POLYTECH NANCY

Control engineering
Problems n°1

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Laplace transform, transfer function and block diagram analysis of linear time-invariant (LTI) dynamic systems

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Exercise 1.1 - Laplace transform of an exponential signal
Consider the following signal:

$$x(t) = e^{-at}\Gamma(t), \quad a = 1$$

1.1.a. Recall the usual name of $\Gamma(t)$ and its definition.

1.1.b. Plot the signal $x(t)$.

1.1.c. Is the signal $x(t)$ causal? Justify your answer.

1.1.d. What is the role of $\Gamma(t)$ in the definition of $x(t)$.

1.1.e. Determine the Laplace transform of $x(t)$ from the definition integral.

Exercise 1.2 - Inverse Laplace transform
Determine the inverse Laplace transform of:

$$Y(s) = \frac{2}{(s+3)(s+5)}$$

Exercise 1.3 - Solution of differential equations
Solve the following differential equations using the Laplace transform:

1.3.a.

$$\dot{y}_1(t) = -2y_1(t), \qquad y_1(0) = 1$$

1.3.b.

$$\dot{y}_2(t) + 2y_2(t) = \Gamma(t), \qquad y_2(0) = 1$$

1.3.c.

$$\ddot{y}_3(t) + 10\dot{y}_3(t) + 16y_3(t) = 10\delta(t), \qquad \dot{y}_3(0) = y_3(0) = 0$$

Exercise 1.4 - Second-order model step responses with numerator dynamics
Consider the following second-order differential equation model

$$3\ddot{y}(t) + 18\dot{y}(t) + 24y(t) = a\dot{u}(t) + 6u(t), \qquad \dot{y}(0) = y(0) = 0; \quad u(0) = 0$$

where a is a real number When the input $u(t)$ is a unit-step function, show that the response $y(t)$ takes the following form in terms of a:

$$y(t) = \left(\frac{1}{4} + \frac{a-3}{6}e^{-2t} + \frac{3-2a}{12}e^{-4t}\right)\Gamma(t)$$

The plot of the response is given in Figure 1.1 for several values of a. Notice that the value of a has an impact on the transient response. In particular, when $a < 0$, the response becomes negative before changing direction and converging to its positive steady-state value while an overshoot in the response occurs when $a > 3$ and the height of the overshoot increases as a increases. The slope of the responses at the origin is also impacted by a non-zero value of a. However, the value of a does not affect the steady-state value of the response.

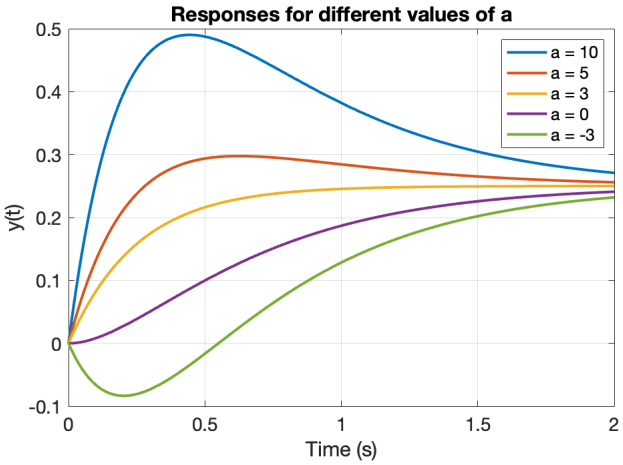


Figure 1.1: Second-order model step responses for different numerator dynamics.

Exercise 1.5 - Transfer function of a mechanical system
Consider a mechanical suspension system shown in Figure 1.2 constituted of a mass, a damper and a spring having a damping and stiffness coefficient of b and k respectively.

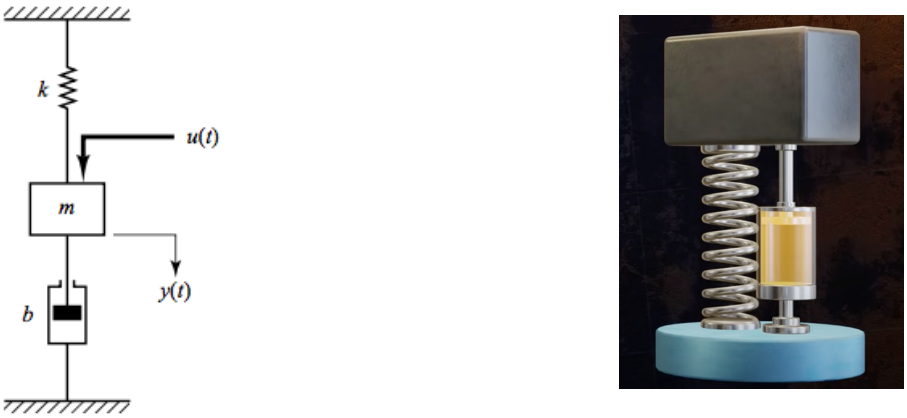


Figure 1.2: Mechanical system.

The differential equation of this mechanical system relating the vertical position $y(t)$ of the mass (system output) and the external force $u(t)$ (system input) applied to the system is:

$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = u(t)$$

1.5.a. Determine the transfer function $G(s) = \frac{Y(s)}{U(s)}$ and represent the system in the form of a block diagram.

1.5.b. Give the system order, the steady-state gain, the pole(s) and zero(s) of the transfer function.

Exercise 1.6 - Equivalent transfer function of simple closed-loop block diagram
Consider the closed-loop block diagram displayed in Figure 1.3.
Derive its equivalent transfer function $T(s) = \frac{Y(s)}{R(s)}$.

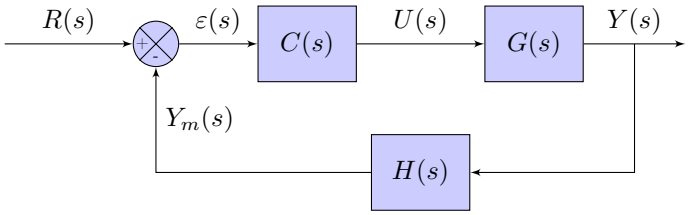


Figure 1.3: Classical block diagram of a simple closed-loop feedback system

Exercise 1.7 - Transfer function of a simple closed-loop block diagram. A case study
Consider the closed-loop block diagram displayed in Figure 1.4.
Determine its equivalent transfer function $T(s) = \frac{Y(s)}{R(s)}$.

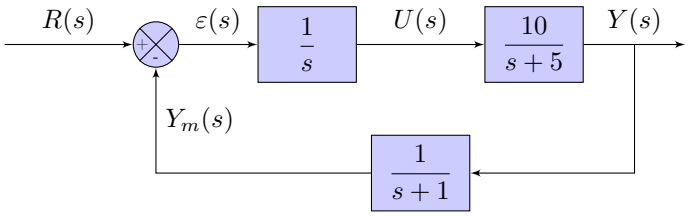


Figure 1.4: Case study of a simple closed-loop feedback system

POLYTECH NANCY

Control engineering
Problems n°2

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Step responses of important systems & System identification from step response test

Exercise 2.1 - Step response of a first-order system
Consider the system described by the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2}{1+10s}$$

1. Determine the steady-state gain K, the time constant T and the pole.
2. Recall the unit step response and calculate its slope at the origin.
3. Calculate the rise times $T_m^{63\%}$ and $T_m^{95\%}$ as well as the settling time $T_r^{5\%}$.
4. Without any calculation, plot precisely the step response and indicate on it its characteristic parameters calculated above.

Exercise 2.2 - Step response of a first-order plus time delay system
Consider a first-order system having a steady-state gain of 2, a time constant of 10 seconds and a pure time delay of 20 seconds.

1. Give the system transfer function $G(s)$.
2. Without any calculation, plot precisely the unit step response and indicate on it its characteristic parameters given above.

Exercise 2.3 - Step response of a dynamical system
Consider a system whose dynamic behavior is governed by the following differential equation:

$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = 10u(t) \qquad \text{with } \dot{y}(0) = 0, \quad y(0) = 0$$

1. Determine the transfer function $G(s) = \frac{Y(s)}{U(s)}$ of the system.
2. Determine the order of the system, the steady-state gain K, the damping ratio z, the undamped natural frequency ω_0, the poles and zeros.
3. Conclude about the type of step response: critical, overdamped or underdamped.
4. Calculate the values of the first and second overshoot $D_{1\%}$ and $D_{2\%}$, the times of the first and second overshoot t_{D_1} and t_{D_2}.
5. Plot the step response and indicate on it the characteristic parameters computed above.

Exercise 2.4 - Identification of a mechanical suspension system from step response test
Consider the position response $y(t)$ of a mechanical suspension system to a unit step input ($u(t) = \Gamma(t)$) plotted in Figure 2.1.

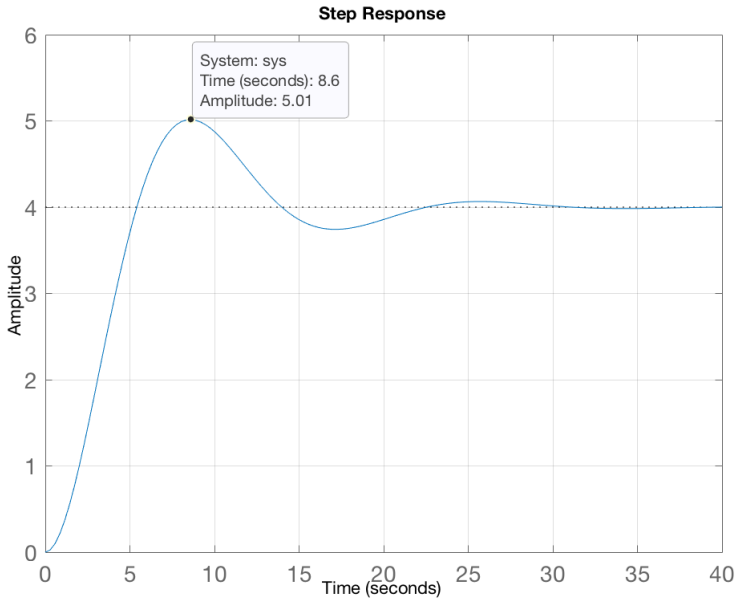


Figure 2.1: Response of a mechanical system to a unit step input

1. Propose a transfer function model form $G(s)$ for the system. Explain your choice.
2. Determine the numerical values of the different parameters of your chosen $G(s)$.
3. Deduce, from your identified transfer function model $G(s)$, the differential equation of the system.

Exercise 2.5 - Step response analysis
Figure 2.2 shows the unit step responses of five different linear time-invariant (LTI) systems. Pair each of the 5 step responses to one of the 7 transfer functions below. Explain your answers.

$$G_1(s) = \frac{0.1}{s+0.1}; \qquad G_2(s) = \frac{4}{s^2+2s+4}; \qquad G_3(s) = \frac{0.5}{s^2-0.1s+2}$$
$$G_4(s) = \frac{-0.5}{s^2+0.1s+2}; \qquad G_5(s) = \frac{1}{s+1}; \qquad G_6(s) = \frac{4}{s^2+0.8s+4}$$
$$G_7(s) = \frac{2}{s^2+s+3}$$

You can use Matlab to verify your solutions.

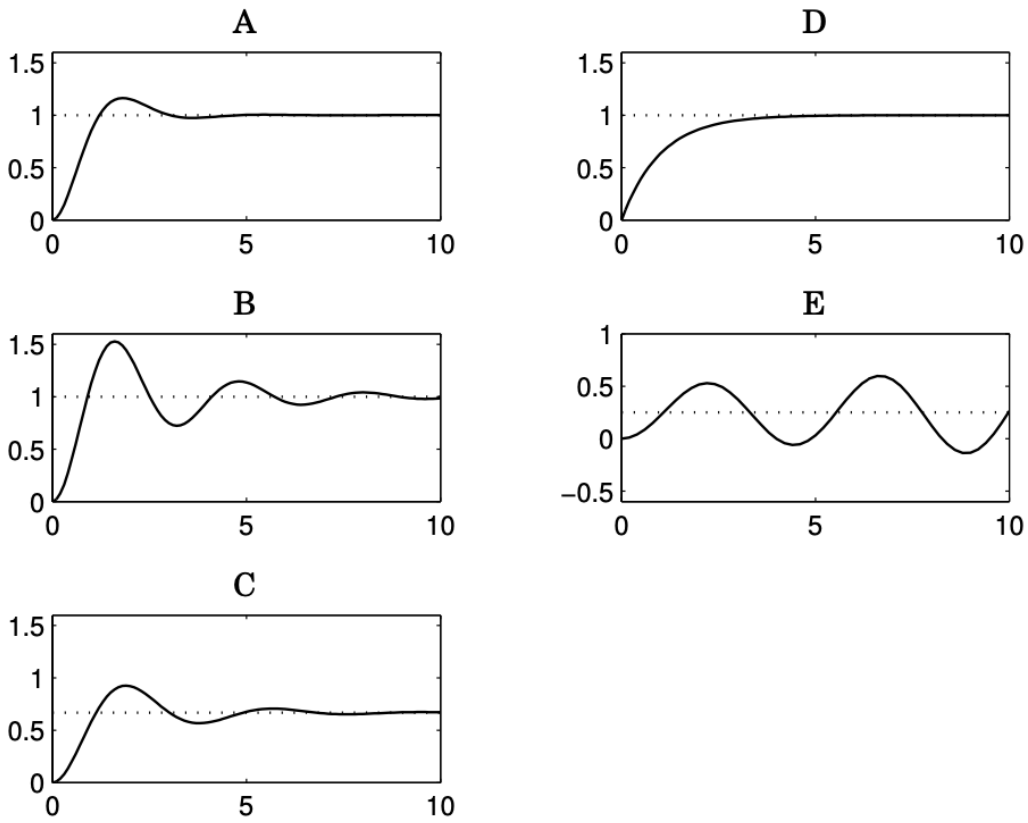


Figure 2.2: Step responses of different LTI systems

Stability and steady-state error analysis

Exercise 3.1 - Stability from the system poles
Determine the poles and zeros of the LTI systems described by the transfer functions below. Conclude about their stability:

$$G_1(s) = \frac{2}{s+2}; \qquad G_2(s) = \frac{2}{s^2+3s+2}; \qquad G_3(s) = \frac{1}{s^2+2s+2}$$
$$G_4(s) = \frac{2}{s^2+4}; \qquad G_5(s) = \frac{2}{s(s+2)}; \qquad G_6(s) = \frac{2}{s^2(s+2)}$$
$$G_7(s) = \frac{2(s^2-2s+2)}{(s+2)(s^2+2s+2)}; \qquad G_8(s) = \frac{200}{(s+2)(s^2-2s+2)}$$

Exercise 3.2 - Links between system poles and step responses
Pair the step responses and pole-zero diagrams in Figure 3.1. Give your solutions so that the pairs of plots that belong to the same system is written in the form Pole-zero-letter–Step-response-letter.

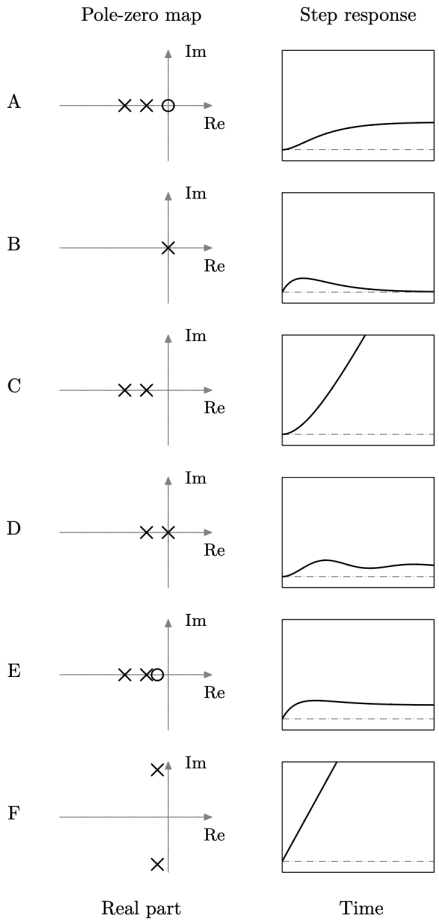


Figure 3.1: Pole-zero diagrams and step responses of different LTI systems. In the pole-zero diagram, imaginary and real parts have equal scaling, `x` marks poles, and `o` marks zeros.

Exercise 3.3 - Stability analysis by using the Routh-Hurwitz criterion
Study the stability of the LTI systems having the characteristic equations below. Specify, in case of instability, the number of unstable poles:

a) $s^3 - 25s^2 + 10s + 450 = 0$

b) $s^3 + 25s^2 + 450 = 0$

c) $s^3 + 25s^2 + 10s + 450 = 0$

d) $s^3 + 25s^2 + 10s + 50 = 0$

Exercise 3.4 - Elevator control system for supertall building
Yokohama's 70-floor Landmark Tower was completed in 1993 and it is still the tallest building in the Greater Tokyo Area with a height of 293m. Its elevators operate at a peak speed of 45 km/h! To reach such a speed without inducing discomfort in passengers, the elevator accelerates for longer periods, rather than more precipitously. Going up, it reaches full speed only at the 27th floor; it begins decelerating 15 floors later. The result is a peak acceleration similar to that of other skyscraper elevators–a bit less than a tenth of the force of gravity. Admirable ingenuity has gone into making this safe and comfortable. Special ceramic brakes had to be developed; iron ones would melt. Computer-controlled systems damp out vibrations. The lift has been streamlined to reduce the wind noise as it speeds up and down.

The closed-loop block diagram for the control of the elevator's vertical position is shown in Figure 3.2.

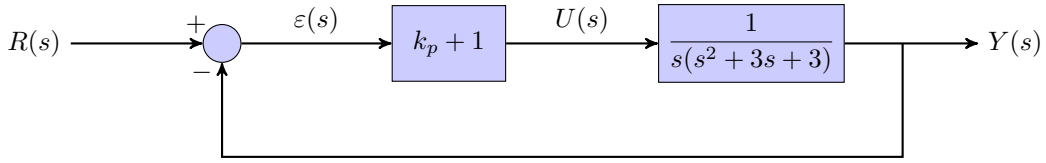


Figure 3.2: Vertical position control system for an elevator

1. Determine the range of values for k_p that ensures the stability of the control vertical position system.
2. By using the final value theorem, calculate the expected steady-state error for a step reference of amplitude A as a function of k_p.

Exercise 3.5 - Mobile robot steering control
The block diagram of the steering control system for a mobile robot is shown in Figure 3.3, where $R(s)$ and $Y(s)$ represent the Laplace transform of the desired and measured heading angle respectively.

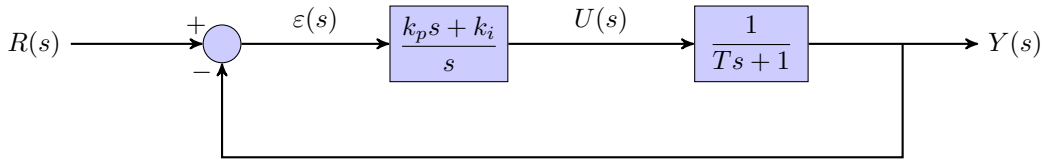


Figure 3.3: Block diagram of the steering control system for a mobile robot

1. Express $\varepsilon(s)$ in terms of $R(s)$, k_p, k_i and T.
2. When $k_p > 0$ and $k_i = 0$, determine the steady-state error for a step reference input $r(t) = A\Gamma(t)$.
3. When $k_p > 0$ and $k_i > 0$, determine the steady-state error for a step reference input $r(t) = A\Gamma(t)$.
4. When $k_p > 0$ and $k_i > 0$, determine the steady-state error for a ramp reference input $r(t) = At\Gamma(t)$.

Model-based PID controller design and tuning for rotary speed control



Figure 4.1: Qube-Servo 2 equipped with the inertia disc

The goal is to design PID-type controllers for the rotary speed of the inertia disc attached to the Quanser Qube-Servo 2, as shown in Figure 4.1. The input and output of the Qube-Servo 2 speed control system are:

- $u(t)$: DC motor voltage in V;
- $\omega(t)$: rotary speed or angular velocity of the inertia disc in rad/s.

1. Control objectives and performance requirements

The performance requirements for the rotary speed control are described in Table 1.

Requirement	Assessment criterion	Level
Tracking of a step setpoint for the rotary speed of the inertia disc	Steady-state error	No steady-state error
	Peak overshoot	$D_{1\%} \approx 4.3\%$
	Settling time at 5 %	$T_r^{5\%} \approx 60$ ms
	Motor input voltage	Limited to ± 10V

Table 1: Performance requirements for the rotary speed control

2. Transfer function modelling of the rotary speed dynamics from a step test

Having a dynamic model of a system is crucial for effective control design. The model is used to design a control according to certain specifications and validate it before deployment on the actual system. However, physical modelling a complex system can be time consuming and, sometimes, not all the system parameters are known. Instead, the model of a system can be obtained experimentally by applying an input to the system and analyzing the corresponding output response.

2.1. For a stable system, a transfer function model can be inferred from a step response test. The rotary speed response (in rad/s) to a step change in the motor voltage is plotted in Figure 4.2. From the step response plot, propose a transfer function model. Justify your choice.

2.2. From Figure 4.2, determine the numerical values of the parameters of your chosen transfer function model.

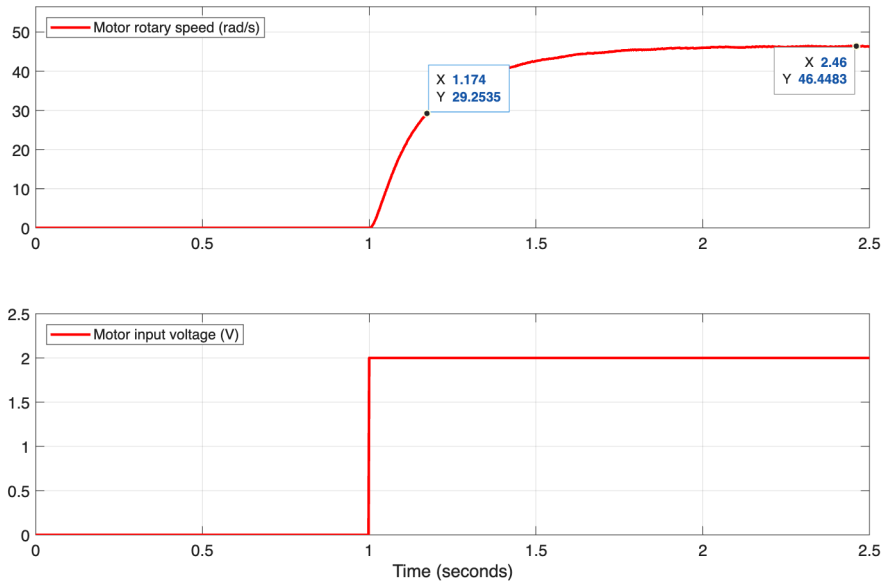


Figure 4.2: Rotary speed response to a step change in the motor voltage

3. Rotary speed control using a simple Proportional (P) feedback controller

The first proposed strategy is a simple proportional (P) feedback controller, shown in Figure 4.3, where $\Omega_r(s) = \mathcal{L}[\omega_r(t)]$, $\Omega(s) = \mathcal{L}[\omega(t)]$ and $U(s) = \mathcal{L}[u(t)]$.

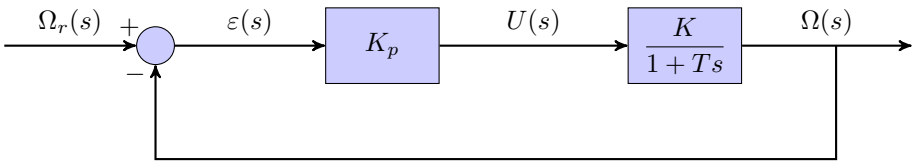


Figure 4.3: Block diagram of the simple P feedback configuration of the rotary speed control

3.1. Determine the closed-loop transfer function $F_{CL}(s)$ in terms of K_p, K and T.

3.2. Determine the range of values for K_p that ensures the stability of the feedback control system.

3.3. Calculate the steady-state tracking error in response to a step $\omega_r(t) = 50\Gamma(t)$ (in rad/s) by using the final value theorem:

$$\lim_{t\to+\infty} \varepsilon(t) = \lim_{s\to 0} s\varepsilon(s)$$
$$\lim_{t\to+\infty} (\omega_r(t) - \omega(t)) = \lim_{s\to 0} s\,(\Omega_r(s) - \Omega(s))$$

3.4. Is it possible to meet the steady-state requirement with this simple P controller?

4. Rotary speed control using a standard PI feedback controller

The second feedback strategy considered is to use a standard PI controller as shown in Figure 4.4, where both proportional and integral terms are applied to the error.

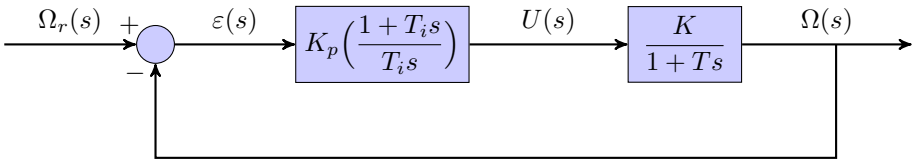


Figure 4.4: Block diagram of the classical PI feedback configuration of the rotary speed control

The proposed design of the PI controller is chosen such that the system time constant is compensated by setting the integral action $T_i = T$ (known as the time-contant compensation method).

4.1. With this setting for $T_i = T$, determine the open-loop transfer function $F_{OL}(s)$ in terms of K_p, K and T.

4.2. Show that the closed-loop transfer function $F_{CL}(s)$ is a standard first-order transfer function without zero. Express its steady-state gain and time constant in terms of K_p, K and T.

4.3. Determine the range of values for K_p that ensures the stability of the closed loop.

4.4. Calculate the steady-state tracking error in response to a step setpoint $\omega_r(t) = 50\Gamma(t)$ (in rad/s).

4.5. Determine the value of K_p that gives $T_r^{5\%} = 60$ ms.

4.6. For this value of K_p, what is the peak overshoot of the closed-loop step response?

4.7. Show that the transfer function $\dfrac{U(s)}{\Omega_r(s)}$ takes the following form:

$$\frac{U(s)}{\Omega_r(s)} = \frac{K_p(1+Ts)}{Ts + K_p K}$$

4.8. For the value of K_p that gives $T_r^{5\%} = 60$ ms, determine the motor voltage $u(0^+)$ just after the step setpoint $\omega_r(t) = 50\,\Gamma(t)$ is applied by using the initial value theorem:

$$\lim_{t\to 0^+} u(t) = \lim_{s\to+\infty} s\,U(s)$$

4.9. Is the motor voltage requirement satisfied?

4.10. Is it possible to meet the peak overshoot and motor voltage requirements with this PI controller structure and design?

5. Rotary speed control using a variant of the standard PI feedback controller

The third strategy is a variant of the standard PI controller in which the proportional term acts on the measured output rather than on the error, as shown in Figure 4.5. This structure is known as an I-P controller (PI controller with proportional action on the measurement). Since the setpoint only enters through the integral term, the control signal does not jump when a step setpoint is applied.

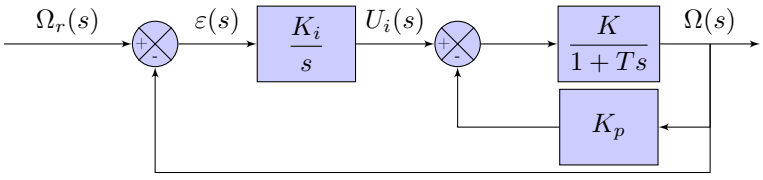


Figure 4.5: Block diagram of the I-P controller (proportional action on the output) for the rotary speed control

5.1. Determine the inner closed-loop transfer function $F_i(s) = \Omega(s)/U_i(s)$ in terms of K_p, K and T.

5.2. Draw the simplified closed-loop block diagram equivalent to Figure 4.5.

5.3. Determine the closed-loop transfer function $F_{CL}(s)$ in terms of K_p, K_i, K and T.

5.4. Determine the range of values for K_p and K_i that ensures the stability of the closed loop.

5.5. Calculate the steady-state tracking error in response to a step setpoint $\omega_r(t) = 50\Gamma(t)$.

5.6. A classical analytical tuning method, called *pole placement by reference-model matching*, consists in choosing the controller gains so that the characteristic polynomial of $F_{CL}(s)$ matches that of a desired second-order reference model $F_{ref}(s)$ derived from the performance requirements. Since $F_{CL}(s)$ has no zero here, this makes $F_{CL}(s)$ identical to $F_{ref}(s)$. From the performance requirements, give the desired second-order transfer function $F_{ref}(s)$.

5.7. Determine the damping ratio z and the undamped natural frequency ω_0 such that the step response of $F_{ref}(s)$ has $D_{1\%} = 4.3\,\%$ and $T_r^{5\%} = 60$ ms. Recall that z and ω_0 are functions of the step response performance indices (settling time $T_r^{5\%}$ and peak overshoot D_1, not expressed in percent)[1]:

$$z = \sqrt{\frac{(\ln(D_1))^2}{\pi^2 + (\ln(D_1))^2}} \qquad \omega_0 = \frac{3}{T_r^{5\%}} \text{ when } z \approx 0.707$$

5.8. Show that the gains K_p and K_i for which $F_{CL}(s) = F_{ref}(s)$ are:

$$F_{CL}(s) = F_{ref}(s) \Rightarrow \begin{cases} K_i = \dfrac{T\omega_0^2}{K} \\ K_p = \dfrac{2zK_i}{\omega_0} - \dfrac{1}{K} \end{cases}$$

5.9. Sketch the closed-loop response to the step setpoint $\omega_r(t) = 50\,\Gamma(t)$. Simulation shows that the motor voltage peaks at about 9.4 V, within the ± 10 V limit.

5.10. Are the performance requirements satisfied with this I-P structure and design?

[1] For $z \approx 0.707$, the peak overshoot (4.3 %) is smaller than 5 %, so $T_r^{5\%}$ is the time at which the response first reaches 95 % of its final value, which gives $\omega_0 T_r^{5\%} \approx 2.9 \approx 3$. The envelope-based approximation $T_r^{5\%} \approx 3/(z\omega_0)$ is conservative and overestimates the settling time by about 40 % in this case.

Model-based PID controller design and tuning for the altitude control of a mini-drone

The goal is to design PID-type controllers for the altitude of the Tello mini-drone shown in Figure 5.1.



Figure 5.1: Tello mini-drone (Ryze Tech, DJI technology)

The input and output of the mini-drone altitude control system are:

- $u(t)$: vertical speed command in % sent to the onboard flight controller;
- $z(t)$: altitude or vertical position along the Z axis of the mini-drone in cm.

The performance requirements for the mini-drone altitude control are described in Table 1.

Requirement	Assessment criterion	Level
Tracking of a step setpoint for the mini-drone altitude	Steady-state error	No steady-state error
	Overshoot	No overshoot allowed (the step response should be critically damped)
	Settling time at 5 %	$T_r^{5\%} \approx 1.75$ s
	Vertical speed command	Limited to $\pm 100\,\%$

Table 1: Performance requirements for the mini-drone altitude control

1. Transfer function modelling of the vertical dynamics from a step test

The determination of a model is a crucial step for the design of a feedback control system. Unfortunately, the manufacturer does not provide a physics-based model relating the vertical speed command $u(t)$ to the altitude $z(t)$ of the Tello mini-drone. The vertical dynamics will therefore be described by a transfer function model determined experimentally from a step test.

Figure 5.2 shows the vertical position (or altitude), the vertical velocity and the vertical speed command recorded when a step of 50 % is applied to the speed command.

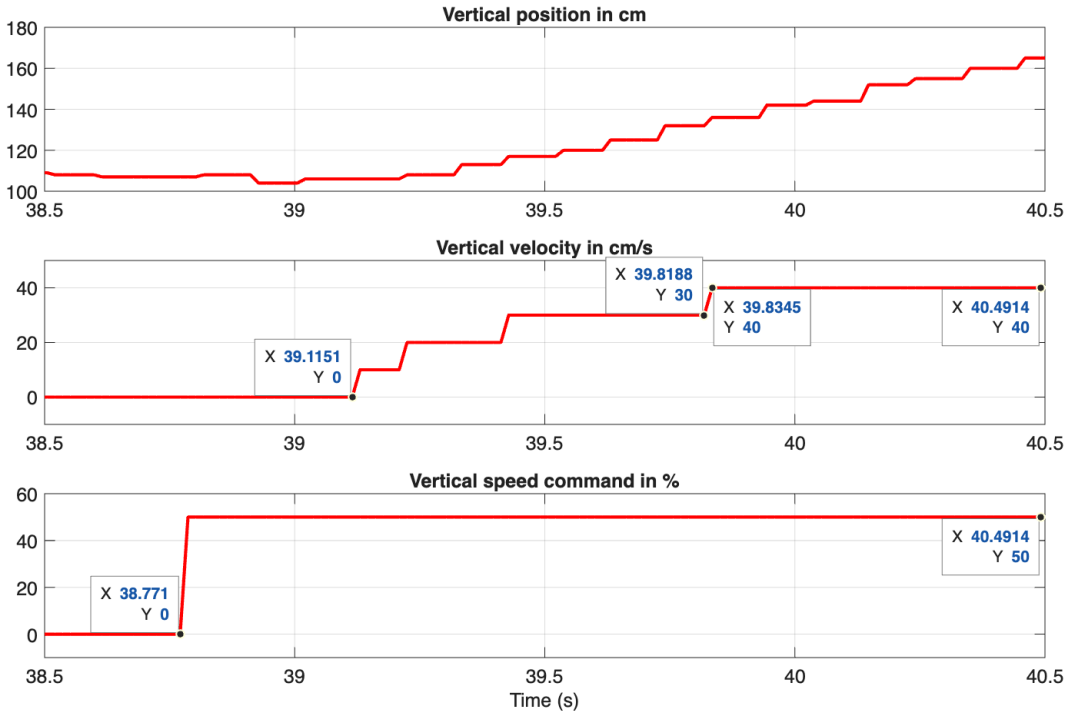


Figure 5.2: Altitude, vertical velocity and vertical speed command recorded during a step test on the Tello mini-drone

Choice of the transfer function model. After a transient, it can be observed from Figure 5.2 that the altitude increases linearly with time (it looks like a ramp): the altitude response to the step is unbounded, which reveals an integrator. The vertical velocity, on the other hand, converges without overshoot towards a constant value, like the step response of a first-order system. The speed command-to-altitude transfer function can therefore be modelled as a first-order plus pure integrator model:

$$\frac{Z(s)}{U(s)} = \frac{K}{s(1+Ts)} \tag{1}$$

where $Z(s) = \mathcal{L}[z(t)]$, $U(s) = \mathcal{L}[u(t)]$, K is the model gain and T is the time constant.

Warning. A time delay of about 0.34 s can be observed on the vertical velocity response of Figure 5.2. It is mainly due to the communication between the control PC and the mini-drone, the onboard processing and the limited resolution of the velocity measurement. For simplicity, this time delay is neglected in the model and in the design of the PID controllers. Since it is not negligible compared with the estimated time constant T, the closed-loop performance obtained on the real mini-drone during the lab may differ from the predicted one: a slower response and an overshoot of the order of 10-15 % could be expected in practice.

Identification of the model parameters. Determining K and T directly from the altitude step response is difficult, since this response is unbounded and its transient response is short compared with the ramp-type response. However, since the vertical velocity $v_z(t)$ is the time derivative of the altitude $z(t)$, both variables are related in the Laplace domain by an integrator, so that (1) can be written as:

$$\frac{Z(s)}{U(s)} = \frac{Z(s)}{V_z(s)} \times \frac{V_z(s)}{U(s)} = \frac{1}{s} \times \frac{K}{1+Ts} \tag{2}$$

where $V_z(s) = \mathcal{L}[v_z(t)]$ is the Laplace transform of the vertical velocity of the mini-drone. The parameters K and T can thus be easily determined from the vertical velocity step response, which is that of a simple first-order model:

$$\frac{V_z(s)}{U(s)} = \frac{K}{1+Ts} \tag{3}$$

The altitude model (1) is then obtained by simply adding the integrator $1/s$.

1.1. From the data displayed on the vertical velocity response of Figure 5.2, determine the numerical values of the parameters K and T of the first-order model (3) by using the identification procedure given in the Appendix ($T = T_r^{5\%}/3$). Since the time delay is neglected, the settling time should be measured from the instant the vertical velocity starts to respond, i.e. about 0.34 s after the step is applied. Note that the vertical velocity is measured with a resolution of 10 cm/s, which limits the accuracy of the estimated time constant.

1.2. Deduce the speed command-to-altitude transfer function model (1).

2. Altitude control using a simple proportional (P) feedback controller

The first proposed feedback strategy is to use a simple P controller as shown in Figure 5.3.

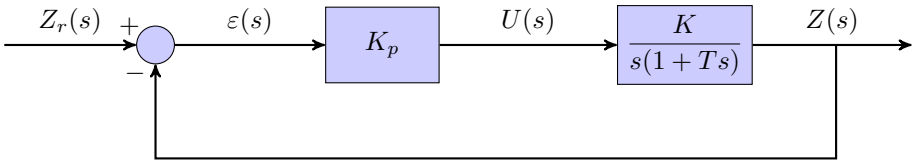


Figure 5.3: Block diagram of the simple P feedback configuration of the mini-drone altitude control

2.1. Determine the closed-loop transfer function $F_{CL}(s)$ in terms of K_p, K and T.

2.2. Determine the range of values for K_p that ensures the stability of the feedback control system.

2.3. Calculate the steady-state tracking error in response to a step $z_r(t) = A\Gamma(t)$ (in cm) by using the final value theorem:

$$\lim_{t\to+\infty} \varepsilon(t) = \lim_{s\to 0} s\,\varepsilon(s)$$
$$\lim_{t\to+\infty} (z_r(t) - z(t)) = \lim_{s\to 0} s\,(Z_r(s) - Z(s))$$

2.4. The pole placement by reference-model matching method introduced in Problems n°4 consists in choosing the controller gain(s) so that $F_{CL}(s)$ behaves like a desired second-order reference model $F_{ref}(s)$ derived from the performance requirements. Since no overshoot is allowed, a critically damped reference model ($z = 1$) is chosen. Show that the reference model takes the form:

$$F_{ref}(s) = \frac{\omega_0^2}{(s+\omega_0)^2}$$

2.5. Show that the critically damped response of the simple P feedback system is obtained for:

$$K_p = \frac{1}{4KT}$$

2.6. For this value of K_p, deduce the undamped natural frequency ω_0 of $F_{CL}(s)$.

2.7. For a critically damped second-order system ($z = 1$), the settling time and the undamped natural frequency are related by

$$T_r^{5\%} \approx \frac{4.75}{\omega_0} \quad \text{when } z = 1$$

Deduce the resulting settling time $T_r^{5\%}$ for the chosen value of K_p.

2.8. Are the performance requirements satisfied with this simple P feedback structure and design?

3. Altitude control using a standard PD feedback controller

The second feedback strategy considered is to use a standard PD controller as shown in Figure 5.4, where both proportional and derivative terms are applied to the error.

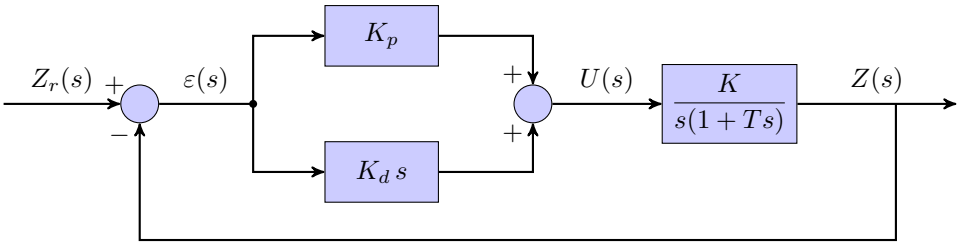


Figure 5.4: Block diagram of the standard PD feedback configuration of the mini-drone altitude control

3.1. Determine the closed-loop transfer function $F_{CL}(s)$ in terms of K_p, K_d, K and T.

3.2. Which part of the step response of a second-order system is affected by the presence of a zero (assumed to be not located at the origin): the transient response or the steady-state response (see, for example, Exercise 1.4 and Figure 1.1)?

3.3. Can the pole placement by reference-model matching method be used to tune K_p and K_d for this standard PD feedback controller?

4. Altitude control using a variant of the standard PD feedback controller

The third strategy is a variant of the standard PD controller in which the derivative term acts on the measured output rather than on the error, as shown in Figure 5.5 (PD controller with derivative action on the measurement).

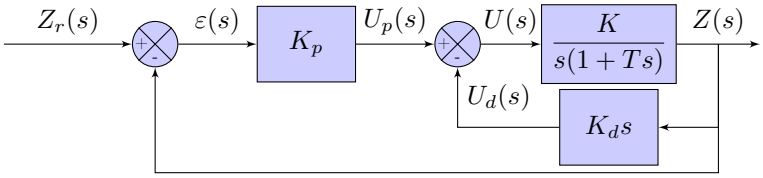


Figure 5.5: Block diagram of the PD controller with derivative action on the output for the mini-drone altitude control

4.1. Determine the inner closed-loop transfer function $F_i(s) = \dfrac{Z(s)}{U_p(s)}$ in terms of K_d, K and T.

4.2. Draw the simplified block diagram equivalent to Figure 5.5.

4.3. Determine the closed-loop transfer function $F_{CL}(s) = \dfrac{Z(s)}{Z_r(s)}$ in terms of K_p, K_d, K and T.

4.4. Determine the range of values for K_p and K_d that ensures the stability of the closed-loop system.

4.5. Calculate the steady-state tracking error in response to a step $z_r(t) = A\Gamma(t)$ by using the final value theorem.

4.6. Using the pole placement by reference-model matching method with the critically damped reference model ($z = 1$), show that:

$$F_{CL}(s) = F_{ref}(s) \Rightarrow \begin{cases} K_p = \dfrac{T\omega_0^2}{K} \\ K_d = \dfrac{2K_p}{\omega_0} - \dfrac{1}{K} \end{cases}$$

Compute their numerical values.

4.7. Show that the transfer function $\dfrac{U(s)}{Z_r(s)}$ takes the following form:

$$\frac{U(s)}{Z_r(s)} = \frac{K_p\, s\,(1+Ts)}{Ts^2 + (1+KK_d)\, s + KK_p}$$

4.8. Determine the speed command $u(0^+)$ just after the step setpoint $z_r(t) = 20\,\Gamma(t)$ (in cm) is applied by using the initial value theorem:

$$\lim_{t\to 0^+} u(t) = \lim_{s\to+\infty} s\,U(s)$$

Simulation shows that $u(0^+)$ is the maximum value of $u(t)$ during the step response. Is the speed command requirement satisfied?

4.9. Are the performance requirements satisfied with this PD structure and design?

To go further. When the time delay τ is not negligible compared with the dominant time constant of the process, it must be taken into account in the controller design. Dedicated methods exist, for example the Smith predictor, which compensates the time delay by using an internal model of the process, and the SIMC tuning rules (Skogestad, 2003), which give simple PI or PD controller settings from a first-order-plus-time-delay model.

For the mini-drone altitude control, the SIMC rules lead to a PD controller with $K_p = 1/(2K\tau)$ and $K_d = K_p T$.

Feedforward plus feedback control & cascade control

A single feedback loop is not always the best answer to a control problem. This problem session introduces two extensions of the basic PID feedback loop that are widely used in practice:

- *feedforward plus feedback control*, illustrated in Exercise 6.1 on the speed control of the Quanser QCar autonomous vehicle;
- *cascade control*, illustrated in Exercise 6.2 on the pitch control of the Quanser AERO equipped with a single rotor, a structure commonly found in drones.

Exercise 6.1 - Feedforward plus feedback speed control of an autonomous vehicle



Figure 6.1: The 1/10-scale autonomous QCar vehicle

The Quanser QCar, as shown in Figure 6.1, is a 1/10-scale autonomous vehicle that you will use in the coming years. Its longitudinal speed is controlled through the drive motor. The input and output of the speed control system are:

- $u(t)$: normalized PWM duty cycle command sent to the drive motor ($u = 0.1$ corresponds to a 10 % duty cycle), named motor command in the following;
- $v(t)$: longitudinal speed of the vehicle in m/s, estimated from the drive motor encoder.

All the experiments have been carried out on the floor. The aim of this exercise is to understand *why* the speed controller of the QCar combines a feedforward term with a PI feedback, and how its gains can be chosen from a simple model.

1. Static characteristic and dynamic model of the QCar longitudinal speed

Table 1 gives the steady-state speed v_{ss} reached by the QCar for five constant values of the motor command u.

Motor command u	0.10	0.125	0.15	0.175	0.20
Steady-state speed v_{ss} (m/s)	0.73	1.02	1.31	1.65	2.00

Table 1: Steady-state speed of the QCar for different values of the motor command

The steady-state speed is *not* proportional to the motor command: as shown in Figure 6.2, the measured points lie on a straight line that does not pass through the origin. The static characteristic is well approximated over the tested range by the affine model:

$$v_{ss} = K\,(u - u_0) \quad \text{with } K \approx 12.7 \text{ m/s and } u_0 \approx 0.04 \tag{1}$$

obtained from the two extreme points of Table 1: $K = \Delta v/\Delta u = 1.27/0.10$. The offset u_0 is a dead zone: below u_0, the motor torque cannot overcome the dry friction and the rolling resistance, and the vehicle does not move.

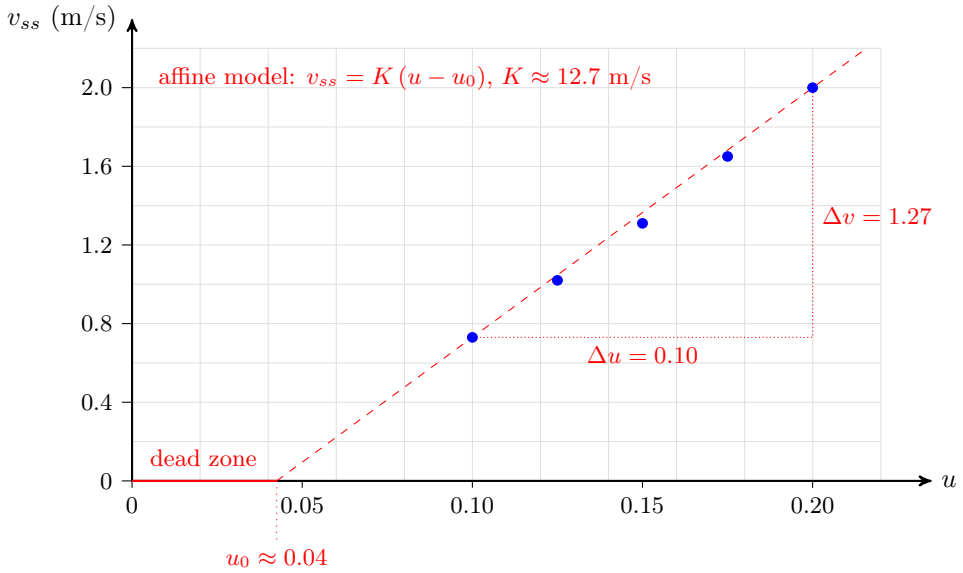


Figure 6.2: Static characteristic of the QCar: measured points of Table 1 (blue) and affine model (1) with its dead zone (red)

1.1. Using the affine model (1), determine the motor command required to drive the QCar at 0.8 m/s and at 1.5 m/s.

1.2. For a stable system, a dynamic model which takes the form of a transfer function can be inferred from a step response test. The QCar speed response to a step change of the motor command is plotted in Figure 6.3. From the step response plot, propose a transfer function model. Justify your choice.

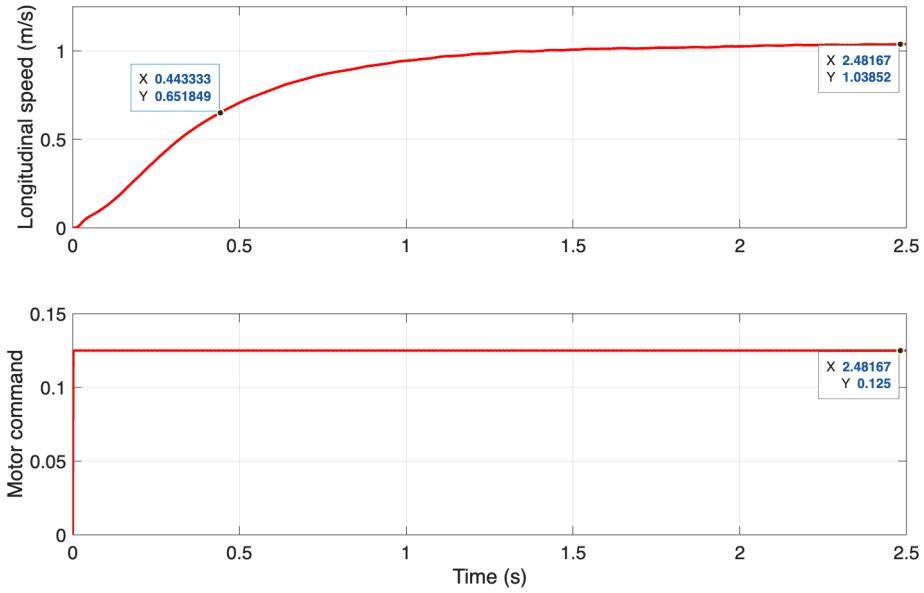


Figure 6.3: Longitudinal speed response of the QCar to a step change of the motor command

Above the dead zone, only the part of the motor command that exceeds u_0 makes the vehicle move. Introducing the *effective motor command* $u_e(t) = u(t) - u_0$, the speed dynamics can be described by a first-order model:

$$G(s) = \frac{V(s)}{U_e(s)} = \frac{K}{1+Ts} \tag{2}$$

where $U(s) = \mathcal{L}[u(t)]$, $U_e(s) = \mathcal{L}[u_e(t)]$ and $V(s) = \mathcal{L}[v(t)]$. The dead zone can be modelled as a constant offset u_0 subtracted from the motor command, with $U_0(s) = \mathcal{L}[u_0\,\Gamma(t)]$.

1.3. Draw the block diagram of the QCar model between the motor command $U(s)$ and the speed $V(s)$, showing the effective motor command $U_e(s)$ and the offset $U_0(s)$. For which values of u is this model valid?

1.4. From Figure 6.3, determine the numerical values of the time constant T (using the time needed to reach 63 % of the final value) and of the gain K of the model (2). Note that the step starts from rest: the gain must be computed with the variation of the effective motor command $u_e(t)$, not with that of $u(t)$. Compare K with the value obtained from the static characteristic.

2. Why combine feedforward and feedback?

The two basic strategies both have a major drawback for the QCar:

- **Feedback only (PI controller).** When the vehicle starts from rest, the motor command is built up by the controller from zero. The QCar remains at rest until the command exceeds the dead zone u_0, and the integral term has to provide the whole steady-state command. The response is therefore delayed and slow, especially at low speed.
- **Feedforward only.** The motor command is computed from the desired speed $v_r(t)$, *without measuring* the speed. The response is immediate, but any model error or disturbance (slope, battery voltage, load, floor surface) leads to a steady-state speed error, since nothing measures and corrects it.

The speed controller provided by Quanser combines both: the *feedforward* term provides immediately a motor command proportional to the desired speed, while the *PI feedback* term corrects the residual error:

$$u(t) = \underbrace{K_{ff}\, v_r(t)}_{\text{feedforward}} + \underbrace{K_p\, \varepsilon(t) + K_i \int_0^t \varepsilon(\tau)\, d\tau}_{\text{PI feedback}} \qquad \text{with } \varepsilon(t) = v_r(t) - v(t)$$

In the speed controller supplied with the vehicle, the default gains are $K_{ff} = 0.1$ (m/s)$^{-1}$, $K_p = 0.3$ (m/s)$^{-1}$ and $K_i = 0.3$ m^{-1}. The integrator is reset to zero whenever the desired speed is equal to zero, so that no integral action is stored while the vehicle is stopped.

The complete control loop, including the model of the QCar of question 1.3, is shown in Figure 6.4. Since the feedforward term is a simple gain, the dead zone is not compensated: the offset $U_0(s)$ acts on the loop as a constant input disturbance, in the same way as the disturbance $D(s)$.

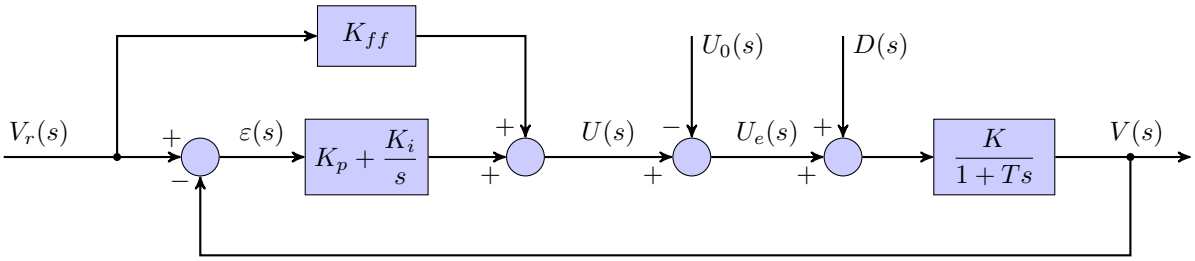


Figure 6.4: Block diagram of the feedforward plus PI feedback speed control of the QCar. The dead zone is modelled by the offset $U_0(s)$; the disturbance $D(s)$, expressed as an equivalent motor command, represents the other effects acting on the vehicle (slope, load, battery voltage, ...)

2.1. Using the affine model (1), which value of K_{ff} would you choose? Compare it with the value used in the default controller, $K_{ff} = 0.1$ (m/s)$^{-1}$. Assume that only this default feedforward term is applied ($K_p = K_i = 0$): using the affine model, determine the steady-state speed reached for $v_r = 0.73$ m/s and for $v_r = 2.00$ m/s. Calculate the relative speed error in each case. Why is the error so large at low speed?

2.2. It can be shown from Figure 6.4 that the speed of the QCar can be written as:

$$V(s) = \frac{K\big((K_{ff}+K_p)\,s + K_i\big)}{Ts^2 + (1+KK_p)\,s + KK_i}\,V_r(s) + \frac{K\,s}{Ts^2 + (1+KK_p)\,s + KK_i}\,\big(D(s) - U_0(s)\big)$$

Does the feedforward gain K_{ff} modify the closed-loop poles?

2.3. Deduce the steady-state speed $v(+\infty)$ for a step setpoint $v_r(t) = v_r\,\Gamma(t)$, a constant disturbance $d(t) = d\,\Gamma(t)$ and the dead zone $u_0\,\Gamma(t)$. Which term of the controller compensates the dead zone?

3. Tuning of the PI feedback

Since the feedforward term provides most of the motor command, the feedback only has to compensate the dead zone, a constant term handled by the integral action, and to correct small residual errors. It can therefore be tuned with a moderate speed of response, which makes it robust to model errors (the local slope of the static characteristic varies by about 20 % over the tested range). The PI controller is written as:

$$C(s) = K_p + \frac{K_i}{s} = K_p\,\frac{1+T_i s}{T_i s} \quad \text{with } T_i = \frac{K_p}{K_i}$$

As in Problems n°4 (section 4), the PI zero is chosen to cancel the pole of the model (2): $T_i = T$. The closed-loop transfer function of the feedback loop is then a first-order transfer function with unit steady-state gain and time constant $T_{CL} = T/(KK_p)$. The closed-loop time constant is chosen equal to the open-loop one, $T_{CL} = T$: the feedback loop is not required to make the vehicle faster, but only to correct the residual errors.[2]

3.1. Show that the PI gains are given by:

$$K_p = \frac{T}{K\,T_{CL}} \qquad K_i = \frac{1}{K\,T_{CL}}$$

3.2. Compute their numerical values for $T_{CL} = T$ and specify their units.

3.3. Compare with the default gains.

[2] A faster feedback loop would amplify the noise of the speed estimated from the encoder and excite the dynamics neglected in the first-order model (see the beginning of the step response of Figure 6.3). The fast part of the response is left to the feedforward term.

Take-away. The feedforward term uses a model of the process to provide immediately the control signal required by the setpoint; it is fast but blind to disturbances and model errors. The feedback term measures the output and corrects what the model does not predict; it guarantees a zero steady-state error but needs an error to act. Combining both gives a fast response with a gently tuned, and therefore robust, feedback loop.

To go further. Combining a static feedforward term with a PI acting on the error may cause a small overshoot on setpoint changes, since the integral term also reacts to the transient error that the feedforward term would have removed on its own. Applying the proportional action to the measurement (I-P structure of Problems n°4) or filtering the setpoint reduces this effect.

Exercise 6.2 - Cascade control of the pitch for a 1DOF aerospace system



Figure 6.5: Quanser AERO in its one-degree-of-freedom configuration with a single rotor

The Quanser AERO, as shown in Figure 6.5, is a reconfigurable experimental platform to experience control design for various aerospace systems, from one-degree-of-freedom (1 DOF) and 2 DOF helicopter to half-quadrotor. It is used here in its 1DOF configuration: the yaw axis is locked and a single rotor, driven by a DC motor, generates the thrust that rotates the body around the pitch axis. Two measurements are available from incremental encoders, but there is only one control input:

- $v_m(t)$: DC motor voltage in V, limited to [0 ; 20] V (the rotor only turns in one direction);
- $\omega_m(t)$: rotary speed of the rotor in rad/s;
- $\theta(t)$: pitch angle of the body in rad.

The two following transfer function models have been identified from experimental data around an operating point:

- rotor dynamics (motor voltage to rotor speed):

$$G_r(s) = \frac{\Omega_m(s)}{V_m(s)} = \frac{K_r}{1 + T_r s} \quad \text{with } K_r \approx 16.2 \text{ rad/(V.s) and } T_r \approx 0.158 \text{ s}$$

- body dynamics (rotor speed to pitch angle):

$$G_b(s) = \frac{\Theta(s)}{\Omega_m(s)} = \frac{K_b\,\omega_b^2}{s^2 + 2z_b\omega_b s + \omega_b^2}$$

with $K_b \approx 0.0223$ rad.s/rad, $\omega_b \approx 1.54$ rad/s and $z_b \approx 0.22$.

The *cascade* control structure is shown in Figure 6.6: an inner loop controls the rotor speed, and an outer loop computes the rotor speed setpoint $\omega_r(t)$ from the pitch angle setpoint $\theta_r(t)$ and the measured pitch angle.

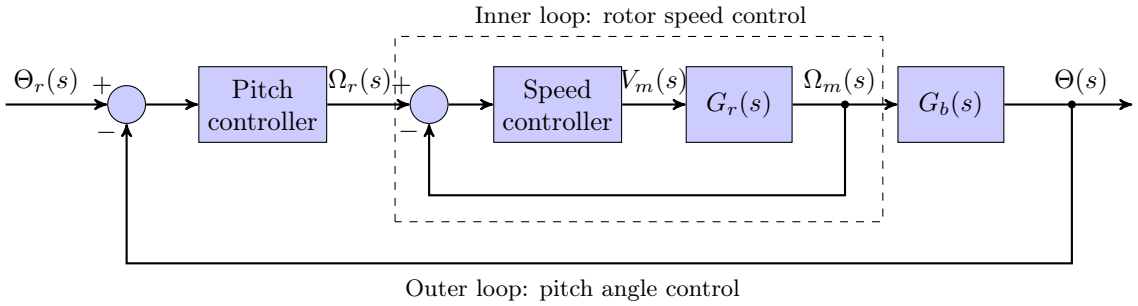


Figure 6.6: Block diagram of the cascade pitch control of the Quanser AERO

The performance requirements for the two control loops are described in Table 2.

Requirement	Assessment criterion	Level
Inner loop: tracking of a step setpoint for the rotor speed	Steady-state error	No steady-state error
	Overshoot	No overshoot allowed (critically damped)
	Settling time at 5 %	$T_r^{5\%} \approx 0.2$ s
	Motor voltage	Limited to [0 ; 20] V
Outer loop: tracking of a step setpoint for the pitch angle	Steady-state error	No steady-state error
	Peak overshoot	$D_{1\%} \approx 4.3$ %
	Settling time at 5 %	$T_r^{5\%} \approx 1$ s

Table 2: Performance requirements for the cascade pitch control of the Quanser AERO

4. Why a cascade structure?

Compared with a single loop computing the motor voltage directly from the pitch error, the cascade structure has two practical advantages:

- **fast rejection of actuator disturbances**: a variation of the supply voltage or of the aerodynamic load on the propeller first modifies the rotor speed; it is corrected by the inner loop within a few tenths of a second, before it has any visible effect on the pitch angle;
- **a simpler design of the pitch controller**: seen from the outer loop, the closed inner loop behaves as an almost perfect actuator ($\Omega_m(s) \approx \Omega_r(s)$), so that the pitch controller can be designed from the body model only, whatever the exact motor parameters.

Both advantages require the inner loop to be much faster than the outer one, which is checked below.

4.1. Calculate the 5 % settling time $T_r^{5\%}$ of the step response of the rotor model $G_r(s)$. Recall that $T_r^{5\%} = 3\,T_m$ for a first-order system.

4.2. Calculate the 5 % settling time $T_r^{5\%}$ of the step response of the body model $G_b(s)$. Recall that, for a lightly damped second-order system ($z < 0.7$), the settling time can be approximated from the envelope of the oscillations:

$$T_r^{5\%} \approx \frac{3}{z_b\,\omega_b}$$

4.3. Calculate the ratio between the settling times of the two subsystems. A cascade control is efficient when the inner subsystem is much faster than the outer one, typically by a factor of 5 to 10. Is a cascade control a good option here?

5. Inner loop: rotor speed control using an I-P controller

The rotor speed is controlled by the I-P controller studied in Problems n°4 (Figure 4.5), with gains $K_{p,\omega}$ and $K_{i,\omega}$:

$$V_m(s) = \frac{K_{i,\omega}}{s}\big(\Omega_r(s) - \Omega_m(s)\big) - K_{p,\omega}\,\Omega_m(s)$$

Recall from Problems n°4 (question 5.8) that the pole placement by reference-model matching method gives:

$$K_{i,\omega} = \frac{T_r\,\omega_0^2}{K_r} \qquad K_{p,\omega} = \frac{2zK_{i,\omega}}{\omega_0} - \frac{1}{K_r}$$

5.1. From the performance requirements of the inner loop, determine z and ω_0 (recall that $\omega_0 \approx 4.75/T_r^{5\%}$ when $z = 1$), then the numerical values of $K_{p,\omega}$ and $K_{i,\omega}$.

5.2. What is the steady-state gain of the inner closed loop? Why can the inner closed loop be approximated by $\Omega_m(s) \approx \Omega_r(s)$ for the design of the outer loop?

6. Outer loop: pitch angle control using an I-PD controller

The pitch angle is controlled by an I-PD controller, in which the integral term acts on the error while the proportional and derivative terms act on the measured pitch angle, as shown in Figure 6.7. As in Problems n°4 and 5, applying the proportional and derivative actions to the output avoids introducing a zero in the closed-loop transfer function and a jump of the control signal when a step setpoint is applied.

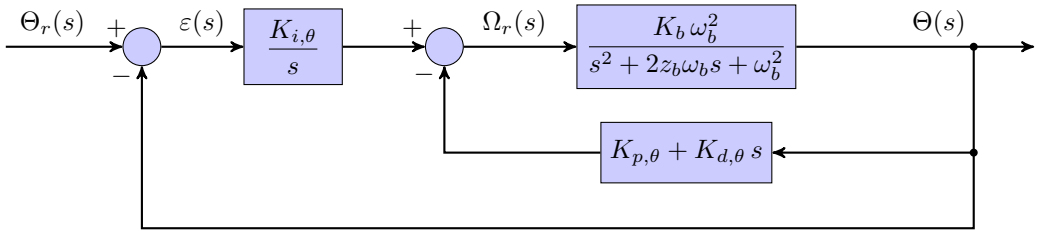


Figure 6.7: Block diagram of the I-PD pitch angle control, the inner loop being approximated by $\Omega_m(s) \approx \Omega_r(s)$

It can be shown that the closed-loop transfer function is:

$$F_{CL}(s) = \frac{\Theta(s)}{\Theta_r(s)} = \frac{K_b\,\omega_b^2\,K_{i,\theta}}{s^3 + \big(2z_b\omega_b + K_b\,\omega_b^2\,K_{d,\theta}\big)\,s^2 + \omega_b^2\big(1 + K_b\,K_{p,\theta}\big)\,s + K_b\,\omega_b^2\,K_{i,\theta}}$$

Since $F_{CL}(s)$ is of third order, the standard second-order reference model cannot be used directly. The desired closed-loop behaviour is chosen as the product of a first-order and a standard second-order transfer function:

$$F_{ref}(s) = \frac{p_0\,\omega_0^2}{(s + p_0)\big(s^2 + 2z\omega_0 s + \omega_0^2\big)}$$

The real pole $-p_0$ is fixed by the designer, while z and ω_0 are deduced from the performance requirements.

6.1. Check that $F_{CL}(s)$ has a unit steady-state gain and no zero.

6.2. Why should p_0 be chosen larger than $z\omega_0$ for the performance requirements to be met?

6.3. Determine z and ω_0 from the performance requirements of the outer loop (see Problems n°4, question 5.7).

6.4. Expand the denominator of $F_{ref}(s)$ and show, by identification with the denominator of $F_{CL}(s)$, that:

$$F_{CL}(s) = F_{ref}(s) \Rightarrow \begin{cases} K_{i,\theta} = \dfrac{p_0\,\omega_0^2}{K_b\,\omega_b^2} \\ K_{p,\theta} = \dfrac{\omega_0^2 + 2z\omega_0\,p_0 - \omega_b^2}{K_b\,\omega_b^2} \\ K_{d,\theta} = \dfrac{p_0 + 2z\omega_0 - 2z_b\omega_b}{K_b\,\omega_b^2} \end{cases}$$

Compute their numerical values for $p_0 = 2\,\omega_0$.

6.5. Simulation of the complete cascade (inner I-P loop of section 5 and outer I-PD loop of section 6) gives $D_1 \approx 3$ % and $T_r^{5\%} \approx 1.2$ s for a step setpoint $\theta_r(t) = 0.2\,\Gamma(t)$ (in rad), the motor voltage varying by about 2 V around its operating point. Are the performance requirements satisfied? Compare the undamped natural frequencies of the inner and outer closed loops: is the rule of thumb of question 4.3 satisfied?

To go further. The choice of p_0 is a trade-off: for $p_0 = \omega_0$, the real pole dominates and the response is slower ($T_r^{5\%} \approx 1.5$ s), whereas for $p_0 = 5\,\omega_0$ the response approaches that of the second-order part ($T_r^{5\%} \approx 1.05$ s) at the price of much larger gains, hence a higher sensitivity to measurement noise and a risk of saturating the motor voltage. Before implementing the cascade on the AERO, the voltage limits and the single direction of rotation of the rotor, the computation of the pitch rate for the derivative action and the dry friction in the pitch pivot must also be considered. Finally, feedforward and cascade control can be combined: the rotor speed required to hold a given pitch angle could be added as a feedforward term to the output of the pitch controller.

Appendices

Useful properties of the Laplace transform

Some useful properties which have found practical use in Control Engineering are summarized below.

Property	signal	Laplace transform
linearity	$ax(t) + by(t)$	$aX(s) + bY(s)$
time delays	$x(t - \tau)$	$e^{-\tau s}X(s)$
convolution	$y(t) = h(t) * u(t)$	$Y(s) = H(s)U(s)$
differentiation	$\dot{x}(t)$	$sX(s) - x(0)$
	$\ddot{x}(t)$	$s^2X(s) - sx(0) - \dot{x}(0)$
integration	$\int_0^t x(\tau)d\tau$	$\frac{X(s)}{s}$
initial value theorem	$\lim_{t\to 0+} x(t) = \lim_{s\to+\infty} sX(s)$	
final value theorem	$\lim_{t\to+\infty} x(t) = \lim_{s\to 0} sX(s)$	if the limit exists

Some Laplace transform pairs

Signal	Laplace transform
$\delta(t)$	1
$\Gamma(t)$	$\frac{1}{s}$
$r(t) = t\Gamma(t)$	$\frac{1}{s^2}$
$t^2\Gamma(t)$	$\frac{2}{s^3}$
$e^{-at}\Gamma(t)$	$\frac{1}{s+a}$
$t^n e^{-at}\Gamma(t)$	$\frac{n!}{(s+a)^{n+1}}$
$\cos(\omega_0 t)\Gamma(t)$	$\frac{s}{s^2+\omega_0^2}$
$\sin(\omega_0 t)\Gamma(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
$e^{-at}\cos(\omega_0 t)\Gamma(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
$e^{-at}\sin(\omega_0 t)\Gamma(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

A few important transfer functions

First-order systems

$$G(s) = \frac{K}{1+Ts}$$

The 2 characteristic parameters of a first order system are:

- K: steady-state gain
$$K = \lim_{s\to 0} G(s)$$
- T: time constant

Characteristic values of a first-order system step response

Rise time at 63%	$T_m^{63\%} = T$
Rise time at 95%	$T_m^{95\%} \approx 3T$
Settling time at 5 %	$T_r^{5\%} \approx 3T$

Second-order systems

$$G(s) = \frac{K}{\frac{s^2}{\omega_0^2} + 2\frac{z}{\omega_0}s + 1} = \frac{K\omega_0^2}{s^2 + 2z\omega_0 s + \omega_0^2}$$

The 3 characteristic parameters of a second-order system are:

- K: steady-state gain
- z : damping ratio ($z > 0$)
- ω_0 : undamped natural frequency

Characteristic values of a underdamped second-order system step response ($z < 1$)

Value of the first overshoot in %	$D_{1\%} = e^{\frac{-\pi z}{\sqrt{1-z^2}}} \times 100$
Time-instant of the first overshoot	$T_{D_1} = \frac{\pi}{\omega_0\sqrt{1-z^2}}$
Value of the n^{th} overshoot in %	$D_{n\%} = -(-D_1)^n \times 100$
Time-instant of the n^{th} overshoot	$T_{D_n} = n\,T_{D_1}$
Pseudo-period	$T_p = \frac{2\pi}{\omega_0\sqrt{1-z^2}}$
Settling time at 5 %	$T_r^{5\%} \approx \frac{3}{\omega_0 z}$ if $z < 1$; $\quad T_r^{5\%} \approx \frac{3}{\omega_0}$ if $z \approx 0.707$
Settling time at x % for a given z	$T_r^{x\%} = \frac{\ln\left(\frac{100}{x\sqrt{1-z^2}}\right)}{\omega_0 z}$
Rise time (100%)	$T_m^{100\%} = \frac{\pi - \operatorname{acos}(z)}{\omega_0\sqrt{1-z^2}}$

Identification of a first-order model

$$G(s) = \frac{K}{1 + Ts}$$

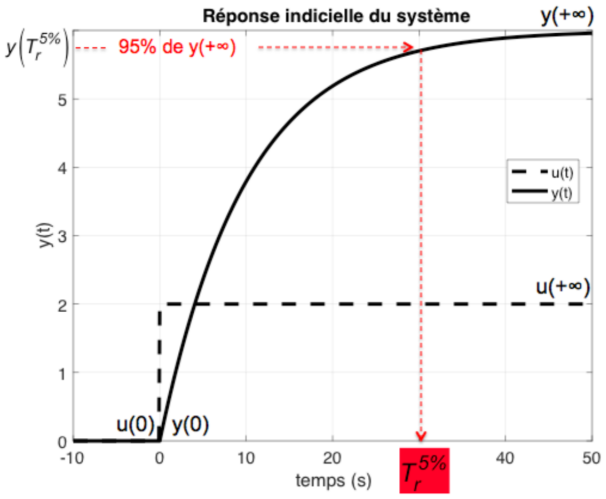


Figure 6.8: Step response of a first-order system

From the step response displayed in Figure 6.8, it is necessary to determine the steady-state gain K and the time constant T. The procedure is as follows:

1. Find the final and initial values of the response and of the step. Deduce K from:

$$K = \frac{y(+\infty) - y(0)}{u(+\infty) - u(0)}$$

2. Find $y(T_r^{5\%})$, deduce from it $T_r^{5\%}$ then T:

$$T = \frac{T_r^{5\%}}{3}$$

Identification of a second-order underdamped model

$$G(s) = \frac{K}{\frac{s^2}{\omega_0^2} + 2\frac{z}{\omega_0}s + 1} = \frac{K\omega_0^2}{s^2 + 2z\omega_0 s + \omega_0^2}$$

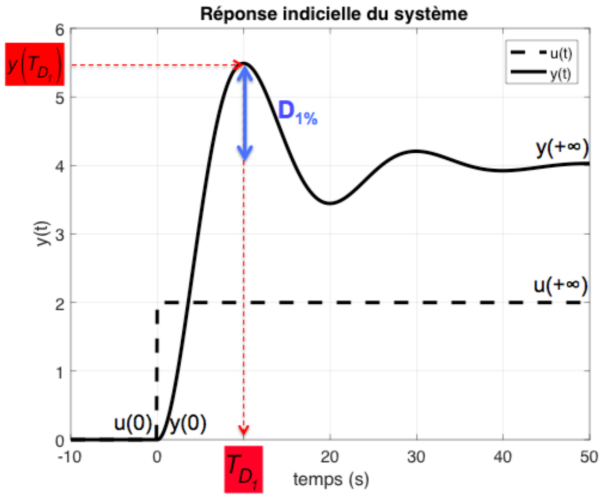


Figure 6.9: Step response of a second-order system

From the step response displayed in Figure 6.9, it is necessary to determine the steady-state gain K, the damping ratio z and the undamped natural frequency ω_0. The procedure is as follows:

1. Find the final and initial values of the response and of the step. Deduce K from:
$$K = \frac{y(+\infty) - y(0)}{u(+\infty) - u(0)}$$

2. Find the final and initial values of the response and that of the first overshoot $y(t_{D_1})$. Deduce from it D_1, then z:
$$D_1 = \frac{y(T_{D_1}) - y(+\infty)}{y(+\infty) - y(0)}$$
$$z = \sqrt{\frac{(\ln(D_1))^2}{(\ln(D_1))^2 + \pi^2}}$$

3. Find the time-instant of the first overshoot T_{D_1}. Deduce from it ω_0:
$$\omega_0 = \frac{\pi}{T_{D_1}\sqrt{1 - z^2}}$$

PID tuning by using empirical rules

A PID controller defined in its so-called *ideal* form is defined as:

$$C(s) = K_p \left(1 + \frac{1}{T_i s} + \frac{T_d s}{1 + \frac{T_d}{N} s} \right) = K_p + K_i \frac{1}{s} + K_d \frac{s}{1 + \frac{T_d}{N} s} \tag{1}$$

The individual effects of these three K_p, K_i and K_d parameters appearing in (1) on the closed-loop performance of stable plants are summarized in Table 1 below. A nice animation that illustrates well the individual effects is available at: `https://en.wikipedia.org/wiki/File:PID_Compensation_Animated.gif`

TABLE 1 Effects of independent P, I, and D tuning on closed-loop response. For example, while K_I and K_D are fixed, increasing K_P alone can decrease rise time, increase overshoot, slightly increase settling time, decrease the steady-state error, and decrease stability margins.

	Rise Time	Overshoot	Settling Time	Steady-State Error	Stability
Increasing K_P	Decrease	Increase	Small Increase	Decrease	Degrade
Increasing K_I	Small Decrease	Increase	Increase	Large Decrease	Degrade
Increasing K_D	Small Decrease	Decrease	Decrease	Minor Change	Improve

Tuning of PID controllers - Take away message

Several tuning rules are available for determining the PID controller parameter values. The proposed rules address different design specifications such as load disturbance rejection or setpoint tracking.

There is no single best method and no one correct answer. You need to approach each problem individually and learn to make your judgment. Generally, all the methods require iterative tuning.

Tuning of the PID controller by using the Ziegler-Nichols rules

Assume a continuous-time process is reasonably well modeled by a **first-order plus time delay transfer function model**:

$$G(s) = \frac{Ke^{-\tau s}}{1 + Ts}$$

The most popular tuning rules are those attributed to Ziegler-Nichols. Their aim is to provide ***satisfactory load disturbance rejection***. Ziegler-Nichols rules have been set to have a one quarter-amplitude decay ratio. A one quarter-amplitude decay ratio is a response where the amount of undershoot decays to one-fourth of the previous amplitude of the overshoot every whole cycle after being upset by a disturbance, as shown in Figure A.1. It may be too oscillatory (underdamped) for some systems.

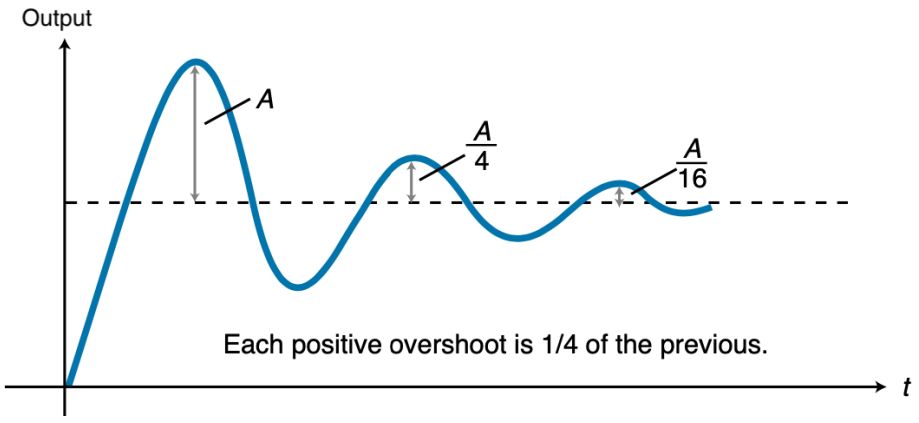


Figure A.1: This response is the desired tuned system response. The closed-loop system reduces the magnitude of each overshoot by one-quarter of the previous one.

Table 3 shows the Ziegler-Nichols rules for P, PI, or PID controllers defined in its so-called *ideal* form (see (1)).

Controller type	K_p	T_i	T_d
P	$\frac{T}{K\tau}$		
PI	$0.9\frac{T}{K\tau}$	3τ	
PID	$1.2\frac{T}{K\tau}$	2τ	0.5τ

Table 3: Ziegler-Nichols disturbance rejection tuning rules for a first-order-plus delay model determined from an open-loop step response

Tuning of the PID controller by using the Chien-Hrones-Reswick rules

As previously said, the Ziegler-Nichols rule aim is to provide satisfactory disturbance rejection. For ***setpoint tracking or servo control***, it can be advantageous to use the empirical rules suggested by Chien-Hrones-Reswick which are given in Table 4 for P, PI, and PID controllers in their *ideal* form (see equation (1)).

Controller type	K_p	T_i	T_d
P	$0.3\frac{T}{K\tau}$		
PI	$0.35\frac{T}{K\tau}$	1.2T	
PID	$0.6\frac{T}{K\tau}$	T	0.5T

Table 4: Chien-Hrones-Reswick setpoint tracking tuning rules for a first-order-plus delay model determined from an open-loop step response

English to French glossary

bandwidth	:	bande passante
crane	:	grue
closed-loop system	:	système bouclé
cut-off frequency	:	fréquence (ou pulsation) de coupure
damped frequency	:	pulsation amortie
damping ratio	:	coefficient d'amortissement
drag	:	traînée
feedback	:	contre-réaction
feedback system	:	système à contre-réaction
hoisting device	:	dispositif de levage
impulse response	:	réponse impulsionnelle
integral wind-up	:	emballement (de l'action) intégral
input	:	entrée
joint	:	articulation
gain	:	gain
heading angle	:	angle de cap
linear time-invariant (LTI)	:	linéaire invariant dans le temps
motor shaft	:	arbre moteur
output	:	sortie
overdamped	:	sur-amorti
overshoot	:	dépassement
pole placement by reference-model matching	:	placement de pôles par identification à un modèle de référence
rise time	:	temps de montée
road grade	:	inclinaison de la route
robot arm joint	:	articulation d'un bras de robot
root locus	:	lieu des racines
setpoint	:	consigne
settling time	:	temps de réponse
steady-state gain	:	gain statique
steady-state response	:	réponse en régime permanent
steering	:	direction
step response	:	réponse indicielle
stream	:	courant
time delay	:	retard pur
time-invariant	:	invariant dans le temps
transient response	:	réponse transitoire
throttle	:	accélérateur
undamped	:	non amorti
undamped natural frequency	:	pulsation propre non amortie
underdamped	:	sous-amorti