Digital control systems

Hugues GARNIER
Floriane COLLIN

Homework Set 1

Due date: Wednesday, March 6, 2024, at 2:00pm.
A pdf file of your scanned solution should be sent by e-mail to Hugues Garnier before the deadline indicated above.

The homework set, to be done in pairs, can be written in French or in English but it should not mix both languages.
For grading purposes, repeat question number for each exercise and give concise answers.

« *I listen, I forget. I see, I remember.* **I do, I understand**. » Confucius.

Exercise 1
Given the z-transform function $X(z)$ below, find the original discrete-time signal by partial-fraction expansion

$$X(z) = \frac{0.5z}{z^2 - 1.2z + 0.35}$$

Exercise 2
A digital controller has been designed to computer-controlled a continuous-time plant and is given

$$C(z) = \frac{U(z)}{\varepsilon(z)} = \frac{z - 0.4}{z^2 - 0.3z + 0.02}$$

Determine the difference equation you would use to implement the digital controller.

Exercise 3
Determine the sampled transfer function $G_{ZOH}(z)$ for the following continuous-time system given by:

$$G(s) = \frac{1}{s(s+1)}$$

from the formula

$$G_{ZOH}(z) = Z\Big(H_0(s)G(s)\Big) = \Big(\frac{z-1}{z}\Big)Z\Big(\frac{G(s)}{s}\Big)$$

when the sampling period is $T_s = 1$s.

Exercise 4
Consider a sampled system constituted by the cascade of a zero-order hold (ZOH) $H_0(s)$, a continuous-time transfer function $G(s)$ and a sampler where:

$$G(s) = \frac{3}{s^2 + 4s + 3}$$

1. Determine the sampled transfer function $G_{ZOH}(z)$ when $T_s = 0.2$s by using the formula given in the Appendix.

2. Check your results with Matlab. Give the Matlab command lines.
3. Calculate the poles and zeros of $G_{ZOH}(z)$.
4. Compare the poles and zeros of $G(s)$ with those of $G_{ZOH}(z)$. What strange result is obtained for the zeros of $G_{ZOH}(z)$?
5. Give the difference equation of the sampled system.
6. Calculate the response $y(k)$ to a unit discrete-time step $u(k) = \Gamma(k)$ for $k = 0, 1, 2, 3, 4$.

Exercise 5
Many systems can be modeled as

$$G(s) = \frac{K}{1 + Ts} e^{-\tau s}$$

Find the ZOH equivalent $G_{ZOH}(z)$ when the time-delay τ is assumed to be a multiple integer of the sampling time $\tau = nT_s$.

Exercise 6
Show that the difference equation:

$$y(k) - 5y(k-1) + 6y(k-2) = \Gamma(k); \quad y(-1) = y(-2) = 0$$

has the following solution

$$y(k) = \left(\frac{1}{2} - (2)^{k+2} + \frac{1}{2}(3)^{k+2}\right) \Gamma(k)$$

Second-order systems
Consider a continuous-time second-order system given by its Laplace transfer function:

$$G(s) = \frac{a \times b}{(s+a)(s+b)}$$

Its equivalent ZOH sampled transfer function is given by:

$$\begin{aligned} G_{ZOH}(z) = Z\Big(H_0(s)G(s)\Big) &= \left(1 - z^{-1}\right) Z\left(\frac{a \times b}{s(s+a)(s+b)}\right) \\ &= \frac{b_1 z^{-1} + b_2 z^{-2}}{(1 + \alpha z^{-1})(1 + \beta z^{-1})} \end{aligned}$$

where

$$\begin{cases} \alpha = -e^{-aT_s} \\ \beta = -e^{-bT_s} \\ b_1 = \dfrac{b \times \alpha - a \times \beta}{b - a} + 1 \\ b_2 = \dfrac{b \times \beta - a \times \alpha}{b - a} + \alpha \times \beta \end{cases}$$