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POLYTECH NANCY

The z-transform

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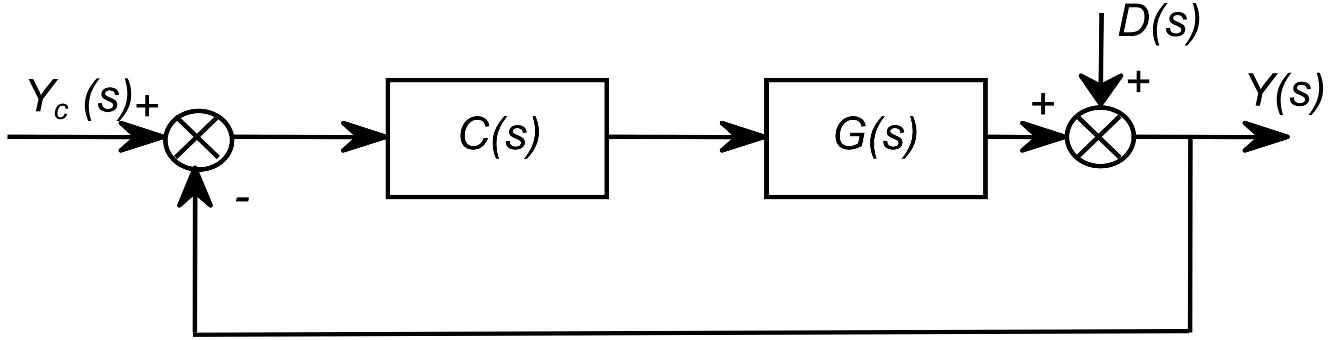
Interest of the Laplace transform *(reminder)*

- To determine the solution of a ***differential equation*** with constant coefficients
  - Ex: solve

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = 0, \qquad y(0) = 0; \dot{y}(0) = 1$$

- Mathematical tool that facilitates ***the analysis of continuous-time control feedback loops***



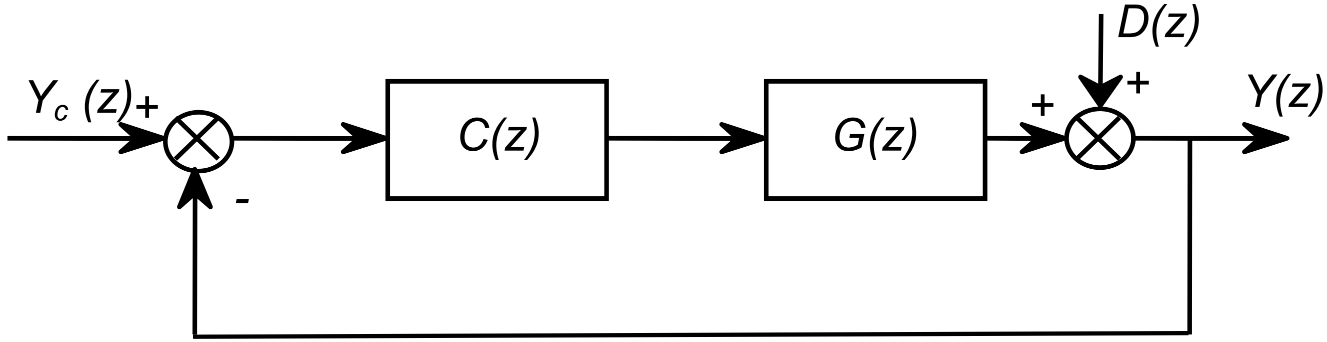
$$Y(s) = \frac{F_{BO}(s)}{1+F_{BO}(s)} Y_c(s) + \frac{1}{1+F_{BO}(s)} D(s)$$

Benefits of the Z-transform

- To determine the solution of a ***difference equation*** with constant coefficients
  - Ex : solve

$$y(k) - 4y(k-1) + 3y(k-2) = \delta(k), \qquad y(-1) = y(-2) = 0$$

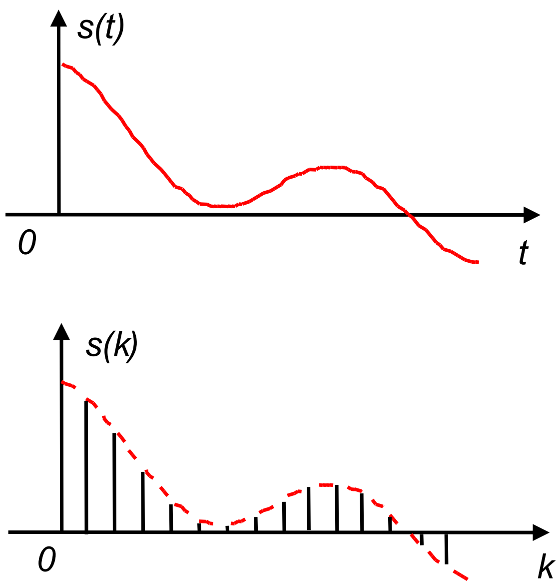
- Mathematical tool that facilitates ***analysis of digital control feedback loops***



$$Y(z) = \frac{F_{BO}(z)}{1+F_{BO}(z)} Y_c(z) + \frac{1}{1+F_{BO}(z)} D(z)$$

Discrete-time signal

- A *discrete-time* signal is denoted $s(k)$ with $k \in Z$
- It is defined only for discrete values of time
- It can be obtained by sampling a continuous-time signal: the values represent the signal samples



Discrete-time signal

- ***Definition***
  - A *discrete-time* signal $s(k)$ is a ***numerical sequence,*** i.e. an ordered list of numbers:

$$s(0)= 1, s(1)= 2, s(2)= 4, \ldots$$

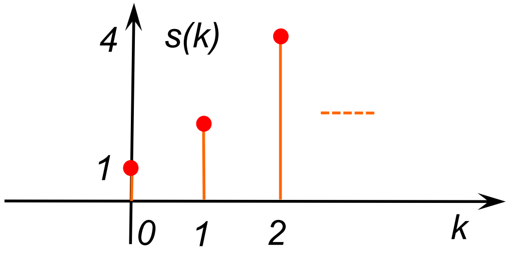
- ***Representation mode***
  - by an **analytical expression**

$$s(k) = 2^k \,, \; k \geq 0$$

  - by a **recurrence relation**

$$s(k) = 2 \times s(k-1), \; k \geq 1 \text{ with } s(0) = 1$$

  - with a **graphical representation**



Difference equations

- ***Definition***

  Let a discrete-time signal s*(0), s(1), ... s(k),...*

  An equation linking the $k^{\text{ième}}$ term to its predecessors is called a **recurrence equation** or **difference equation**

$$s(k) - 4s(k-1) + 3s(k-2) = \delta(k) \qquad s(-1) = s(-2) = 0$$

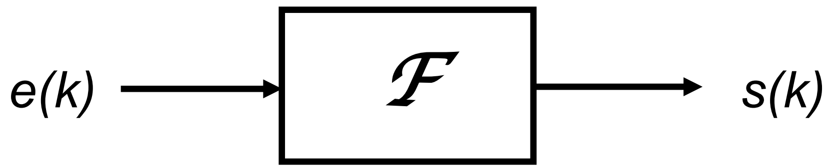
- **Solving a difference equation** consists in determining the solution (the digital signal) *s(k)* that satisfies the equation

$$s(k) = \frac{1}{2}(3^{k+1} - 1) \quad k \geq 0$$

- There are several methods for solving difference equations
  - One of them is based on the ***z-transform***

Discrete-time system

- A discrete-time system is defined as an operator between ***two discrete-time signals.*** It is described by a difference equation



$$s(k)=\mathcal{F}(s(k-1), (s(k-2), \ldots, e(k), e(k-1), \ldots)$$

$\mathcal{F}$ **:** is a ***discrete-time system*** which will be assumed hereafter to be:

- linear
- time-invariant
- causal

Tool for analyzing the characteristics of a linear discrete-time system

- A linear discrete-time system can be described mathematically by :
  - a difference equation
  - a convolution product
  - its transfer function

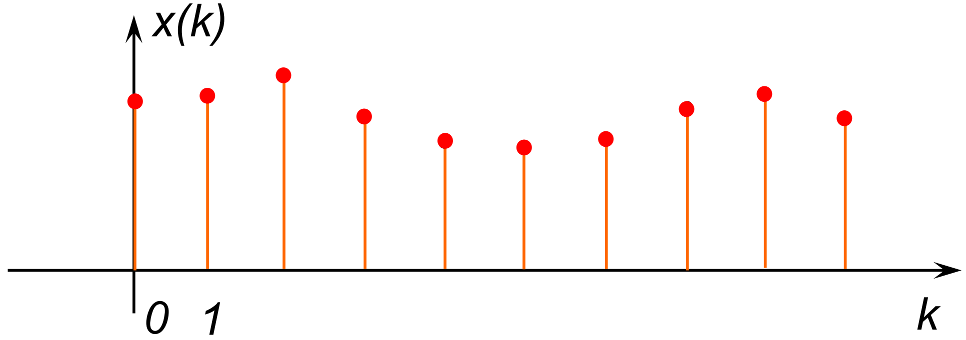
- ***The mathematical tool*** used to facilitate its analysis is :

the z-transform

z-transform

- Let *x(k)* be a causal discrete-time signal. ***The z-transform*** is defined by :

$$Z\big(x(k)\big)=X(z)=\sum_{k=0}^{+\infty} x(k)\, z^{-k}$$



where

  – z is the z-transform variable

  – $z = r\,e^{j\theta} = \alpha + j\beta$

- $X(z)$ *is* said to be the z-transform of the signal $x(k)$

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Link between Fourier transform and z-transform

- The z-transform defined above is in fact the monolateral z-transform

  In fact, there exists the two-sided z-transform defined by :

$$X(z) = \sum_{k=-\infty}^{+\infty} x(k)\, z^{-k}$$

- There is a relationship between the bilateral z-transform and the Fourier transform of a discrete-time signal:

$$X(f) = X(z)\Big|_{z=e^{j2\pi f T_e}} = \sum_{k=-\infty}^{+\infty} x(k)\, e^{-j2\pi f k T_e}$$

  – With T_e the signal sampling period

Link between Laplace transform and transformed into Z

For a <u>sampled signal</u> (ideally), the ***Laplace transform*** is given by :

$$X_e(s) = \sum_{k=0}^{+\infty} x(k)e^{-ksT_e} = \sum_{k=0}^{+\infty} x(k)\left(e^{-sT_e}\right)^k$$

En posant $z=e^{sT_e}$

$$X(z) = Z(x_e(t)) = \sum_{k=0}^{+\infty} x(k)\, z^{-k}$$

The ***Z-transform*** can therefore be seen as *the* ***Laplace transform*** applied to a sampled signal *(ideally)* in which the change of variable :

$$z=e^{sT_e}$$

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Whole series *(reminders)*

- ***Definition***

  An ***integer series*** of variable x *is* any sum (finite or infinite) of the elements of a numerical sequence with general term $\boldsymbol{u_k = a\, x_k^k}$

  where a_k is a real number and k is a natural ***number***

  $$u_0 + u_1 + u_2 + \dots$$

  On any interval where it is convergent, the sum of an integer series is a function. An integer series is therefore a function of the form :

  $$a_0 + a_1 x + a_2 x^2 + \dots = \sum_{k=0}^{+\infty} a_k\, x^k$$

- ***Remarks***
  - an integer series does not necessarily converge for all x
  - there exists an integer R called the ***radius or region of*** convergence such that the whole series converges for $|x| < R$ and diverges for $|x| > R$

Convergence region

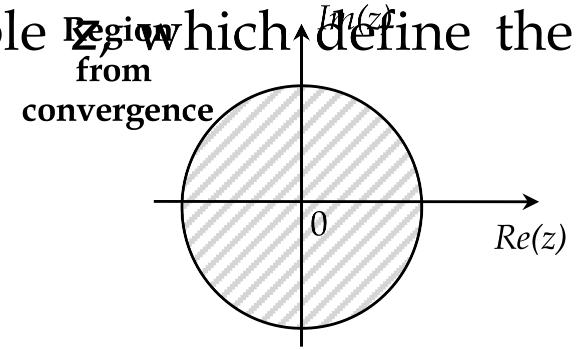
- The Z-transform of a signal *x(k)* is defined by an integer series *(infinite sum of the terms of a numerical sequence)*

$$X(z) = \sum_{k=0}^{+\infty} x(k)\, z^{-k} = \sum_{k=0}^{+\infty} x(k) \left(z^{-1}\right)^{k}$$

- When this sum is finite, the series is said to be convergent and *X(z)* exists.

- This is the case for certain values of the variable *z*, which define the Convergence Region (RdC).

$$\text{RdC} = \left\{ z = \alpha + j\beta \text{ telles que } \sum_{k=0}^{+\infty} \left| x(k) z^{-k} \right| < \infty \right\}$$



- RdC contains no pole of *X(z)*. It corresponds, in general, for causal signals outside a circle $|z| > a$
- For physical signals (which have a finite existence time) :

RdC= the whole complex plane with the possible exclusion of $z=0$ or $z=\infty$

Convergence region - Example

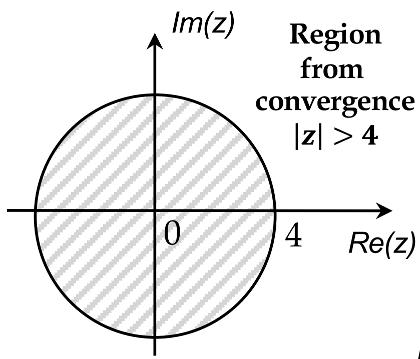
- Consider the digital signal whose first values are :
  x(0) = 1, x(1) = 4, x(2) = 16, x(3) = 64, x(4) = 256 ...

  Using the definition of the *z-transform*, determine *X(z)*. Under what condition does the resulting series converge? Assuming this condition is met, give the value of *X(z)*.

$$X(z) = \sum_{k=0}^{+\infty} x(k)\, z^{-k} = x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots$$

$$X(z) = 1 + 4z^{-1} + 4^2 z^{-2} + \dots = \left(\frac{4}{z}\right)^0 + \left(\frac{4}{z}\right)^1 + \left(\frac{4}{z}\right)^2 + \dots = \sum_{k=0}^{+\infty} \left(\frac{4}{z}\right)^k$$

$$X(z) = \sum_{k=0}^{+\infty} \left(\frac{4}{z}\right)^k = \frac{1}{1-\frac{4}{z}} = \frac{z}{z-4} \quad si \quad |z| > 4$$



Reminder: *sum of a geometric sequence of reason q* $\sum_{k=0}^{+\infty} q^k = \frac{1}{1-q}$ si $|q| < 1$

z-transform - Example

- Let be the discrete-time signal with finite existence time defined by :

x(0) = 1, x(1) = 2, x(2) = 3, x(k>2) = 0

Using the definition of the Z-transform, determine *X(z)*

$$X(z) = \sum_{k=0}^{+\infty} x(k)\, z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots = 1 + 2z^{-1} + 3z^{-2}$$

$$X(z) = \frac{1 + 2z^{-1} + 3z^{-2}}{1}$$

$$X(z) = \frac{z^2 + 2z + 3}{z^2}$$

Common form of a z-transform

- The Z-transform can be written in 2 forms:
  - ***Integer series (impractical form)***: obtained from the definition

    We obtain an integer series in negative power of z weighted by $x(k)$

$$X(z)=\sum_{k=0}^{+\infty} x(k)\,z^{-k}=x(0)+x(1)z^{-1}+x(2)z^{-2}+\ldots+x(i)z^{-i}+\ldots$$

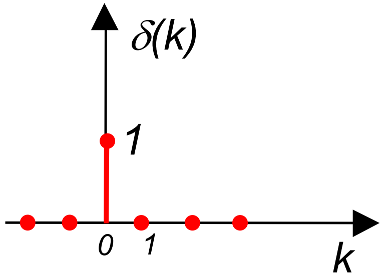
  - ***Rational function (most common form)*** in positive power of z or in negative power of z $(z)^{-i}$

$$Exemple : X(z)=\frac{1}{1-2z^{-1}+z^{-2}} \qquad ou \qquad X(z)=\frac{z^2}{z^2-2z+1}$$

Usual discrete-time signals

- ***Unit impulse or Kronecker impulse***

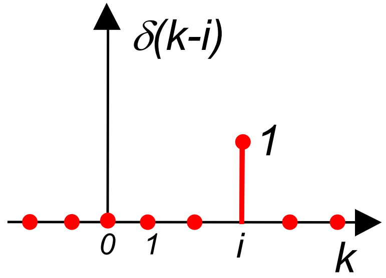
$$\delta(k) = \begin{cases} 1 & \text{pour} \quad k = 0 \\ 0 & \text{pour} \quad k \neq 0 \end{cases}$$



It should not be confused with the Dirac pulse $\delta(t)$, which is a continuous-time signal. It's much easier to manipulate!

- ***Delayed Kronecker impulse***

$$\delta(k-i) = \begin{cases} 1 & \text{pour} \quad k = i \\ 0 & \text{pour} \quad k \neq i \end{cases}$$

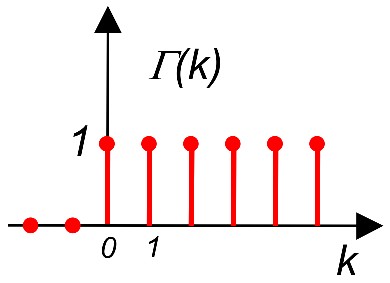


- Any signal $s(k)$ can be written : $s(k) = \sum_{i=-\infty}^{+\infty} s(i)\delta(k-i)$

Usual discrete-time signals

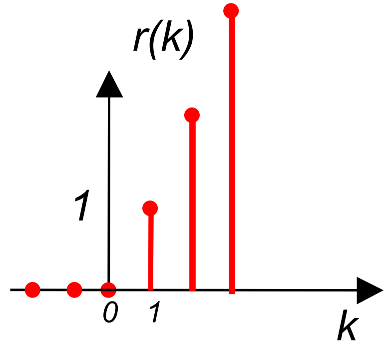
- ***Unit level***

$$\Gamma(k) = \begin{cases} 1 \;\; pour \;\; k \geq 0 \\ 0 \;\; pour \;\; k < 0 \end{cases}$$



- ***Ramp unit***

$$r(k) = \begin{cases} k \;\; pour \;\; k \geq 0 \\ 0 \;\; pour \;\; k < 0 \end{cases}$$

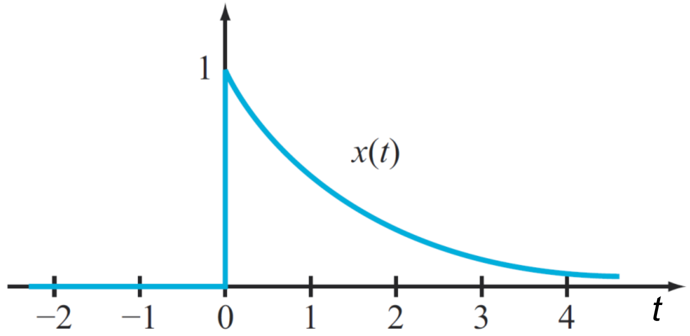


Usual discrete-time signals

- *Reminder*: causal continuous-time exponential signal (null if $t<0$)

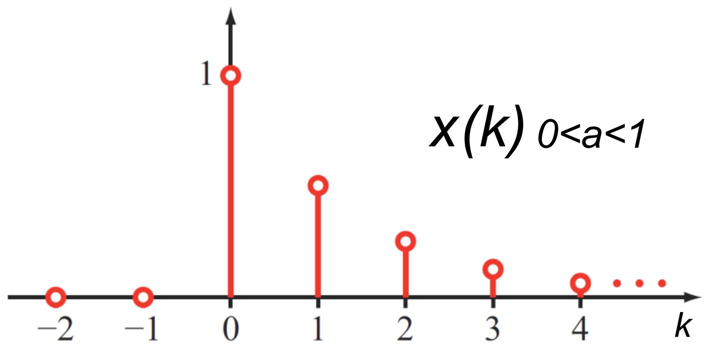
$$x(t) = e^{-\alpha t}\Gamma(t)$$

makes the signal causal



- **Discrete-time** *(or exponential base a)* and causal ***geometric signal***

$$x(k) = a^k\,\Gamma(k)$$



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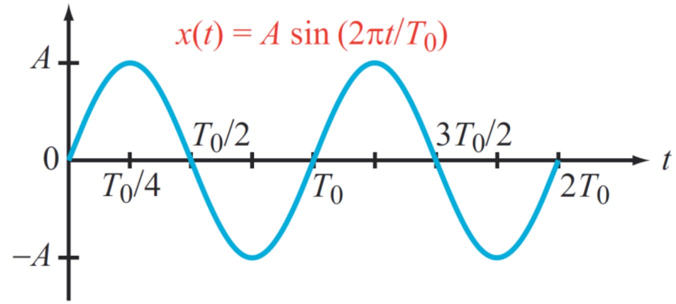
Discrete-time sinusoidal signal

- <u>Reminder</u>: *continuous-time* sinusoidal signal

$$x(t) = A\sin(\omega_o t + \varphi_o)$$

période: $T_o = \dfrac{2\pi}{\omega_o}$, ω_o en rad/s



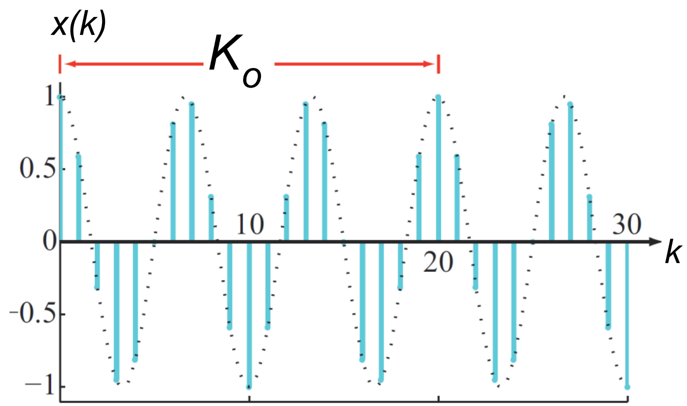
- ***Discrete-time sinusoidal signal***

$$x(k) = A\sin(\Omega_o k + \phi_o)$$

$x(k)$ périodique de période $K_o = \dfrac{2\pi}{\Omega_o}, K_o \in Z$

si $\Omega_o = \dfrac{M}{N}2\pi$ avec M et N entiers, Ω_o en rad

K_o plus petit entier >0 tel que $\dfrac{M}{N}$ entier



A discrete-time sinusoidal signal is not always periodic!

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z-transform of standard signals

- ***Impulse unit***

$$Z\left(\delta(k)\right)=1$$

$$Z\left(\delta(k)\right)=\sum_{k=0}^{\infty}\delta(k)\ z^{-k}=\delta(0)z^{-0}+\delta(1)z^{-1}+\ldots=1$$

- ***Unit level***

$$Z\left(\Gamma(k)\right)=\frac{1}{1-z^{-1}}=\frac{z}{z-1}$$

$$Z\left(\Gamma(k)\right)=\sum_{k=0}^{\infty}1\ z^{-k}=\sum_{k=0}^{\infty}\left(z^{-1}\right)^{k}$$

$$Z\left(\Gamma(k)\right)=\frac{1}{1-z^{-1}}\quad \text{si}\quad \left|z^{-1}\right|<1\ \Leftrightarrow\ \left|z\right|>1$$

Sum of a geometric sequence of reason z^{-1}

$$\sum_{k=0}^{\infty}q^{k}=\frac{1}{1-q}\quad si\quad \left|q\right|<1$$

z-transform of standard signals

- ***Geometric signal (causal)***

$$Z\left(a^k \Gamma(k)\right) = \frac{z}{z-a}$$

$$si\ a = e^{bT_e}, \qquad Z\left(\left(e^{bT_e}\right)^k \Gamma(k)\right) = \frac{z}{z-e^{bT_e}}$$

- ***Sine signals (causal)***

$$Z\left(\sin\left(\omega_o kT_e\right)\Gamma(k)\right) = \frac{z \sin\left(\omega_o T_e\right)}{z^2 - 2\cos\left(\omega_o T_e\right)z + 1}$$

$$Z\left(\cos\left(\omega_o kT_e\right)\Gamma(k)\right) = \frac{z\left(z-\cos\left(\omega_o T_e\right)\right)}{z^2 - 2\cos\left(\omega_o T_e\right)z + 1}$$

Table of z-transforms

$x(k)$	$X(z)$
$\delta(k)$	1
$\delta(k-i)$	z^{-i}
$\Gamma(k)$	$\frac{z}{z-1}$
$r(k)=kT_e\Gamma(k)$	$\frac{zT_e}{(z-1)^2}$
$(kT_e)^2\,\Gamma(k)$	$\frac{z(z+1)T_e^2}{(z-1)^3}$
$a^k\Gamma(k)$	$\frac{z}{z-a}$

Table of z-transforms

$x(k)$	$X(z)$
$(kT_e)a^k\Gamma(k)$	$\dfrac{azT_e}{(z-a)^2}$
$(kT_e)^2 a^k\Gamma(k)$	$\dfrac{az(z+a)T_e^2}{(z-a)^3}$
$\sin(\omega_o kT_e)\Gamma(k)$	$\dfrac{z\sin(\omega_o T_e)}{z^2-2\cos(\omega_o T_e)z+1}$
$\cos(\omega_o k_e T)\Gamma(k)$	$\dfrac{z\,(z-\cos(\omega_o T_e))}{z^2-2\cos(\omega_o T_e)z+1}$

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Properties *of* the z-transform

transform

- ***Linearity***

$$Z\left(a\,x(k)+b\,y(k)\right)=a\,X(z)+bY(z)$$

- ***Time delay***

$$Z\left(x(k-i)\right)=z^{-i}X(z)$$

$$Z\left(x(k-1)\right)=z^{-1}X(z)$$

$$Z\left(x(k-2)\right)=z^{-2}X(z)$$

- ***Time advance***

$$Z\left(x(k+i)\right)=z^{i}\left[X(z)-\sum_{k=0}^{i-1}x(k)\,z^{-k}\right]$$

$$Z\left(x(k+1)\right)=zX(z)-zx(0)$$

Initial conditions of the *x(k)* signal

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Properties of the z-transform

- ***Time convolution product***

$$x(k) * y(k) = \sum_{i=0}^{+\infty} x(i)\, y(k-i)$$

$$Z\big(x(k) * y(k)\big) = X(z) \times Y(z)$$

- ***Initial value theorem*** Final ***value theorem***

$$\lim_{k \to 0} \big(x(k)\big) = \lim_{z \to +\infty} \big(X(z)\big)$$

$$\lim_{k \to +\infty} \big(x(k)\big) = \lim_{z \to 1} \big(\big(z-1\big) X(z)\big)$$

If limits exist

Inverse z-transform

- ***The inverse z-transform recovers the signal samples***

$$Z^{-1}\left(S(z)\right) = s(k)$$

- ***Several methods are available:***

1. *Decomposition into sums of rational functions and use of tables*

$$S(z) = A_1 \frac{z}{z-a_1} + A_2 \frac{z}{z-a_2} + \dots$$

$$s(k) = Z^{-1}\left(S(z)\right) = Z^{-1}\left(A_1 \frac{z}{z-a_1} + A_2 \frac{z}{z-a_2} + \dots\right)$$

$$s(k) = A_1 Z^{-1}\left[\frac{z}{z-a_1}\right] + A_2 Z^{-1}\left[\frac{z}{z-a_2}\right] + \dots$$ *because the inverse Z-transform is linear*

$$s(k) = A_1 (a_1)^k \,\Gamma(k) + A_2 (a_2)^k \,\Gamma(k) + \dots$$ car $Z\left(a^k \,\Gamma(k)\right) = \frac{z}{z-a}$

Please note that this is not the usual breakdown into simple elements. Presence of a z in the numerator!

Inverse z-transform - Example

Find the original *x(k)* of the transform below:

$$X(z) = \frac{z^2}{z^2 - 3z + 2}$$

Derive the values of $x(0)$, $x(1)$, $x(2)$ and $x(3)$.

Inverse z-transform - Example

Find the original *x(k)* of the transform below:

$$X(z) = \frac{z^2}{z^2 - 3z + 2} = \frac{z^2}{(z-1)(z-2)} = A_1 \frac{z}{z-1} + A_2 \frac{z}{z-2}$$

$$A_1 = \lim_{z \to 1} \frac{z\text{-}1}{z} X(z) = \lim_{z \to 1} \frac{z\text{-}1}{z} \frac{z^2}{(z-1)(z-2)} = -1$$

$$A_2 = \lim_{z \to 2} \frac{z\text{-}2}{z} X(z) = \lim_{z \to 2} \frac{z\text{-}2}{z} \frac{z^2}{(z-1)(z-2)} = 2$$

$$x(k) = Z^{-1}\left(X(z)\right) = -Z^{-1}\left[\frac{z}{z-1}\right] + 2Z^{-1}\left[\frac{z}{z-2}\right]$$

$$x(k) = -\Gamma(k) + 2 \times 2^k \Gamma(k) = (-1 + 2^{k+1})\Gamma(k)$$

$k = 0, \quad x(0) = 1; \qquad k = 2, \quad x(2) = 7$

$k = 1, \quad x(1) = 3; \qquad k = 3, \quad x(3) = 15$

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Inverse z-transform

2. *Polynomial division*

$$X(z)=\frac{B(z)}{A(z)}=\frac{b_0 z^{n_b}+b_1 z^{n_b-1}+\ldots+b_{n_b}}{a_0 z^{n_a}+a_1 z^{n_a-1}+\ldots+a_{n_a}}$$

Simply divide *B(z)* by *A(z)* defined as a positive power of *z* to obtain a series in decreasing power of z^{-1} whose coefficients are the *x(k)* values we're looking for.

$$B(z) \quad \Big| \quad A(z)$$

$$X(z)=x(0)+x(1)z^{-1}+x(2)z^{-2}+x(3)z^{-3}+\ldots$$

By identification, we deduce : $\quad x(0),\quad x(1),\quad x(2),\quad x(3),\cdots$

Inverse Z-transform

2. *Polynomial division - Example*

$$X(z)=\frac{z^2}{z^2-3z+2}$$

$$\begin{array}{l|l} z^2 & z^2-3z+2 \\ \hline -z^2+3z-2 & 1+3z^{-1}+7z^{-2}+15z^{-3}+\ldots+x(i)z^{-i}+\ldots \\ \quad 3z+2 & \\ \quad \vdots & \end{array}$$

$$X(z)=x(0)+x(1)z^{-1}+x(2)z^{-2}+x(3)z^{-3}+\ldots+x(i)z^{-i}+\ldots$$

By identification, we deduce : $x(0)=1,\quad x(1)=3,\quad x(2)=7,\quad x(3)=15,\cdots$

Solving difference equations using the z-transform

- The resolution procedure is as follows:

1. Apply the z-transform to the 2 members of the difference equation in $x(k)$

2. Calculate $X(z)$ using the properties of the z-transform

3. Decompose $X(z)$ into simple rational functions

4. Use the transform table to obtain $x(k)$ by inverse transform

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Benefits of the Z-transform

- Determines the solution of a difference equation
  - Solve

$$s(k) - 4s(k-1) + 3s(k-2) = \delta(k) \qquad s(-1) = s(-2) = 0$$

$$s(k) = \frac{1}{2}(3^{k+1} - 1) \quad k \geq 0$$

- Facilitates the analysis of discrete-time systems
  - Determine the step response of a digital system. This is equivalent to solving a difference equation

$$s(k) = -\frac{1}{2}s(k-1) + e(k), \qquad e(k) = \Gamma(k)$$

$$s(k) = \frac{1}{3}\left(2 + \left(-0,5\right)^{k}\right)\Gamma(k) \qquad Z^{-1}\left(\frac{z}{z+0,5}\right) = Z^{-1}\left(\frac{z}{z-(-0,5)}\right) = (-0,5)^{k}\,\Gamma(k)$$