POLYTECH NANCY

**Control engineering**
**Problems n°1**

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# Laplace transform, transfer function and block-diagram analysis of linear time-invariant (LTI) dynamic systems

**Exercise 1.1 - Laplace transform of an exponential signal**
Consider the following signal:

$$x(t) = e^{-at}\Gamma(t), \quad a = 1$$

**1.1.a.** Recall the usual name of $\Gamma(t)$ and its definition.

**1.1.b.** Plot the signal $x(t)$.

**1.1.c.** Is the signal $x(t)$ causal ? Justify your answer.

**1.1.d.** What is the role of $\Gamma(t)$ in the definition of $x(t)$.

**1.1.e.** Determine the Laplace transform of $x(t)$ from the definition integral.

**Exercise 1.2 - Laplace transform of a delayed impulse function**
Consider the following signal:

$$y(t) = \delta(t - \tau), \quad \tau > 0$$

**1.2.a.** Recall the usual name of $\delta(t)$ and its definition.

**1.2.b.** Plot the signal $y(t)$.

**1.2.c.** Determine the Laplace transform of $y(t)$ by using the Laplace transform properties (see Appendices).

**Exercise 1.3 - Inverse Laplace transform**
Determine the inverse Laplace transform of:

$$Y(s) = \frac{2}{(s+3)(s+5)}$$

**Exercise 1.4 - Solution of differential equations**
Solve the following differential equations using the Laplace transform:

**1.4.a.**

$$\dot{y}_1(t) = -2y_1(t), \qquad y_1(0) = 1$$

**1.4.b.**

$$\dot{y}_2(t) + 2y_2(t) = \Gamma(t), \qquad y_2(0) = 1$$

**1.4.c.**

$$\ddot{y}_3(t) + 10\dot{y}_3(t) + 16y_3(t) = 10\delta(t), \qquad \dot{y}_3(0) = y_3(0) = 0$$

**Exercise 1.5 - Transfer function of a mechanical system**
Consider a mechanical suspension system shown in Figure 1.1 constituted of a mass, a damper and a spring having a damping and stiffness coefficient of $b$ and $k$ respectively.
The differential equation of this mechanical system relating the vertical position $y(t)$ of the mass (system output) and the external force $u(t)$ (system input) applied to the system is:

$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = u(t)$$

**1.5.a.** Determine the transfer function $G(s) = \frac{Y(s)}{U(s)}$ and represent the system in the form of a block diagram.

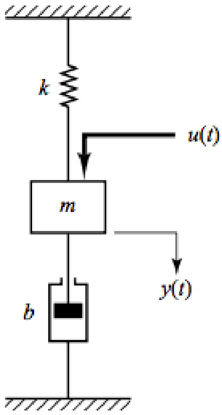


*Figure* 1.1: Mechanical system.

**1.5.b.** Give the system order, the steady-state gain, the pole(s) and zero(s) of the transfer function.

**Exercise 1.6 - Equivalent transfer function of simple closed-loop block-diagram**
Consider the closed-loop block diagram displayed in Figure 1.2.
Derive its equivalent transfer function $T(s) = \frac{Y(s)}{R(s)}$.

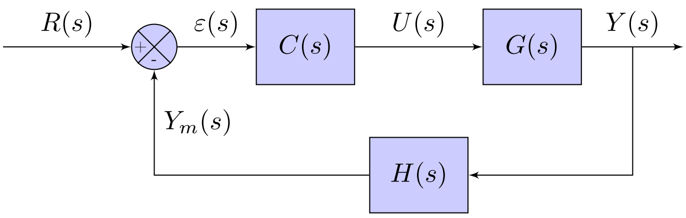


*Figure* 1.2: Classical block-diagram of a simple closed-loop feedback system

**Exercise 1.7 - Transfer function of a simple closed-loop block-diagram. A case study**
Consider the closed-loop block diagram displayed in Figure 1.3.
Determine its equivalent transfer function $T(s) = \frac{Y(s)}{R(s)}$.

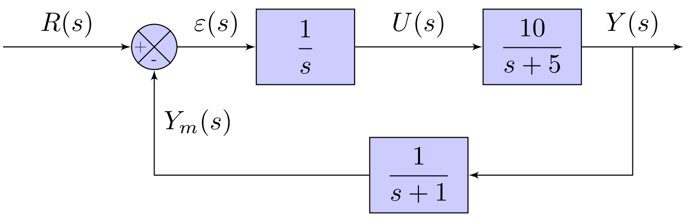


*Figure* 1.3: Case study of a simple closed-loop feedback system

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# Step responses of important systems & System identification from step response test

**Exercise 2.1 - Step response of a first-order system**
Consider the system described by the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2}{1+10s}$$

1. Determine the steady-state gain $K$, the time-constant $T$ and the pole.
2. Recall the unit step response and calculate its slope at the origin.
3. Calculate the rise-times $T_m^{63\%}$ and $T_m^{95\%}$ as well as the settling-time $T_r^{5\%}$.
4. Without any calculation, plot precisely the step response and indicate on it its characteristic parameters calculated above.

**Exercise 2.2 - Step response of a first-order plus time-delay system**
Consider a first-order system having a steady-state gain of 2, a time-constant of 10 seconds and a pure time-delay of 20 seconds.

1. Give the system transfer function $G(s)$.
2. Without any calculation, plot precisely the unit step response and indicate on it its characteristic parameters given above.

**Exercise 2.3 - Step response of a dynamical system**
Consider a system whose dynamic behavior is governed by the following differential equation:

$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = 10u(t) \qquad \text{with } \dot{y}(0) = 0, \quad y(0) = 0$$

1. Determine the transfer function $G(s) = \frac{Y(s)}{U(s)}$ of the system.
2. Determine the order of the system, the steady-state gain $K$, the damping ratio $z$, the undamped natural frequency $\omega_0$, the poles and zeros.
3. Conclude about the type of step response: critical, overdamped or underdamped.
4. Calculate the values of the first and second overshoot $D_{1\%}$ and $D_{2\%}$, the times of the first and second overshoot $t_{D_1}$ and $t_{D_2}$.
5. Plot the step response and indicate on it the characteristic parameters computed above.

**Exercise 2.4 - Dominant behavior identification of a thermal system from step response test**
Consider the temperature response $y(t)$ (in °Celsius) to a step input ($u(t) = 35 \times \Gamma(t-20)$) of a dynamic thermal system plotted in Figure 2.1.

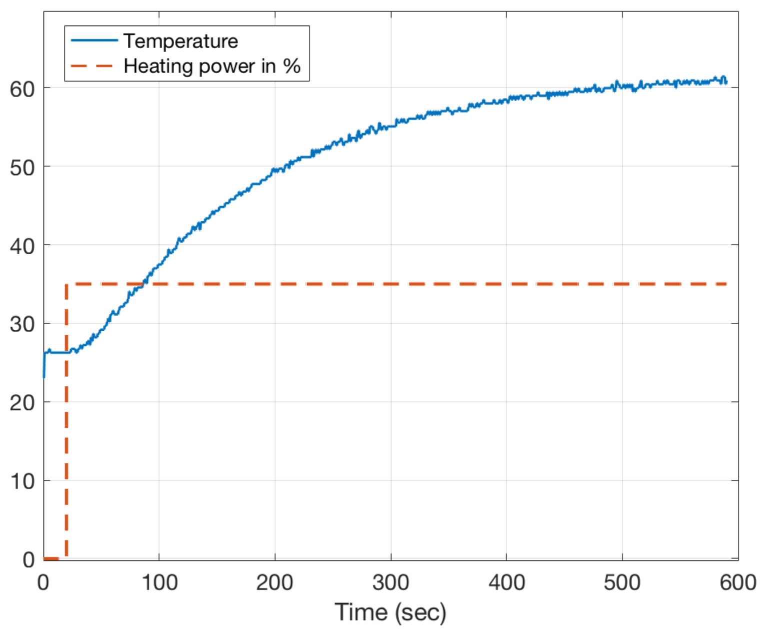


*Figure* 2.1: Step and temperature response of the thermal system

1. Propose a transfer function model form $G(s)$ for the system. Explain your choice.
2. Determine the numerical values of the different parameters of your chosen $G(s)$.
3. Deduce, from your identified transfer function model $G(s)$, the differential equation of the system.

**Exercise 2.5 - Identification of a mechanical suspension system from step response test**
Consider the position response $y(t)$ of a mechanical suspension system to a unit step input ($u(t) = \Gamma(t)$) plotted in Figure 2.2.

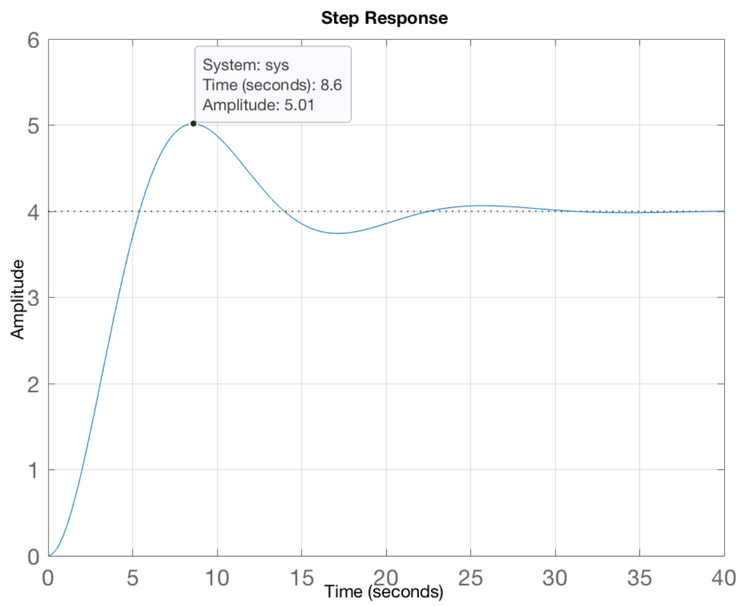


*Figure* 2.2: Response of a mechanical system to a unit step input

1. Propose a transfer function model form $G(s)$ for the system. Explain your choice.
2. Determine the numerical values of the different parameters of your chosen $G(s)$.
3. Deduce, from your identified transfer function model $G(s)$, the differential equation of the system.

**Exercise 2.6 - Step response analysis**
Figure 2.3 shows the unit step responses of five different linear time-invariant (LTI) systems. Pair each of the 5 step responses to one of the 7 transfer functions below. Explain your answers.

$$G_1(s) = \frac{0.1}{s+0.1}; \qquad G_2(s) = \frac{4}{s^2+2s+4}; \qquad G_3(s) = \frac{0.5}{s^2-0.1s+2}$$
$$G_4(s) = \frac{-0.5}{s^2+0.1s+2}; \qquad G_5(s) = \frac{1}{s+1}; \qquad G_6(s) = \frac{4}{s^2+0.8s+4}$$
$$G_7(s) = \frac{2}{s^2+s+3}$$

You can use Matlab to verify your solutions.

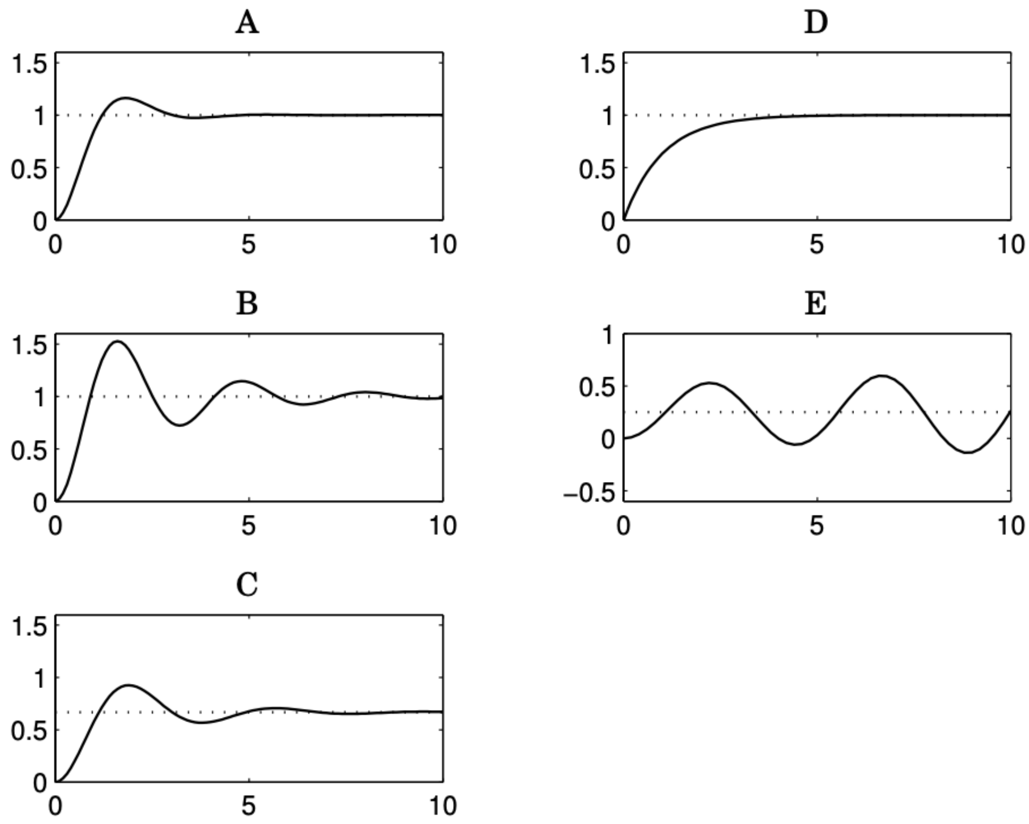


*Figure* 2.3: Step responses of different LTI systems

# Stability and steady-state error analysis

**Exercise 3.1 - Stability from the system poles**
Determine the poles and zeros of the LTI systems described by the transfer functions below. Conclude about their stability:

$$G_1(s) = \frac{2}{s+2}; \qquad G_2(s) = \frac{2}{s^2+3s+2}; \qquad G_3(s) = \frac{1}{s^2+2s+2}$$
$$G_4(s) = \frac{2}{s^2+4}; \qquad G_5(s) = \frac{2}{s(s+2)}; \qquad G_6(s) = \frac{2}{s^2(s+2)}$$
$$G_7(s) = \frac{2(s^2-2s+2)}{(s+2)(s^2+2s+2)}; \qquad G_8(s) = \frac{200}{(s+2)(s^2-2s+2)}$$

**Exercise 3.2 - Links between system poles and step responses**
Pair the step responses and pole-zero diagrams in Figure 3.1. Give your solutions so that the pairs of plots that belong to the same system is written in the form Pole-zero-letter–Step-response-letter.

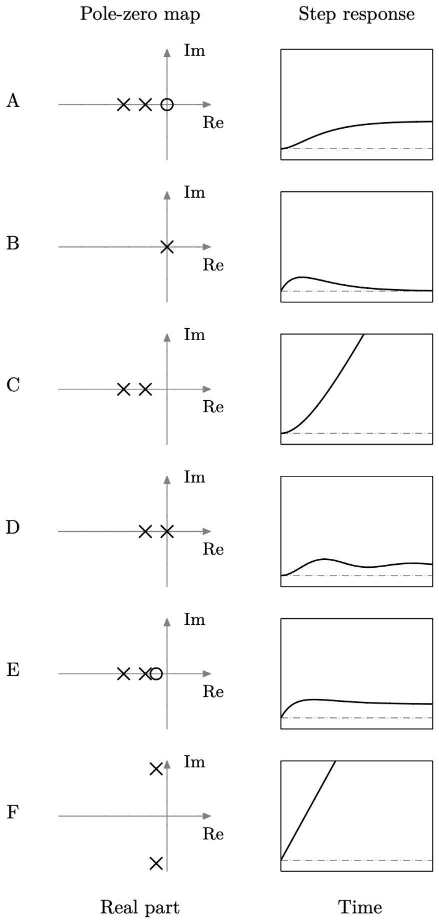


*Figure* 3.1: Pole-zero diagrams and step responses of different LTI systems. In the pole-zero diagram, imaginary and real parts have equal scaling, `x` marks poles, and `o` marks zeros.

**Exercise 3.3 - Stability analysis by using the Routh-Hurwitz criterion**
Study the stability of the LTI systems having the characteristic equations below. Specify, in case of instability, the number of unstable poles:

a) $s^3 - 25s^2 + 10s + 450 = 0$

b) $s^3 + 25s^2 + 450 = 0$

c) $s^3 + 25s^2 + 10s + 450 = 0$

d) $s^3 + 25s^2 + 10s + 50 = 0$

**Exercise 3.4 - Elevator control system for supertall building**
Yokohama's 70-floor Landmark Tower was completed in 1993 and it is still the tallest building in the Greater Tokyo Area with a height of 293m. Its elevators operate at a peak speed of 45 km/h! To reach such a speed without inducing discomfort in passengers, the elevator accelerates for longer periods, rather than more precipitously. Going up, it reaches full speed only at the 27th floor; it begins decelerating 15 floors later. The result is a peak acceleration similar to that of other skyscraper elevators–a bit less than a tenth of the force of gravity. Admirable ingenuity has gone into making this safe and comfortable. Special ceramic brakes had to be developed; iron ones would melt. Computer-controlled systems damp out vibrations. The lift has been streamlined to reduce the wind noise as it speeds up and down.

The closed-loop block-diagram for the control of the elevator's vertical position is shown in Figure 3.2.

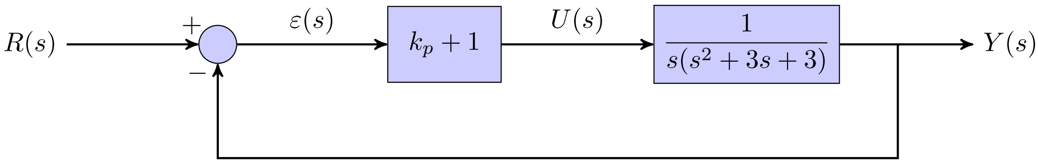


*Figure* 3.2: Vertical position control system for an elevator

1. Determine the range of values for $k_p$ that ensures the stability of the control vertical position system.
2. By using the final value theorem, calculate the expected steady-state error for a step reference of amplitude $A$ as a function of $k_p$.

**Exercise 3.5 - Mobile robot steering control**
The block diagram of the steering control system for a mobile robot is shown in Figure 3.3, where $R(s)$ and $Y(s)$ represent the Laplace transform of the desired and measured heading angle respectively.

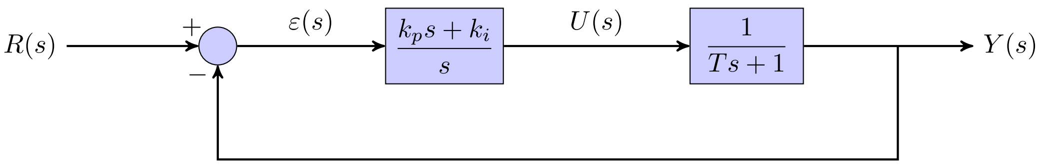


*Figure* 3.3: Block diagram of the steering control system for a mobile robot

1. Express the steady-state error in terms of $R(s)$, $k_p$, $k_i$ and $T$.
2. When $k_p > 0$ and $k_i = 0$, determine the steady-state error for a step reference input $r(t) = A\Gamma(t)$.
3. When $k_p > 0$ and $k_i > 0$, determine the steady-state error for a step reference input $r(t) = A\Gamma(t)$.
4. When $k_p > 0$ and $k_i > 0$, determine the steady-state error for a ramp reference input $r(t) = At\Gamma(t)$.

## Cruise control of a vehicule

Automatic cruise control is an excellent example of a feedback control system found nowadays in most vehicles. The purpose of the cruise control system is to track the desired speed set by the driver and to maintain the vehicle speed constant despite external disturbances, such as changes in wind or road grade. This is usually accomplished by measuring the vehicle speed, comparing it to the desired or reference speed, and automatically adjusting the throttle according to a control law.

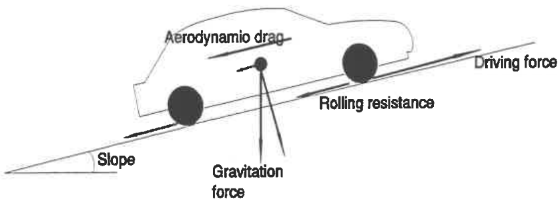


*Figure* 4.1: Forces acting on a car

We consider here a simple model of a vehicle represented in Figure 4.1. The vehicle, of mass $m$, is acted on by a driving force, $f(t)$ (in N) which represents the force generated at the road/tire interface and $\phi(t)$ (in rad) denotes the angle of the road with the horizontal axis.
For this simplified model it is assumed that we can control the force $f(t)$ directly and will neglect the dynamics of the powertrain, tires, etc., that go into generating the force. The resistive forces due to rolling resistance and wind drag, are assumed to vary linearly with the vehicle velocity, $v(t)$ (in m/s) through the damping coefficient $b$, and act in the direction opposite the car motion.
Summing forces in the horizontal direction and applying Newton's second law, we arrive at the following differential equation:

$$m\dot{v}(t) + bv(t) = f(t) - mg\sin(\phi(t)) \tag{1}$$

where $g$ is the acceleration due to gravity.
For this problem, we assume that the physical parameters of the system are:

$$m = 1000 \text{ kg}; \qquad b = 100 \text{ Ns/m}; \qquad g = 10 \text{ m/s}^2$$

1. **Modelling**
   - **1.a.** Define the output $y(t)$, the input $u(t)$ and the disturbance variable $d(t)$ and their unit.
   - **1.b.** Is the model describing the vehicle dynamics linear? Explain your answer.
   - **1.c.** Assuming that the slope remains small, linearize the model.
   - **1.d.** Let $Y(s)$, $U(s)$ and $D(s)$ denote the Laplace transforms of $y(t)$, $u(t)$ and $d(t)$ respectively. Show that equation (1) can be written in the Laplace domain as:

     $$Y(s) = G(s)U(s) + G_D(s)D(s)$$

     where $G(s) = \dfrac{K}{1+Ts}$ and $G_D(s) = \dfrac{K_D}{1+Ts}$.
     Express the value of the two steady-state gains $K$ and $K_D$ along with the time-constant $T$ in terms of $m$, $b$ and $g$.
   - **1.e.** Compute the poles of each model and conclude about the stability of both transfer function.
   - **1.f.** Represent the system in the form of a block-diagram.

In the following study, we will assume first that the road is flat $\phi(t) = 0$ and therefore that there is no disturbance acting on the control loop. The performance specification for the cruise control in response to a step on the speed setpoint are the following:

- Settling-time at 5 % $\leq$ 10 s
- Overshoot $\leq$ 10%
- Steady-state error $\leq$ 2%

2. **Proportional feedback (P) control**
   Let $R(s)$ denote the Laplace transform of the speed reference (or setpoint) $r(t)$. We want to drive at a constant speed of $r(t) = 25\Gamma(t)$ (25 m/s = 90 km/h).

**2.a.** We choose first to implement a simple feedback proportional (P) controller where the throttle is automatically adjusted according to the following control law:

$$u(t) = k_p\varepsilon(t) \qquad \text{where } k_p > 0$$
$$\varepsilon(t) = r(t) - y(t)$$

Determine the controller transfer function $C(s) = \dfrac{U(s)}{\varepsilon(s)}$.

**2.b.** Represent the closed-loop block diagram of the cruise control.

**2.c.** Calculate the open-loop $F_{OL}(s)$ and closed-loop transfer function $F_{CL}(s)$.

**2.d.** Determine the range of values for $k_p$ that ensures the stability of the closed-loop P control.

**2.e.** By using the final value theorem, determine the steady-state error

$$\lim_{t\to+\infty} \varepsilon(t) = \lim_{t\to+\infty} (r(t) - y(t))$$

in terms of $k_p$ for the following setpoint $r(t) = 25\Gamma(t)$ (25 m/s = 90 km/h).

**2.f.** Compute the steady-state error when $k_p$=900. Is the requirement for the steady-state error satisfied?

**2.g.** Determine the value of $k_p$ to satisfy the performance specification. What are the practical limits to this proportional gain value and therefore this simple P control.

3. **Proportional and integral (PI) feedback control**

**3.a.** We now choose to implement a proportional integral (PI) controller, given by the following transfer function:

$$C(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s}$$

**3.b.** Determine the new open-loop $F_{OL}(s)$ and closed-loop transfer function $F_{CL}(s)$.

**3.c.** Determine the range of values for $k_p$ and $k_i$ that ensures the stability of the closed-loop PI control.

**3.d.** By using the final value theorem, determine the steady-state error

$$\lim_{t\to+\infty} \varepsilon(t) = \lim_{t\to+\infty} (r(t) - y(t))$$

for the following setpoint $r(t) = 25\Gamma(t)$ (25 m/s = 90 km/h).

**3.e.** By neglecting the presence of the zero, determine the value of $k_p$ and $k_i$ to have a percent overshoot of $D_{1\%} = 10$ % and a settling time at 5% equal to $T_r^{5\%} = 10$s. The following formula could be useful:

$$z = \sqrt{\frac{(\ln(D_1))^2}{\pi^2 + (\ln(D_1))^2}}$$
$$\omega_0 \approx \frac{3}{T_r^{5\%} z}$$

4. **Adaptive cruise control**

   Adaptive cruise control (ACC) is an intelligent form of cruise control with "automatic distance control" that slows down and speeds up automatically to keep pace with the car in front of you.

   The driver sets the desired speed, just as with traditional cruise control, then a radar sensor watches for traffic ahead, locks on to the car in a lane, and instructs the car to stay 2 to 4 seconds behind the car ahead.
   Regardless of the technology, ACC should work day and night, but its abilities are hampered by heavy rain, fog, or snow.

   Adaptive cruise control is one of 20 terms used to describe its functions.

   **4.a** Look on the Internet and find out 5 alternative names given to adaptive cruise control.

   **4.b** Look on youtube to find and watch a short video presenting how adaptive cruise control works.

Control engineering
Problems n°5
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# Temperature control for battery systems

## Exercise 5.1 - Empirical PID tuning rules

Several tuning rules are available for determining the PID controller parameter values when the system can be approximated as a first-order plus time-delay transfer function model determined from an open-loop step response (See Appendix). The proposed rules address different design specifications such as disturbance rejection or setpoint tracking mode.

1. The temperature response $\theta(t)$ of a continuous-time thermal process to a step input $(u(t) = 35 \times \Gamma(t-20))$ when there is no disturbance has been recorded and is plotted in Figure 5.1.

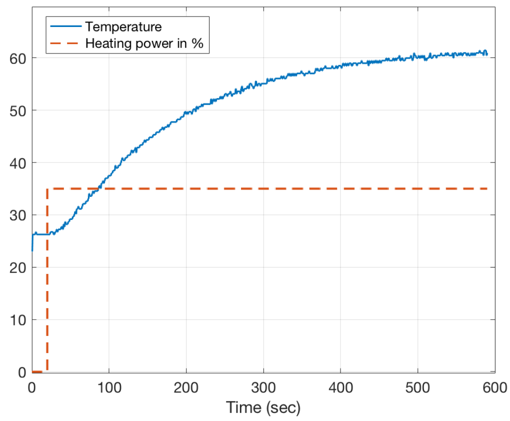


*Figure* 5.1: Step and temperature response of a thermal system

From the step response, determine the numerical parameter values of a first-order transfer function plus time-delay model (Exercise 2.4 revisited!).
We assume in the following that the thermal system can be reasonably well modeled by the first-order plus delay transfer function model

$$G(s) = \frac{1}{1+140s}e^{-10s}$$

2. Find the PI controller that results by applying the Ziegler-Nichols tuning rules to the process when the design specification is disturbance rejection.

3. Find the PI controller that results by applying the Chien-Hrones-Reswick tuning rules to the process when the design specification is setpoint tracking.

## Exercise 5.2 - Temperature control

Extreme temperature changes result in many failures of electronic circuits or battery systems. Temperature control feedback systems can reduce the change of temperature by using a heater to overcome outdoor low temperatures. The analysis aims at illustrating some of the control tuning and performance issues associated with temperature control.

### 1. System modelling

The differential equation model of the thermal system takes the following form

$$T\frac{d\theta(t)}{dt} + \theta(t) = Ku(t-\tau) + K_D d(t) \quad (1)$$

where

- $\theta(t)$ is the temperature in °Celsius
- $u(t)$ is the heating power in % such that $u(t) \in [0 - 100\%]$
- $d(t)$ represents the effects of a cold airstream

**1.a.** Define the output, the input and the disturbance variable.

**1.b.** Let $\Theta(s)$, $U(s)$ and $D(s)$ denote the Laplace transforms of $\theta(t)$, $u(t)$ and $d(t)$ respectively. Show that equation (1) can be written in the Laplace domain as:

$$\Theta(s) = G(s)U(s) + G_D(s)D(s)$$

where $G(s) = \dfrac{Ke^{-\tau s}}{1+Ts}$ and $G_D(s) = \dfrac{K_D}{1+Ts}$.

**1.c.** Represent the system in the form of a block diagram. We assume in the following that the numerical values of the model parameters are:

- $K = 1$
- $T = 140$ s
- $\tau = 10$ s
- $K_D = 0.5$

## 2. Performance analysis of a PI feedback control in servo mode

We suppose first that there is no disturbance ($d(t) = 0$). The block diagram of the temperature servo-control is displayed in Figure 5.2.

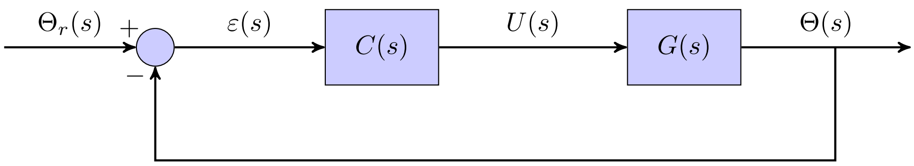


*Figure* 5.2: Block-diagram of the temperature control in servo mode

The design specifications for the temperature control in servo mode (or setpoint tracking mode) are the following:

- the steady-state error in response to a step on the temperature setpoint should be zero.
- the closed-loop system response to a step setpoint should be critically overdamped. No overshoot is tolerated.

The controller is chosen to be a PI controller of the following ideal form:

$$C(s) = K_p \frac{1+T_i s}{T_i s}$$

As the time-delay $\tau$ is much smaller than the time-constant $T$, the following approximation can be made

$$e^{-\tau s} = \frac{1}{e^{\tau s}} \approx \frac{1}{1+\tau s}$$

The transfer function model of the system then becomes

$$G(s) = \frac{K}{(1+\tau s)(1+Ts)}$$

When the system can be well approximated as an overdamped second-order transfer function model with two different time-constants, one approach to design the PI controller consists in setting the integral time-constant value $T_i$ to compensate the dominant (larger) time-contant of the system ($T$ here). This PI tuning rule is known as the compensation method of the dominant time-constant of the system.

**2.a.** The design of the PI controller is chosen such that the dominant time-constant of the system is compensated which results in setting the integral action $T_i = T$. With this setting for $T_i$, determine the open-loop transfer function $F_{OL}(s)$ in terms of $K$, $K_p$, $\tau$, $T$.

**2.b.** Express the closed-loop transfer function $F_{CL}(s)$ in terms of $K$, $K_p$, $\tau$, $T$.

**2.c.** Determine the range of values for $K_p$ that ensures the stability of the feedback control system.

**2.d.** Calculate the steady-state tracking error in response to a step $\theta_r(t) = A\Gamma(t)$, *i.e.* :

$$\lim_{t\to+\infty} \varepsilon(t) = \lim_{t\to+\infty} (\theta_r(t) - \theta(t))$$

**2.e.** Determine the value for $K_p$ that makes the closed-loop transfer function to have a damping ratio of $z = 1$.

**2.f.** Compare the designed PI controller with the one obtained by applying the empirical Chien-Hrones-Reswick tuning rules obtained in Exercise 5.1.

## 3. Performance analysis of a PI feedback control in regulation mode

We now investigate the performance of the feedback control in presence of a disturbance $d(t)$.
The design specifications for the temperature control in regulation mode (setpoint set to a constant value) are the following:

- the closed-loop system should be able to reject a constant cold airstream.

The controller is chosen to be a PI controller of the following ideal form:

$$C(s) = K_p \frac{1 + T_i s}{T_i s}$$

Let us assume that the temperature setpoint now equals to zero ($\theta_r(t) = 0$). The block-diagram of the regulation control is displayed in Figure 5.3. The PI controller $C(s)$ is the one that results from the dominant time-constant compensation method.

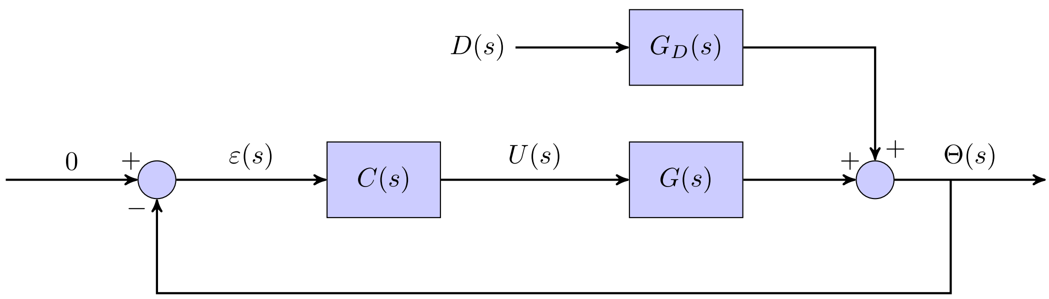


*Figure* 5.3: Block-diagram of the feedback control in regulation mode

**3.a.** Show that $\Theta(s)$ can be written as:

$$\Theta(s) = F_D(s)D(s)$$

where $F_D(s) = \dfrac{G_D(s)}{1 + C(s)G(s)}$

**3.b.** Express the transfer function $F_D(s)$ in terms of $K$, $K_p$, $\tau$, $T$.

**3.c.** At time-instant $t_0$, a constant stream of cold air is acting on the system. Compute the steady-state value of the temperature $\theta(t)$ in response to a step $d(t) = \alpha\Gamma(t - t_0)$, *i.e.*

$$\lim_{t\to+\infty} \theta(t)$$

**3.d.** Conclude about the ability of the PI feedback control to reject a constant stream of cold air.

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# Positional control of a DC servo-motor

One of the most common devices for actuating a control system is DC motors. A DC motor is any of a class of rotary electrical motors that converts direct current electrical energy into mechanical energy. Small DC motors are commonly used in tools, toys, appliances and in computer-related equipment such as disk drives or printers. Larger DC motors are currently used in propulsion of electric vehicles, robot systems, elevators, cranes and hoisting devices. Large DC motors are also widely used in the industry such as in drives for steel rolling mills. DC motors that are used in servo systems are called DC servo-motors. In DC servo-motors, the rotor inertias have been made very small. They are frequently used in robot control systems and other angular position or speed control system.
We study the angular position (positional) control of a DC servo-motor in this problem as they appear for each rotary joint of robotic arms such as the Canadarm2 as shown in Figure 6.1 for example.

*Figure* 6.1: Astronaut Stephen Robinson anchored to the end of Canadarm2 during STS-114, 2005. By NASA - http://spaceflight.nasa.gov/gallery/images/shuttle/sts-114/html/s114e6647.html

## 1. DC servo-motor modelling

The inductance in the armature circuit of a DC servo-motor is usually small and can therefore be neglected. In this case, the motor voltage-to-angular position transfer function for the DC servo-motor takes the following form

$$G(s) = \frac{\Theta(s)}{U(s)} = \frac{K}{s(1+Ts)} \tag{1}$$

- $\Theta(s) = \mathscr{L}\left[\theta(t)\right]$, where $\theta(t)$ is the angular position of the motor shaft in rad;
- $U(s) = \mathscr{L}\left[u(t)\right]$, where $u(t)$ is the applied motor voltage in V;
- $K$ is the motor gain in rad/(V.s);
- $T$ is the motor time-constant in second.

**1.a.** Define the input and the output of the DC servo-motor and their unit.

**1.b.** From equation (1), it can be seen that the transfer function involves a pure integrator term $\frac{1}{s}$. The transfer function model can also be seen as the cascade of a pure integrator and a simple first-order model.
It is indeed well-known that the angular velocity $\omega(t)$ is the time-derivative of the angular position $\theta(t)$

$$\omega(t) = \frac{d\theta(t)}{dt}$$

Both variables are linked in the Laplace domain by a pure integrator (or pure derivator) so that (1) can be expressed as:

$$\frac{\Theta(s)}{U(s)} = \frac{\Theta(s)}{\Omega(s)} \times \frac{\Omega(s)}{U(s)} = \frac{1}{s} \times \frac{K}{1+Ts} \tag{2}$$

$\Omega(s) = \mathscr{L}\left[\omega(t)\right]$, where $\omega(t)$ is the motor angular velocity (or speed) in rad/s.
From the analysis above, complete the block-diagram in Figure 6.2.

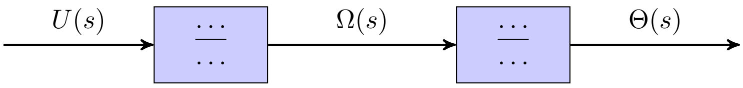
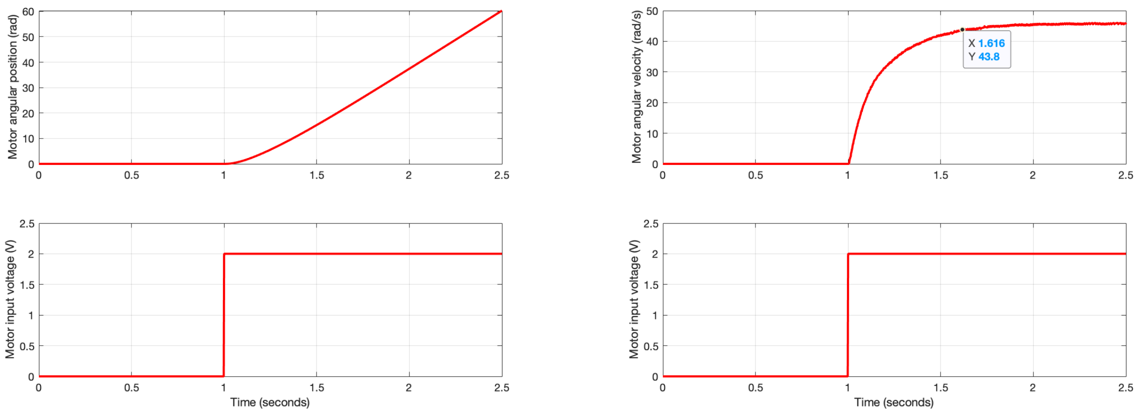



*Figure* 6.2: Block-diagram of the DC motor

**1.c.** Identifying a system having a pure integrator from a step response is tricky since the response is diverging. The angular position response (in rad) to a step input of amplitude 2V sent to the motor voltage is plotted in Figure 3(a). The response of the motor starts out slowly due to the time constant, but once that is out of the way the motor position ramps at a constant velocity. It is easier when the motor speed is also measured to identify the response between the motor speed and the input voltage since the voltage-to-angular velocity transfer function has the well-known first-order model form whose parameters can be easily estimated from the step response (the a priori knowledge about the pure integrator is then added in the final model):

$$\frac{\Omega(s)}{U(s)} = \frac{K}{1+Ts}, \tag{3}$$

The angular velocity response (in rad/s) to a step input of amplitude 2V sent to the motor voltage is plotted in Figure 3(b). From the step response plot, determine the parameters of the first-order model.




*(a)* Angular position response

*(b)* Angular velocity response

*Figure* 6.3: Angular position and velocity responses to a step motor input voltage

The performance requirements for the angular position control design are described in Table 1.

| Requirement | Assessment criteria | Level |
|---|---|---|
| Control the position | position reference tracking | No steady-state error |
| | motor input voltage | limited to the range [-5V ; +5 V] |
| | Settling time at 5 % | As short as possible |
| | Overshoot | less than or equal to 5% |
| | Disturbance rejection | Rejection of load effects |

Table 1: Performance requirements for angular position control

We assume in the following that the numerical values of the model parameters are:

- $K = 23$ rad/(V.s);
- $T = 0.2$ s

## 2. Servo-motor control using simple proportional feedback

Figure 6.4 shows a simple proportional feedback configuration of the positional servo system. This basic configuration has been used in industry for many years.

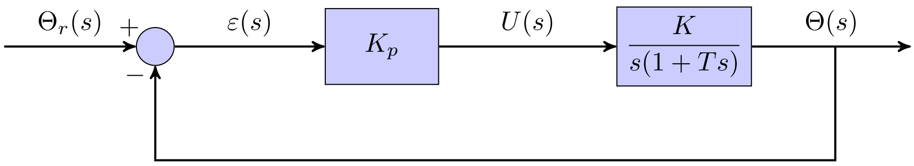



*Figure* 6.4: Block-diagram of the simple proportional feedback configuration of the positional servo system

**2.a.** Determine the open-loop transfer function $F_{OL}(s)$ in terms of $K$, $K_p$ and $T$.

**2.b.** Express the closed-loop transfer function $F_{CL}(s)$ in terms of $K$, $K_p$ and $T$.

**2.c.** Determine the range of values for $K_p$ that ensures the stability of the feedback control system.

**2.d.** Calculate the steady-state tracking error in response to a step $\theta_r(t) = A\Gamma(t)$, *i.e.* :

$$\lim_{t \to +\infty} \varepsilon(t) = \lim_{t \to +\infty} (\theta_r(t) - \theta(t))$$

**2.e.** Determine the value for $K_p$ that makes the closed-loop transfer function to have a damping ratio of $\frac{\sqrt{2}}{2}$.

**2.f.** Calculate the percent overshoot, the peak time and settling time at 5 % (see abacus in the Appendix). Plot the shape of the closed-loop P control response to a unit step setpoint. Are the performance requirements satisfied?

## 3. Servo-motor control using proportional and derivative feedback

We now consider the performance of a proportional and derivative (PD) control which involves a velocity feedback loop as shown in Figure 6.5. This control system represents a high-speed, high precision positional servo system. The positional servomotor systems of this type are used frequently in today's angular position control systems.

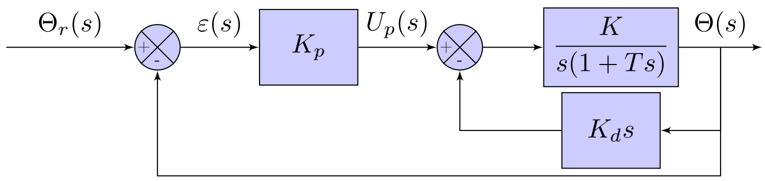



*Figure* 6.5: Block-diagram of the PD feedback configuration of the positional servo system

**3.a.** What is the advantage of implementing the derivative term on the output rather than on the error signal $\varepsilon(s)$?

**3.b.** Determine the internal closed-loop transfer function $F_i(s) = \frac{\Theta(s)}{U_p(s)}$ in terms of $K$, $K_d$ and $T$.

**3.c.** Plot the simplified closed-loop block-diagram.

**3.d.** Determine the closed-loop transfer function $F_{CL}(s)$ in terms of $K$, $K_p$, $K_d$ and $T$.

**3.e.** Determine the range of values for $K_p$ and $K_d$ that ensure the stability of the feedback control system.

**3.f.** Calculate the steady-state tracking error in response to a step $\theta_r(t) = A\Gamma(t)$, *i.e.* :

$$\lim_{t\to+\infty} \varepsilon(t) = \lim_{t\to+\infty} (\theta_r(t) - \theta(t))$$

**3.g.** Determine the value for $K_p$ and $K_d$ that make the closed-loop transfer function step response to have a percent overshoot of 4.3 % and a settling time at 5% of 0.05s (see abacus in the Appendix).

**3.h.** Calculate the peak time and plot the shape of the closed-loop PD control response to a unit step setpoint. Are the performance requirements satisfied? Compare your plot with the closed-loop PD control response to a unit step setpoint displayed in Figure 6.6.

**3.i.** In practice, a pure derivative cannot be implemented because it will give very large amplification of the measurement noise. The gain of the derivative term must thus be limited at high frequencies. This is usually done by approximating the pure derivative term as

$$T_d s \approx \frac{T_d s}{1 + \frac{T_d s}{N}} \tag{4}$$

The low-pass filter transfer function on the right approximates the pure derivative well at low frequencies but the gain is limited to $N$ at high frequencies. $N$ is typically chosen in the range 3 to 20.

Modify the block-diagram of the closed-loop control to make appear the low pass-filter.

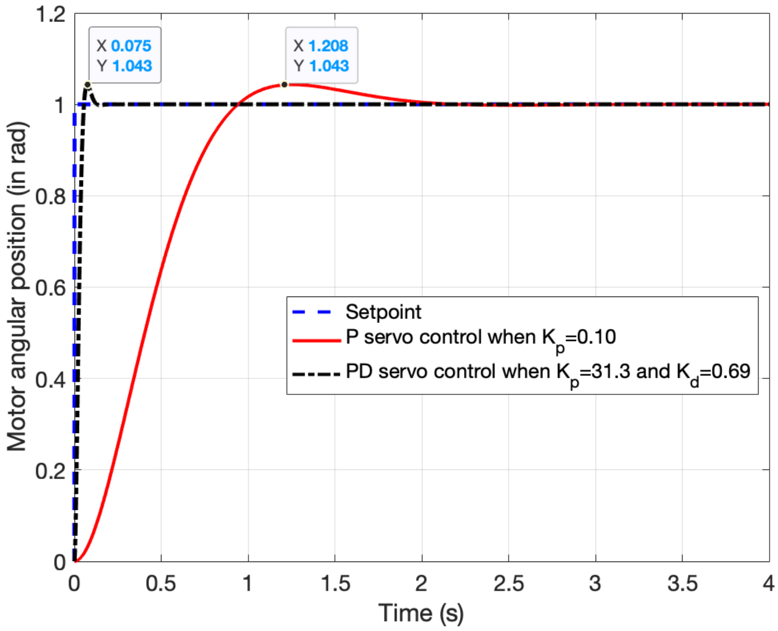



*Figure* 6.6: Closed-loop P and PD control responses to a unit step setpoint. Both controls satisfy the overshoot requirement but the PD control response is much faster.

# Appendices

### Useful properties of the Laplace transform

Some useful properties which have found practical use in Control Engineering are summarized below.

| Property | signal | Laplace transform |
|---|---|---|
| linearity | $ax(t) + by(t)$ | $aX(s) + bY(s)$ |
| time-delays | $x(t - \tau)$ | $e^{-\tau s}X(s)$ |
| convolution | $y(t) = h(t) * u(t)$ | $Y(s) = H(s)U(s)$ |
| differentiation | $\dot{x}(t)$ | $sX(s) - x(0)$ |
| | $\ddot{x}(t)$ | $s^2X(s) - sx(0) - \dot{x}(0)$ |
| integration | $\int_0^t x(\tau)d\tau$ | $\frac{X(s)}{s}$ |
| initial value theorem | $\lim_{t\to 0+} x(t) = \lim_{s\to +\infty} sX(s)$ | |
| final value theorem | $\lim_{t\to +\infty} x(t) = \lim_{s\to 0} sX(s)$ | if the limit exists |

**Some Laplace transform pairs**

| Signal | Laplace transform |
|---|---|
| $\delta(t)$ | $1$ |
| $\Gamma(t)$ | $\frac{1}{s}$ |
| $r(t) = t\Gamma(t)$ | $\frac{1}{s^2}$ |
| $t^2\Gamma(t)$ | $\frac{2}{s^3}$ |
| $e^{-at}\Gamma(t)$ | $\frac{1}{s+a}$ |
| $t^n e^{-at}\Gamma(t)$ | $\frac{n!}{(s+a)^{n+1}}$ |
| $\cos(\omega_0 t)\Gamma(t)$ | $\frac{s}{s^2+\omega_0^2}$ |
| $\sin(\omega_0 t)\Gamma(t)$ | $\frac{\omega_0}{s^2+\omega_0^2}$ |
| $e^{-at}\cos(\omega_0 t)\Gamma(t)$ | $\frac{s+a}{(s+a)^2+\omega_0^2}$ |
| $e^{-at}\sin(\omega_0 t)\Gamma(t)$ | $\frac{\omega_0}{(s+a)^2+\omega_0^2}$ |

# A few important transfer functions

## First-order systems

$$G(s) = \frac{K}{1+Ts}$$

The 2 characteristic parameters of a first order system are:

- $K$: steady-state gain
$$K = \lim_{s \to 0} G(s)$$
- $T$: time-constant

### Characteristic values of a first-order system step response

| | |
|---|---|
| Rise-time at 63% | $T_m^{63\%} = T$ |
| Rise-time at 95% | $T_m^{95\%} \approx 3T$ |
| Settling-time at 5 % | $T_r^{5\%} \approx 3T$ |

## Second-order systems

$$G(s) = \frac{K}{\dfrac{s^2}{\omega_0^2} + 2\dfrac{z}{\omega_0}s + 1} = \frac{K\omega_0^2}{s^2 + 2z\omega_0 s + \omega_0^2}$$

The 3 characteristic parameters of a second-order system are:

- $K$: steady-state gain
- $z$ : damping ratio ($z > 0$)
- $\omega_0$ : undamped natural frequency

### Characteristic values of a underdamped second-order system step response ($z < 1$)

| | |
|---|---|
| Value of the first overshoot in % | $D_{1\%} = e^{\frac{-\pi z}{\sqrt{1-z^2}}} \times 100$ |
| Time-instant of the first overshoot | $T_{D_1} = \dfrac{\pi}{\omega_0\sqrt{1-z^2}}$ |
| Value of the $n^{\text{th}}$ overshoot in % | $D_{n\%} = -(-D_1)^n \times 100$ |
| Time-instant of the $n^{\text{th}}$ overshoot | $T_{D_n} = n\, T_{D_1}$ |
| Pseudo-period | $T_p = \dfrac{2\pi}{\omega_0\sqrt{1-z^2}}$ |
| Settling-time at 5 % | $T_r^{5\%} \approx \dfrac{3}{\omega_0 z}$ for $z \approx 0.707$ |
| *This can lead to a rough estimate if $z \neq 0.707$. Use then the abacus given next page for a better estimate obtained from the formula below* | |
| Settling-time at $x$ % | $T_r^{x\%} = \dfrac{\ln\left(\dfrac{100}{x\sqrt{1-z^2}}\right)}{\omega_0 z}$ |
| Rise-time (100%) | $T_m^{100\%} = \dfrac{\pi - \text{acos}(z)}{\omega_0\sqrt{1-z^2}}$ |

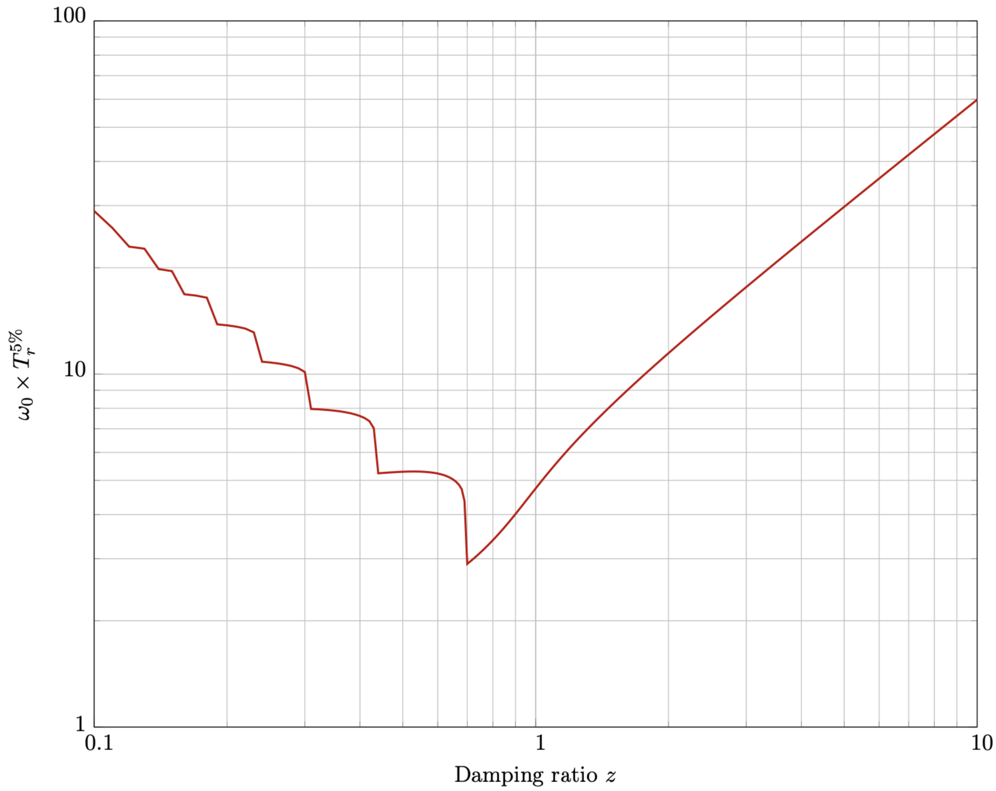



*Figure* 6.7: Abacus plotting the product of the undamped natural frequency $\omega_0$ by the settling time at 5%, $T_r^{5\%}$, versus the damping ratio $z$ for second-order system step response.

# Model identification from step responses

## Identification of a first-order model

$$G(s) = \frac{K}{1+Ts}$$

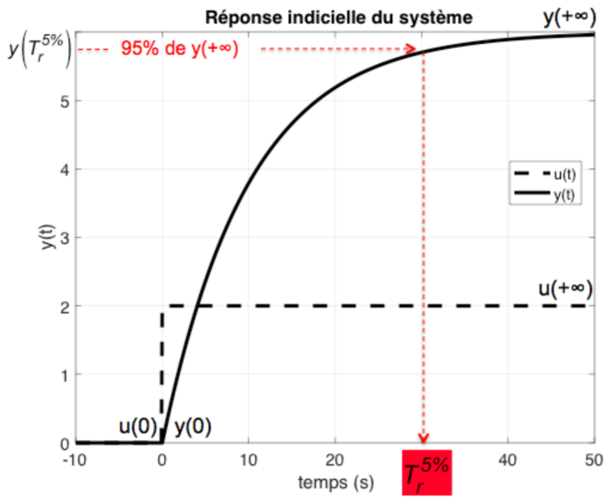



*Figure* 6.8: Step response of a first-order system

From the step response displayed in Figure 6.8, it is necessary to determine the steady-state gain $K$ and the time constant $T$. The procedure is as follows:

1. Find the final and initial values of the response and of the step. Deduce $K$ from:

$$K = \frac{y(+\infty) - y(0)}{u(+\infty) - u(0)}$$

2. Find $y(T_r^{5\%})$, deduce from it $T_r^{5\%}$ then $T$:

$$T = \frac{T_r^{5\%}}{3}$$

## Identification of a second-order underdamped model

$$G(s) = \frac{K}{\dfrac{s^2}{\omega_0^2} + 2\dfrac{z}{\omega_0}s + 1} = \frac{K\omega_0^2}{s^2 + 2z\omega_0 s + \omega_0^2}$$

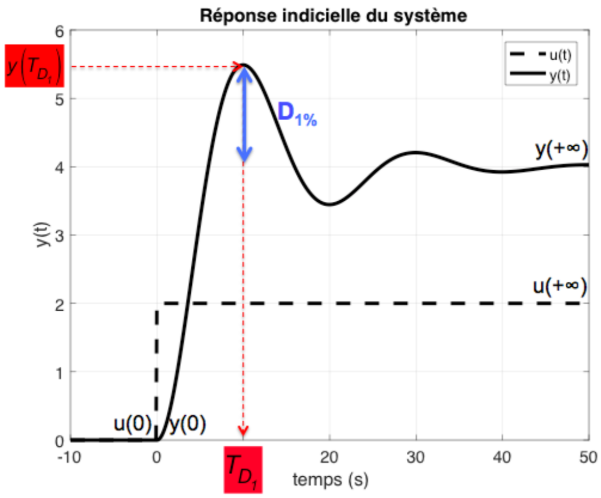



*Figure* 6.9: Step response of a second-order system

From the step response displayed in Figure 6.9, it is necessary to determine the steady-state gain $K$, the damping ratio $z$ and the undamped natural frequency $\omega_0$. The procedure is as follows:

1. Find the final and initial values of the response and of the step. Deduce $K$ from:

$$K = \frac{y(+\infty) - y(0)}{u(+\infty) - u(0)}$$

2. Find the final and initial values of the response and that of the first overshoot $y(t_{D_1})$. Deduce from it $D_1$, then $z$:

$$D_1 = \frac{y(T_{D_1}) - y(+\infty)}{y(+\infty) - y(0)}$$

$$z = \sqrt{\frac{(\ln(D_1))^2}{(\ln(D_1))^2 + \pi^2}}$$

3. Find the time-instant of the first overshoot $T_{D_1}$. Deduce from it $\omega_0$:

$$\omega_0 = \frac{\pi}{T_{D_1}\sqrt{1 - z^2}}$$

## Identification of a first-order plus time-delay model by the Broïda method

Broïda has suggested to approximate the underdamped step response of any $n$-th order system by a first-order plus time-delay model

$$G(s) = \frac{Ke^{-\tau s}}{1 + Ts}$$

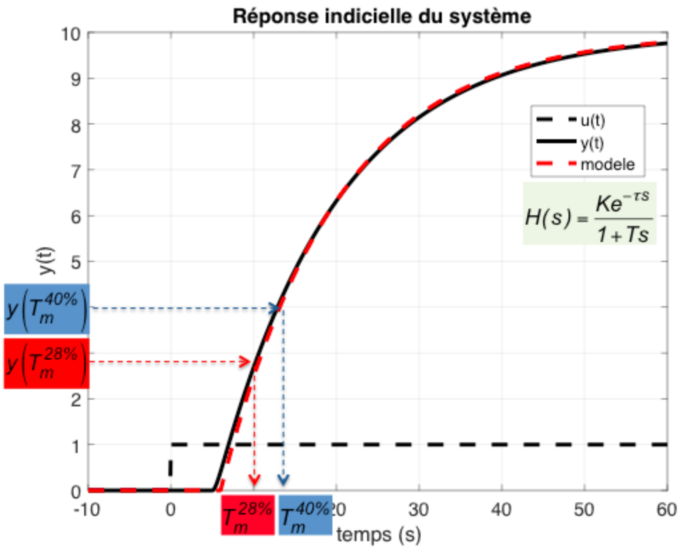



*Figure* 6.10: Underdamped step response of any $n$-th order system approximated by a first-order plus time-delay model

From the step response displayed in Figure 6.10, it is necessary to determine the steady-state gain $K$, the time-constant $T$ and the pure time-delay $\tau$. The procedure is as follows:

1. Find the final and initial values of the response and of the step. Deduce $K$ from:

$$K = \frac{y(+\infty) - y(0)}{u(+\infty) - u(0)}$$

2. Find $y(T_m^{28\%})$ and $y(T_m^{40\%})$, deduce $T_m^{28\%}$ and $T_m^{40\%}$ then calculate :

$$\tau = 2,8T_m^{28\%} - 1,8T_m^{40\%}$$
$$T = 5,5\left(T_m^{40\%} - T_m^{28\%}\right)$$

**PID tuning by using empirical rules**

A PID controller defined in its so-called *ideal* form is defined as:

$$C(s) = K_p\left(1 + \frac{1}{T_i s} + \frac{T_d s}{1 + \frac{T_d}{N}s}\right) = K_p + K_i\frac{1}{s} + K_d\frac{s}{1 + \frac{T_d}{N}s} \tag{1}$$

The individual effects of these three $K_p$, $K_i$ and $K_d$ parameters appearing in (1) on the closed-loop performance of stable plants are summarized in Table 1 below.

TABLE 1 **Effects of independent P, I, and D tuning on closed-loop response.**
**For example, while $K_I$ and $K_D$ are fixed, increasing $K_P$ alone can decrease rise time, increase overshoot, slightly increase settling time, decrease the steady-state error, and decrease stability margins.**

| | Rise Time | Overshoot | Settling Time | Steady-State Error | Stability |
|---|---|---|---|---|---|
| Increasing $K_P$ | Decrease | Increase | Small Increase | Decrease | Degrade |
| Increasing $K_I$ | Small Decrease | Increase | Increase | Large Decrease | Degrade |
| Increasing $K_D$ | Small Decrease | Decrease | Decrease | Minor Change | Improve |

### Tuning of the PID controller by using the Ziegler-Nichols rules

Assume a continuous-time process is reasonably well modeled by the first-order plus time-delay transfer function model:

$$G(s) = \frac{Ke^{-\tau s}}{1 + Ts}$$

Several tuning rules are available for determining the PID controller parameter values. The proposed rules address different design specifications such as load disturbance rejection or setpoint tracking.

The most popular tuning rules are those attributed to Ziegler-Nichols. Their aim is to provide *satisfactory load disturbance rejection.*

Table 2 shows the Ziegler-Nichols rules for P, PI, or PID controllers defined in its so-called *ideal* form (see (1)).

| Controller type | $K_p$ | $T_i$ | $T_d$ |
|---|---|---|---|
| P | $\frac{T}{K\tau}$ | | |
| PI | $0.9\frac{T}{K\tau}$ | $3\tau$ | |
| PID | $1.2\frac{T}{K\tau}$ | $2\tau$ | $0.5\tau$ |

Table 2: Ziegler-Nichols disturbance rejection tuning rules for a first-order-plus delay model determined from an open-loop step response

### Tuning of the PI controller by using the Chien-Hrones-Reswick rules

As previously said, the Ziegler-Nichols rule aim is to provide *satisfactory disturbance rejection.* For setpoint tracking (or servo control), it can be advantageous to use the empirical rules suggested by Chien-Hrones-Reswick which are given in Table 3 for P, PI, and PID controllers in their *ideal* form (see equation (1)).

| Controller type | $K_p$ | $T_i$ | $T_d$ |
|---|---|---|---|
| P | $0.3\frac{T}{\tau}$ | | |
| PI | $0.35\frac{T}{\tau}$ | 1.2T | |
| PID | $0.6\frac{T}{\tau}$ | T | 0.5T |

Table 3: Chien-Hrones-Reswick setpoint tracking tuning rules for a first-order-plus delay model determined from an open-loop step response

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## English to French glossary

| | | |
|---|---|---|
| bandwidth | : | bande passante |
| crane | : | grue |
| closed-loop system | : | système bouclé |
| cut-off frequency | : | fréquence (ou pulsation) de coupure |
| damped frequency | : | pulsation amortie |
| damping ratio | : | coefficient d'amortissement |
| drag | : | traînée |
| feedback | : | contre-réaction |
| feedback system | : | système à contre-réaction |
| hoisting device | : | dispositif de levage |
| impulse response | : | réponse impulsionnelle |
| integral wind-up | : | emballement (de l'action) intégral |
| input | : | entrée |
| gain | : | gain |
| heading angle | : | angle de cap |
| linear time-invariant (LTI) | : | linéaire invariant dans le temps |
| motor shaft | : | arbre moteur |
| output | : | sortie |
| overdamped | : | sur-amorti |
| overshoot | : | dépassement |
| rise time | : | temps de montée |
| road grade | : | inclinaison de la route |
| robot arm joint | : | articulation d'un bras de robot |
| root locus | : | lieu des racines |
| setpoint | : | consigne |
| settling time | : | temps de réponse |
| steady-state gain | : | gain statique |
| steady-state response | : | réponse en régime permanent |
| steering | : | direction |
| step response | : | réponse indicielle |
| stream | : | courant |
| time-delay | : | retard pur |
| time-invariant | : | invariant dans le temps |
| transient response | : | réponse transitoire |
| throttle | : | accélérateur |
| undamped | : | non amorti |
| undamped natural frequency | : | pulsation propre non amortie |
| underdamped | : | sous-amorti |