Sampled systems

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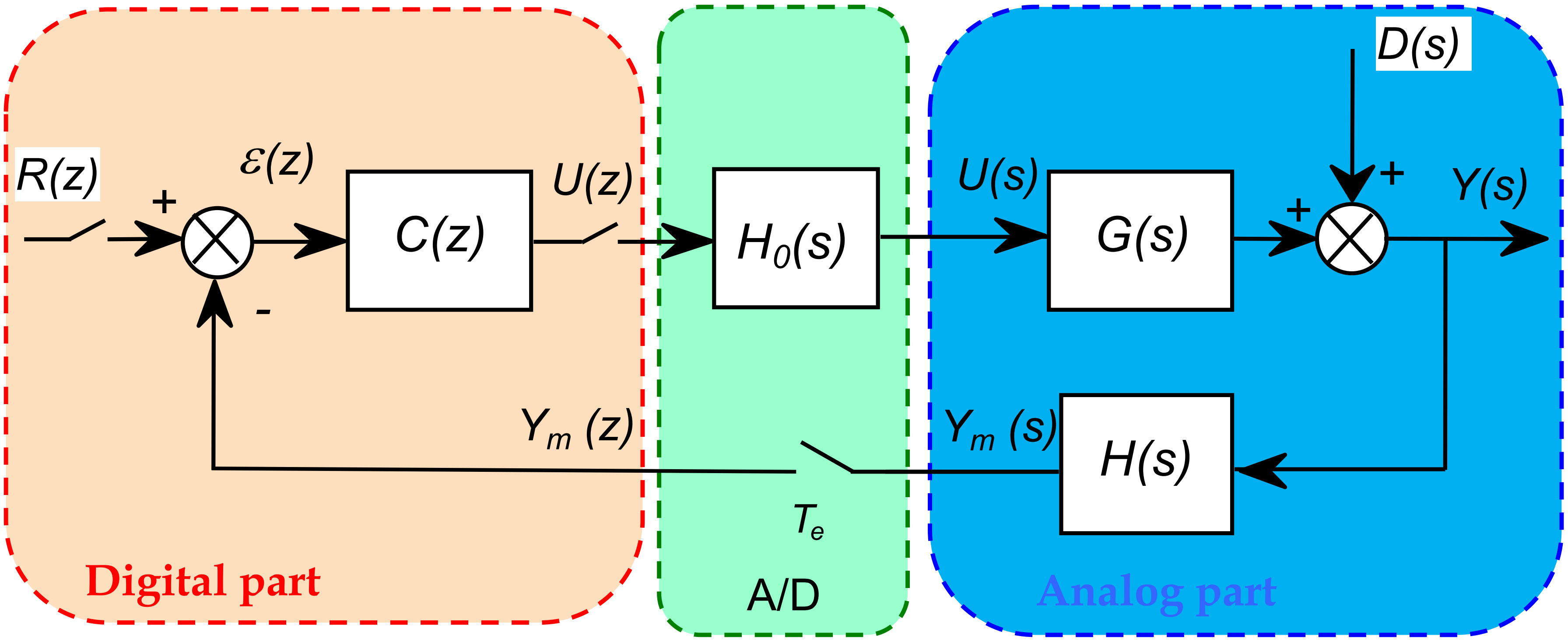
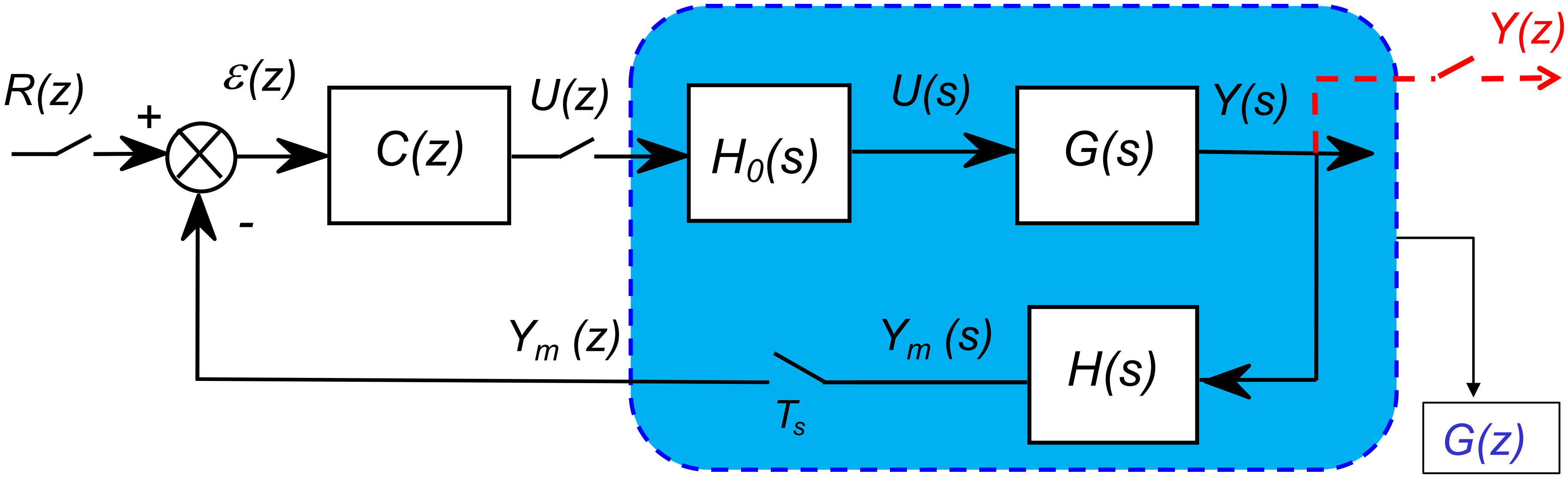
Reminder - Digital control block-diagram

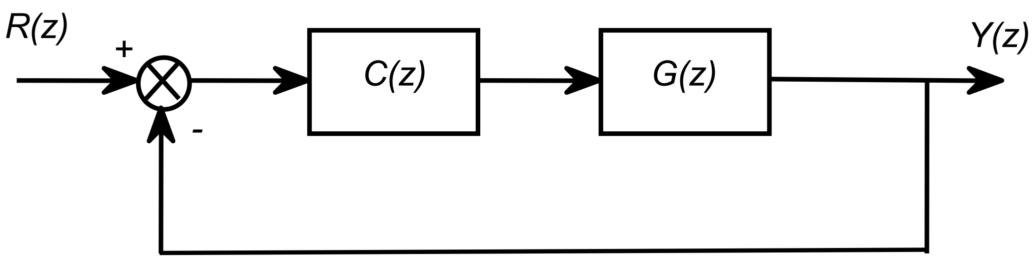


- A digital control strategy consists of two parts:
  - the first is analog
  - the second is digital
- To understand their interaction, it is easier to convert the analog part into digital

Digital control block-diagram

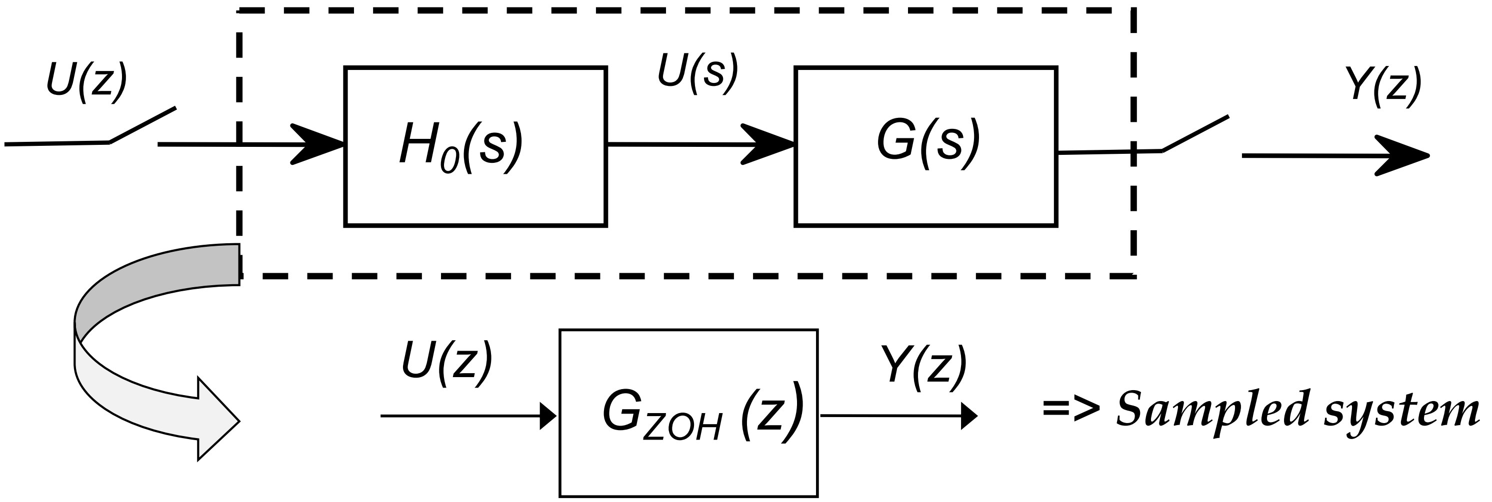


- Analysis of digital control performance in the discrete domain involves:
  - the addition of a ***dummy*** **sampler** at the output
  - the determination of the z-transform transfer function $G(z)$ consisting of the cascading of the zero-order hold $H_0(s)$, $G(s)$ and the sampler called the **sampled** ***system***

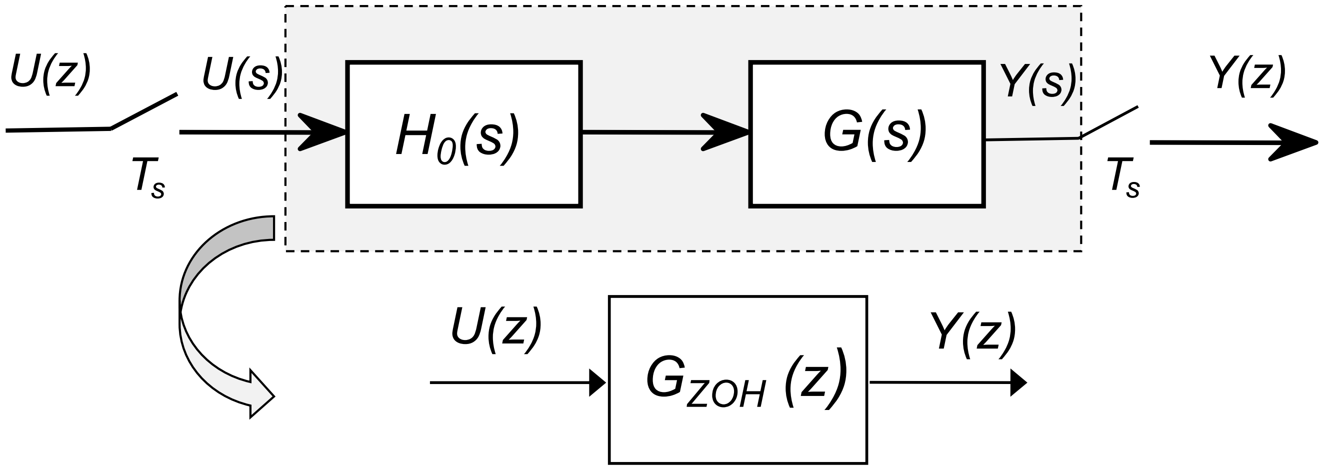


Sampled system

- A **sampled system** consists of the cascading of
  - the zero-order hold modelled by $H_0(s)$
  - the continuous-time system modelled by $G(s)$ ($H(s)=1$ is assumed)
  - the sampler
- The input/output signals of the sampled system are ***z-transforms of discrete-time signals***



Transfer function of a sampled system



$$Y(z) = Z\left(y(kT_s)\right) = Z\left(y(t)\big|_{t=kT_s}\right) \qquad y(t) = L^{-1}\left(Y(s)\right) \qquad Y(s) = H_0(s)G(s)\,U(s)$$

$$Y(z) = Z\left(L^{-1}\left(H_0(s)G(s)\right)\Big|_{t=kT_s}\right)U(z)$$

$$\frac{Y(z)}{U(z)} = G_{ZOH}(z) = Z\left(L^{-1}\left(H_0(s)G(s)\right)\Big|_{t=kT_s}\right)$$

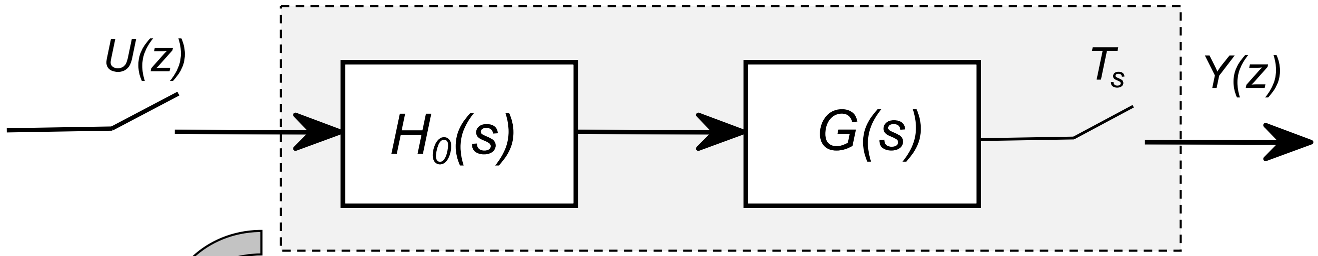
To simplify notation

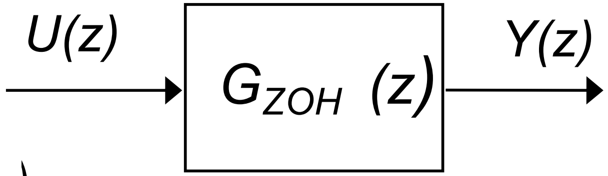
$$G_{ZOH}(z) = Z\left(H_0(s)G(s)\right)$$

Caution! $G_{ZOH}(z)$ is not equal to $Z(G(s))$!!!

$$G_{ZOH}(z) \neq Z\left(G(s)\right)$$

Transfer function of a sampled system





$$G_{ZOH}(z) = Z\left(H_0(s)G(s) \right)$$

$$G_{ZOH}(z) = Z\left(\left(\frac{1-e^{-T_s s}}{s}\right)G(s)\right)$$

$$G_{ZOH}(z) = Z\left(\frac{G(s)}{s} - e^{-T_s s}\frac{G(s)}{s}\right)$$

$$G_{ZOH}(z) = Z\left(\frac{G(s)}{s}\right) - Z\left(e^{-T_s s}\frac{G(s)}{s}\right)$$

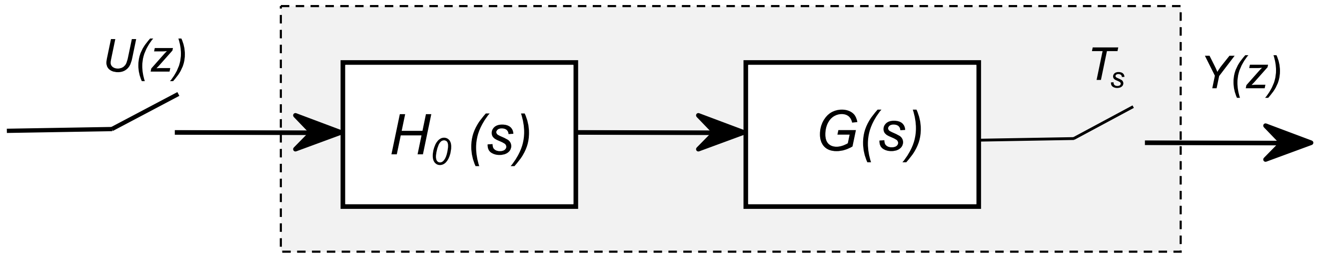
$$G_{ZOH}(z) = Z\left(\frac{G(s)}{s}\right) - z^{-1}Z\left(\frac{G(s)}{s}\right)$$

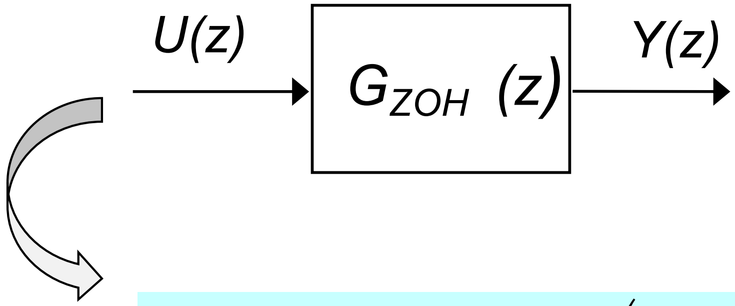
Recall : $H_0(s) = \dfrac{1-e^{-T_s s}}{s}$

$$G_{ZOH}(z) = \left(1-z^{-1}\right)Z\left(\frac{G(s)}{s}\right)$$

$$G_{ZOH}(z) = \frac{z-1}{z}Z\left(\frac{G(s)}{s}\right)$$

Transfer function of a sampled system

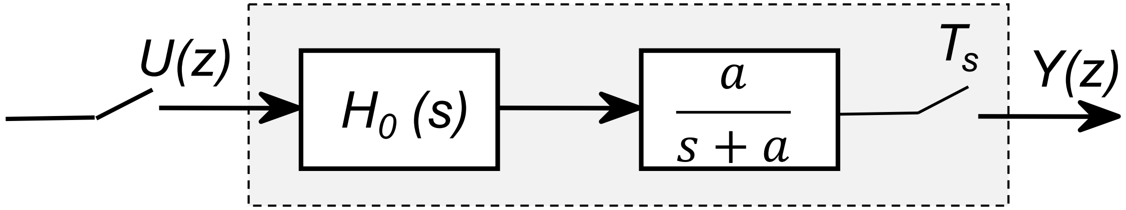




$$G_{ZOH}(z) = \left(1 - z^{-1}\right) Z\left(\frac{G(s)}{s}\right)$$

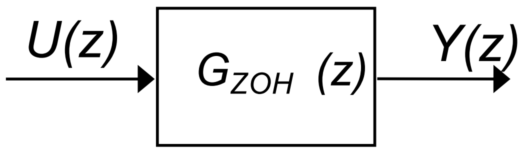
$$G_{ZOH}(z) = \frac{z-1}{z} Z\left(\frac{G(s)}{s}\right)$$

Sampled transfer function calculation Example 1



$$G_{ZOH}(z) = (1 - z^{-1}) Z\left(\frac{G(s)}{s}\right)$$

$$G_{ZOH}(z) = \frac{z-1}{z} Z\left(\frac{G(s)}{s}\right)$$



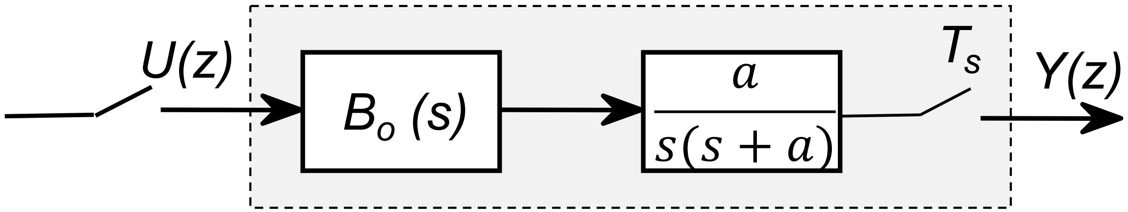
It is necessary to decompose into simple elements G(s)/s then use the table of z-transforms

Let $G(s) = \frac{a}{s+a}$; then $\underset{ZOH}{G}(z) = (1 - z^{-1})\, z \left\{ \frac{a}{s(s+a)} \right\} = (1 - z^{-1}) z \left\{ \frac{1}{s} - \frac{1}{s+a} \right\}$.

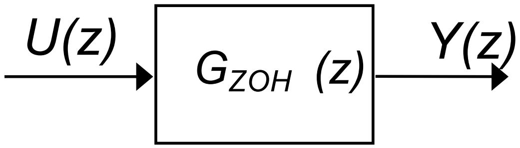
By using the z-transform table, we obtain: $\underset{ZOH}{G}(z) = \frac{z-1}{z}\left(\frac{z}{z-1} - \frac{z}{z-e^{-aT}} \right) = \frac{1-e^{-aT}}{z-e^{-aT}}$.

As an example, let $G(s) = \frac{1}{s+1}$, $T = 0.2s$. Then, the pulse transfer function is obtained as: $\underset{ZOH}{G}(z) = \frac{0.181}{z-0.819}$, or $\underset{ZOH}{G}(z) = \frac{1-e^{-0.2}}{z-e^{-0.2}}$.

Sampled transfer function calculation
Example 2



$$G_{ZOH}(z) = \left(1 - z^{-1}\right) Z\left(\frac{G(s)}{s}\right)$$
$$G_{ZOH}(z) = \frac{z-1}{z} Z\left(\frac{G(s)}{s}\right)$$



It is necessary to decompose into simple elements G(s)/s and then use the table of z-transforms

Let $G(s) = \frac{a}{s(s+a)}$; then $G_{ZOH}(z) = (1 - z^{-1})\, z\left\{\frac{a}{s^2(s+a)}\right\} = (1 - z^{-1}) z\left\{\frac{1}{s^2} - \frac{1/a}{s} + \frac{1/a}{s+a}\right\}$.

By using the z-transform table, we obtain: $G_{ZOH}(z) = \frac{z-1}{z}\left[\frac{Tz}{(z-1)^2} - \frac{1}{a}\left(\frac{z}{z-1} - \frac{z}{z-e^{-aT}}\right)\right] = \frac{aT(z-e^{-aT}) - (z-1)(1-e^{-aT})}{a(z-1)(z-e^{-aT})}$.

As an example, let $G(s) = \frac{1}{s(s+1)}$, $T = 0.2s$; then, the pulse transfer function is obtained as: $G_{ZOH}(z) = \frac{0.0187(z+0.936)}{(z-1)(z-0.819)}$.

1 zero of discretization

Denominator degree of G(s): n=2; numerator degree of G(s): m=0

Caution! *G(s)* has no zero, but G_{ZOH} *(z)* has an artificial zero called the discretization zero (when *n-m* ≥ *2*).
The order *n* is retained (here *n* =2).

Sampled transfer functions of some common systems

$G(s)$	$G_{ZOH}(z)$	
$\dfrac{1}{s}$	$\dfrac{T_s\ z^{-1}}{1-z^{-1}}$	
$\dfrac{K}{1+Ts}$	$\dfrac{b_1 z^{-1}}{1+a_1 z^{-1}}$	$a_1 = -e^{-T_s/T}$ $b_1 = K(1+a_1)$
$\dfrac{K}{1+Ts}e^{-\tau s}$ $\tau = n\ T_s$	$\dfrac{b_1 z^{-1}}{1+a_1 z^{-1}} z^{-n}$	$a_1 = -e^{-T_s/T}$ $b_1 = K(1+a_1)$

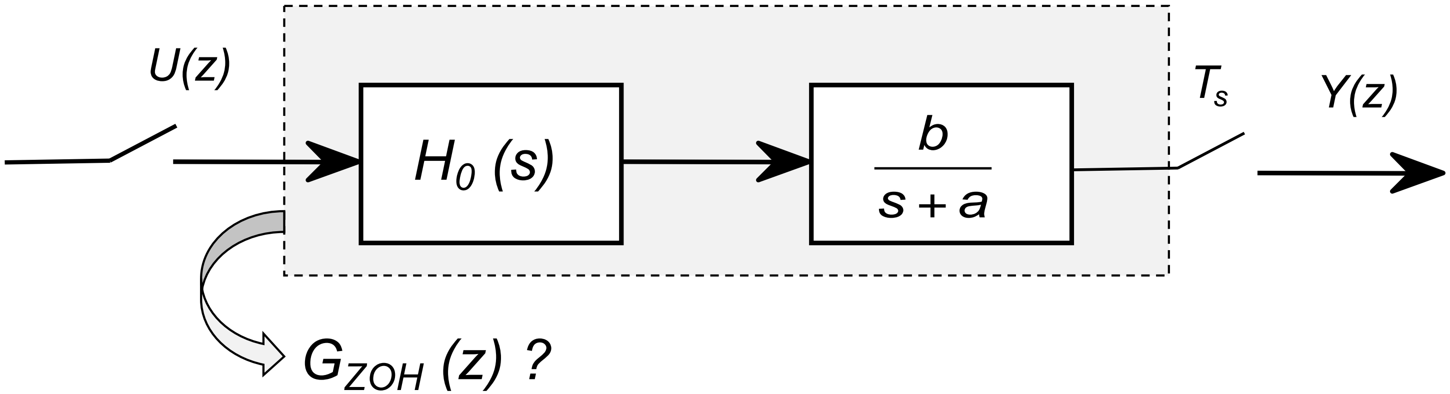
Today, it is easy to use Matlab!

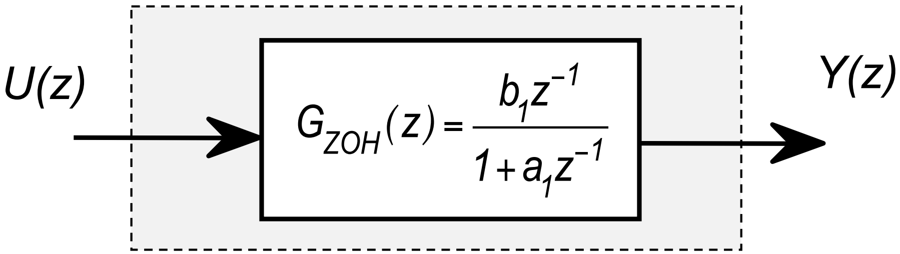
Command Window

```
>> help c2d
 c2d  Converts continuous-time dynamic system to discrete time.

    SYSD = c2d(SYSC,TS,METHOD) computes a discrete-time model SYSD with
    sample time TS that approximates the continuous-time model SYSC.
    The string METHOD selects the discretization method among the following:
       'zoh'           Zero-order hold on the inputs
       'foh'           Linear interpolation of inputs
       'impulse'       Impulse-invariant discretization
       'tustin'        Bilinear (Tustin) approximation.
       'matched'       Matched pole-zero method (for SISO systems only).
       'least-squares' Least-squares minimization of the error between
                       frequency responses of the continuous and discrete
                       systems (for SISO systems only).
       'damped'        Damped Tustin approximation based on TRBDF2 formula
                       (sparse models only).
    The default is 'zoh' when METHOD is omitted. The sample time TS should
    be specified in the time units of SYSC (see "TimeUnit" property).
```

Sampled transfer function of a first-order system

- Determine the sampled transfer function of the first-order system



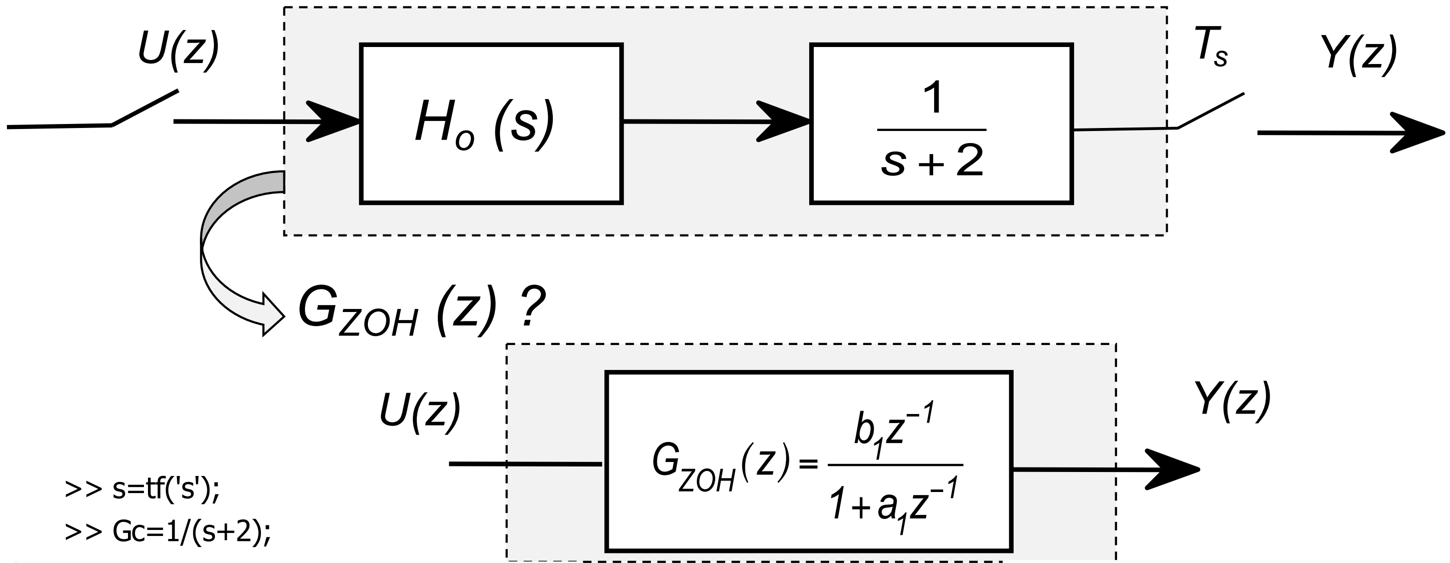


$$\begin{cases} a_1 = -e^{-a\,T_s} \\ b_1 = \frac{b}{a}(1 + a_1) \end{cases}$$

The parameters of $G_{ZOH}(z)$ depend on T_s

Sampled transfer function - Example

- Determine the sampled transfer function of the system below when $T_s = 0.1s$ and $T_s = 0.01s$




```
>> s=tf('s');
>> Gc=1/(s+2);
```

Command Window

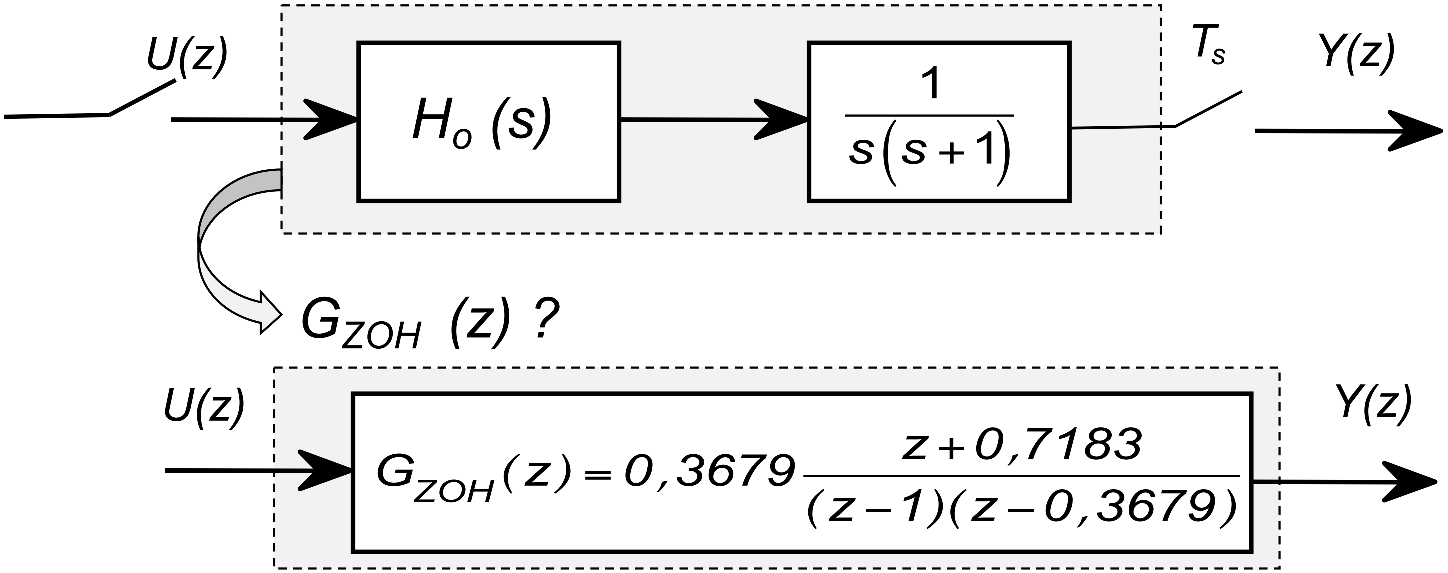
```
>> Ts=0.1;
>> Gd=c2d(Gc ,Ts, 'zoh')

Gd =

   0.09063
  ----------
  z - 0.8187

Sample time: 0.1 seconds
Discrete-time transfer function.
```

Command Window

```
>> Ts=0.01;
>> Gd=c2d(Gc ,Ts, 'zoh')

Gd =

   0.009901
  ----------
  z - 0.9802

Sample time: 0.01 seconds
Discrete-time transfer function.
```

Sampled transfer function - Example

- Determine the sampled transfer function of the system below when T_s =1s



- Matlab verification

```
s=tf('s');
G=1/(s*(s+1));
Ts=1;
Gd=c2d(G,Ts,'zoh')
```

Command Window

```
>> Gd=c2d(  G, Ts,  zoh)

Gd =

    0.3679 z + 0.2642
  ----------------------
  z^2 - 1.368 z + 0.3679

Sample time: 1 seconds
Discrete-time transfer function.
```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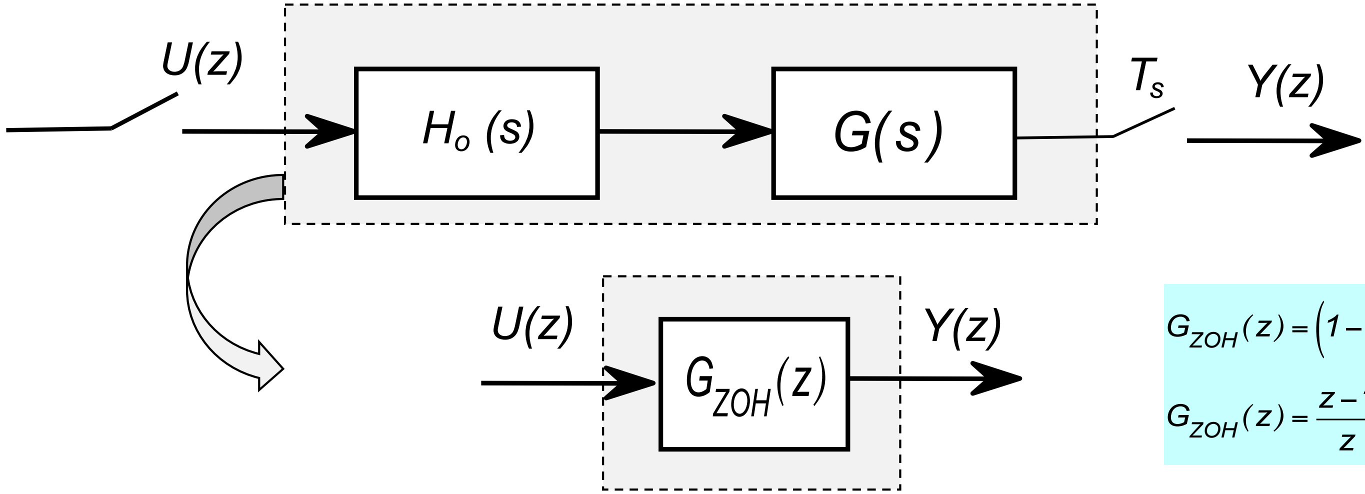
Properties of sampled transfer functions



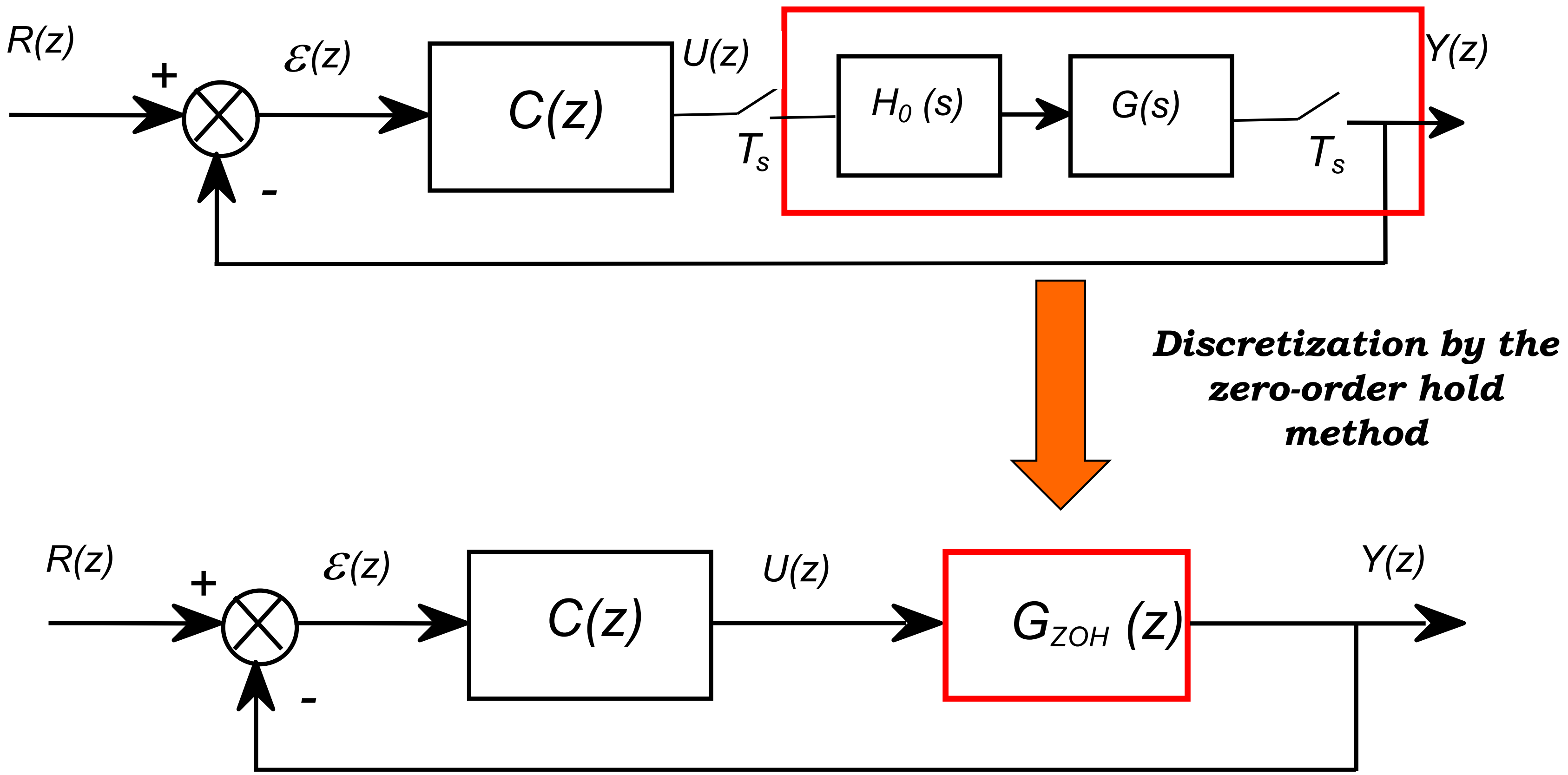
$$G_{ZOH}(z) = \left(1 - z^{-1}\right) Z\left(\frac{G(s)}{s}\right)$$

$$G_{ZOH}(z) = \frac{z-1}{z} Z\left(\frac{G(s)}{s}\right)$$

- ✓ A continuous **linear** system remains **linear** after discretization by ZOH
- ✓ The model order is preserved
- ✓ The parameters of the sampled transfer function are dependent on the sampling period T_s
- ✓ The poles p_d of the sampled transfer function can be computed from the poles p_c of the continuous-time model by the formula

$$p_d = e^{p_c T_s}$$

Digital control block-diagram



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