April 2021.
You are asked to send your solution (typed or handwritten) in a pdf file by e-mail to your Tutorial instructor: Floriane Collin or Hugues Garnier before the deadline indicated above.
Your solution, to be worked in pairs, can be written in French or in English but it should not mix both languages.
For grading purposes, repeat question number for each exercise and give concise answers.

« *I listen, I forget. I see, I remember.* **I do, I understand**. » Confucius.

Digital control systems - Final exam from 2020/2021 - 1h45

Instructions:

1. Do not forget to write your name above and include all these pages with your copy.
2. You can answer in French or in English but do not mix both languages.
3. The only material you can consult is your personal A4 page double-sided.
4. You may use a hand calculator but with no communication capabilities.
5. The 8 exercises must be solved on your copy in the given order.
6. Good luck !

Multiple choice questions *(there is one correct answer from the choices only. Wrong answers will not be penalized.)*

A sampled system is stable if all the poles of the closed-loop transfer function lie outside the unit circle of the z-plane ?

☐ *True*

☐ *False*

What is the role of a Zero-Order Hold in a digital closed-loop block diagram ?

☐ it guarantees the appropriate choice of the sampling period

☐ it avoids the aliasing effect

☐ it interpolates the discrete-time sequence generated by the digital controller

☐ it samples the continuous-time signals

It is recommended to design a digital controller via the approximation of a PID analog controller when the sampling period is

☐ slow in comparison with the settling time of the closed-loop system

☐ fast in comparison with the settling time of the closed-loop system

1 Inverse z-transform

1. Determine the inverse z-transform by using the partial fraction expansion method of

$$X(z) = \frac{0.5z}{(z-0.5)(z-0.7)}$$

2. Give the first four terms of the discrete-time sequence and the final value $x(+\infty)$.

2 Difference equation of a digital controller

A digital controller has been designed to computer-controlled a continuous-time plant and is given

$$C(z) = \frac{U(z)}{\varepsilon(z)} = \frac{z-0.4}{z^2-0.3z+0.02}$$

Determine the difference equation you would use to implement the digital controller.

3 Stability of a sampled transfer function

A sampled transfer function is given as

$$G_{ZOH}(z) = \frac{z-1}{z^2+0.7z+1}$$

Determine whether the system is stable or not. Justify your answer.

4 Analysis of a sampled transfer function

Consider the following sampled transfer function

$$G_{ZOH}(z) = \frac{Y(z)}{U(z)} = \frac{0.5}{z-0.8}$$

1. Find its order, its steady-state gain and its pole.
2. Determine the system response to a unit step, i.e. when $u(k) = \Gamma(k)$.
3. Determine the corresponding continuous-time transfer function $G(s)$ when the sampling period is $T_s = 0.1$s.

5 Sampled version of a double integrator system

Consider a sampled system constituted by the cascade of a zero-order hold (ZOH) $B_o(s)$, a continuous-time simple double integrator system $G(s)$ and a sampler where:

$$G(s) = \frac{1}{s^2}$$

1. Determine the poles and zero of $G(s)$.
2. State whether the continuous-time system is stable or not ?
3. Determine the sampled transfer function $G_{ZOH}(z)$ when $T_s = 1$s.
4. Calculate the poles and zero of $G_{ZOH}(z)$.
5. What strange result is obtained for the zeros of $G_{ZOH}(z)$?
6. Investigate whether the sampled system is stable or not ?
7. Give the difference equation of the sampled system.

6 Digital form of an analog PD controller

A PD controller in its so-called *parallel* form is described by the following transfer function:

$$C(s) = \frac{U(s)}{\varepsilon(s)} = K_p + K_d s$$

1. Determine the control equation in the continuous-time domain
2. Use the following backward Euler approximation to find a digital version $C(z)$ of the continuous-time controller $C(s)$
   $$s = \frac{1 - z^{-1}}{T_s}$$
   where T_s denotes the sampling period.
3. Deduce from $C(z)$ the control equation in the discrete-time domain.

7 Closed-loop control system analysis from code

The implementation code of a digital PID controller is given below.

```
e:=yc-y
i:=i+e*Ts/Ti
u:=Kp*(e+i)
```

1. Determine from the code the type of PID controller (P, PI, PD, PID)?
   Explain your answer.
2. Determine the transfer function of the digital controller $C(z)$.
3. Deduce the transfer function of the analog controller $C(s)$.
4. Determine the approximation method used to implement the analog controller: backward Euler, forward Euler, Tustin.
   Explain your answer.

8 Deadbeat digital control of a first-order continuous-time system

The goal in this exercise is to design a so-called deadbeat digital controller for a simple first-order system. A deadbeat feedback control system has the remarkable property that its step response reaches the desired steady-state after n samples only where n is the order of the sampled transfer function $G_{ZOH}(z)$ of the continuous-time system.

Consider a sampled system constituted by the cascade of a zero-order hold (ZOH) $B_o(s)$, a continuous-time transfer function $G(s)$ and a sampler where

$$G(s) = \frac{1}{s+1}$$

1. Represent the block-diagram of the sampled system.
2. Show by using the appendix that the sampled transfer function $G_{ZOH}(z)$, for a general value of the sampling period T_s, is
$$G_{ZOH}(z) = \frac{1 - e^{-T_s}}{z - e^{-T_s}}$$
3. It is assumed there is no disturbance acting on the system. A unity feedback digital control with $C(z)$ is implemented for the continuous-time system. Represent the "fully" digital closed-loop system with the sampled transfer function $G_{ZOH}(z)$.
4. Recall (or derive it if you do not remember) the transfer function of closed-loop system denoted as $F_{CL}(z)$ as a function of $C(z)$ and $G_{ZOH}(z)$.
5. Express $C(z)$ in terms of $F_{CL}(z)$ and $G_{ZOH}(z)$.
6. The design method of a deadbeat controller consists in imposing the desired closed-loop transfer function to be $F_{CL}(z) = z^{-n}$, where n is the order of the sampled transfer function $G_{ZOH}(z)$. Use the result above to show that the deadbeat controller for the first-order system is given by:
$$C(z) = \frac{z - e^{-T_s}}{(1 - e^{-T_s})(z-1)}$$
7. Plot the sampled response of the deadbeat feedback digital control system when the setpoint is a discrete-time unit step.

Appendices

Table of common unilateral Laplace and z-transforms

$f(t)$	$F(s)$	$f(k)$	$F(z)$	ROC
$\delta(t)$	1	$\delta(k)$	1	All z
$\delta(t-t_0)$	e^{-st_0}	$\delta(k-i)$	z^{-i}	$z \neq 0$
$\Gamma(t)$	$\frac{1}{s}$	$\Gamma(k)$	$\frac{z}{z-1}$	$\|z\| > 1$
$t\Gamma(t)$	$\frac{1}{s^2}$	$kT_s\Gamma(k)$	$\frac{zT_s}{(z-1)^2}$	$\|z\| > 1$
$t^2\Gamma(t)$	$\frac{2}{s^3}$	$(kT_s)^2\Gamma(k)$	$\frac{z(z+1)T_s^2}{(z-1)^3}$	$\|z\| > 1$
$e^{-at}\Gamma(t)$	$\frac{1}{s+a}$	$a^k\Gamma(k)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$

Useful properties of the unilateral z-transform

Property	signal	z-transform
linearity	$ax(k)+by(k)$	$aX(z)+bY(z)$
delays (or shifts)	$x(k-1)$	$z^{-1}X(z)$
	$x(k-i)$	$z^{-i}X(z)$
accumulation	$\sum_{i=0}^{k} x(i)$	$\frac{1}{1-z^{-1}}X(z)$
final value theorem	$\lim_{k\to+\infty} x(k) = \lim_{z\to 1}(z-1)X(z)$	if the limit exists

Sampled transfer function of continuous-time first order systems
Consider a continuous-time first-order system given by its Laplace transfer function:

$$G(s) = \frac{b}{s+a}$$

Its equivalent ZOH (zero-order hold) sampled transfer function is given by:

$$G_{ZOH}(z) = Z\Big(B_o(s)G(s)\Big) = \left(1-z^{-1}\right) Z\left(\frac{b}{s(s+a)}\right) = \frac{b_1 z^{-1}}{1+a_1 z^{-1}}$$

where

$$\begin{cases} a_1 = -e^{-aT_s} \\ b_1 = \dfrac{b}{a}(1+a_1) \end{cases}$$

and T_s denotes the sampling period.