Control engineering
Problems n°1
H. Garnier
F. Collin

Laplace transform, transfer function and block-diagram analysis of linear time-invariant (LTI) dynamic systems

Exercise 1.1 - Laplace transform of an exponential signal
Consider the following signal:

$$x(t) = e^{-at}\Gamma(t), \quad a = 1$$

1.1.a. Recall the usual name of $\Gamma(t)$ and its definition.

1.1.b. Plot the signal $x(t)$.

1.1.c. Is the signal $x(t)$ causal ? Justify your answer.

1.1.d. What is the role of $\Gamma(t)$ in the definition of $x(t)$.

1.1.e. Determine the Laplace transform of $x(t)$ from the definition integral.

Exercise 1.2 - Laplace transform of a delayed Dirac delta function
Consider the following signal:

$$y(t) = \delta(t - \tau), \quad \tau > 0$$

1.2.a. Recall the usual name of $\delta(t)$ and its definition.

1.2.b. Plot the signal $y(t)$.

1.2.c. Determine the Laplace transform of $y(t)$ by using the Laplace transform properties (see Appendices).

Exercise 1.3 - Inverse Laplace transform
Determine the inverse Laplace transform of:

$$Y(s) = \frac{2}{(s+3)(s+5)}$$

Exercise 1.4 - Solution of differential equations
Solve the following differential equations using the Laplace transform:

1.4.a.

$$\dot{y}_1(t) = -2y_1(t), \qquad y_1(0) = 1$$

1.4.b.

$$\dot{y}_2(t) + 2y_2(t) = \Gamma(t), \qquad y_2(0) = 1$$

1.4.c.

$$\ddot{y}_3(t) + 10\dot{y}_3(t) + 16y_3(t) = 10\delta(t), \qquad \dot{y}_3(0) = y_3(0) = 0$$

Exercise 1.5 - Transfer function of a mechanical system
Consider a mechanical suspension system shown in Figure 1.1 constituted of a mass, a damper and a spring having a damping and stiffness coefficient of b and k respectively.
The differential equation of this mechanical system relating the vertical position $y(t)$ of the mass (system output) and the external force $u(t)$ (system input) applied to the system is:

$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = u(t)$$

1.5.a. Determine the transfer function $G(s) = \frac{Y(s)}{U(s)}$ and represent the system in the form of a block diagram.

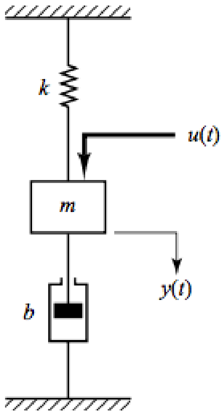


Figure 1.1: Mechanical system.

1.5.b. Give the system order, the steady-state gain, the pole(s) and zero(s) of the transfer function.

Exercise 1.6 - Equivalent transfer function of simple closed-loop block-diagram
Consider the closed-loop block diagram displayed in Figure 1.2.
Derive its equivalent transfer function $T(s) = \frac{Y(s)}{R(s)}$.

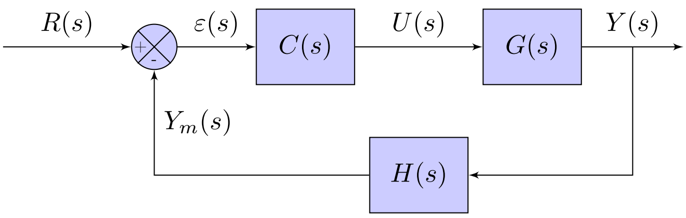


Figure 1.2: Classical block-diagram of a simple closed-loop feedback system

Exercise 1.7 - Transfer function of a simple closed-loop block-diagram. A case study
Consider the closed-loop block diagram displayed in Figure 1.3.
Determine its equivalent transfer function $T(s) = \frac{Y(s)}{R(s)}$.

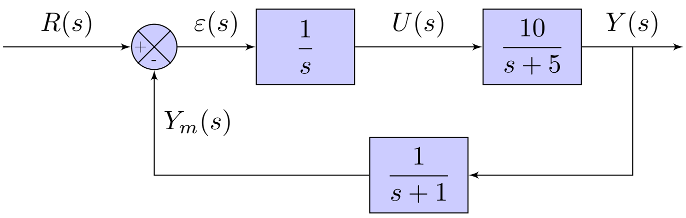


Figure 1.3: Case study of a simple closed-loop feedback system

H. Garnier
F. Collin

Step responses of important systems & System identification from step response test

Exercise 2.1 - Step response of a first-order system
Consider the system described by the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2}{1+10s}$$

1. Determine the steady-state gain K, the time-constant T and the pole.
2. Recall the unit step response and calculate its slope at the origin.
3. Calculate the rise-times $T_m^{63\%}$ and $T_m^{95\%}$ as well as the settling-time $T_r^{5\%}$.
4. Without any calculation, plot precisely the step response and indicate on it its characteristic parameters calculated above.

Exercise 2.2 - Step response of a first-order plus time-delay system
Consider a first-order system having a steady-state gain of 2, a time-constant of 10 seconds and a pure time-delay of 20 seconds.

1. Give the system transfer function $G(s)$.
2. Without any calculation, plot precisely the unit step response and indicate on it its characteristic parameters given above.

Exercise 2.3 - Step response of a dynamical system
Consider a system whose dynamic behavior is governed by the following differential equation:

$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = 10u(t) \qquad \text{with } \dot{y}(0) = 0, \quad y(0) = 0$$

1. Determine the transfer function $G(s) = \frac{Y(s)}{U(s)}$ of the system.
2. Determine the order of the system, the steady-state gain K, the damping ratio z, the undamped natural frequency ω_0, the poles and zeros.
3. Conclude about the type of step response: critical, overdamped or underdamped.
4. Calculate the values of the first and second overshoot $D_{1\%}$ and $D_{2\%}$, the times of the first and second overshoot t_{D_1} and t_{D_2}.
5. Plot the step response and indicate on it the characteristic parameters computed above.

Exercise 2.4 - Dominant behavior identification of a thermal system from step response test
Consider the temperature response $y(t)$ (in $^\circ$Celsius) to a step input ($u(t) = 35 \times \Gamma(t - \tau_u)$) with $\tau_u = 20$ seconds of a dynamic thermal system plotted in Figure 2.1.

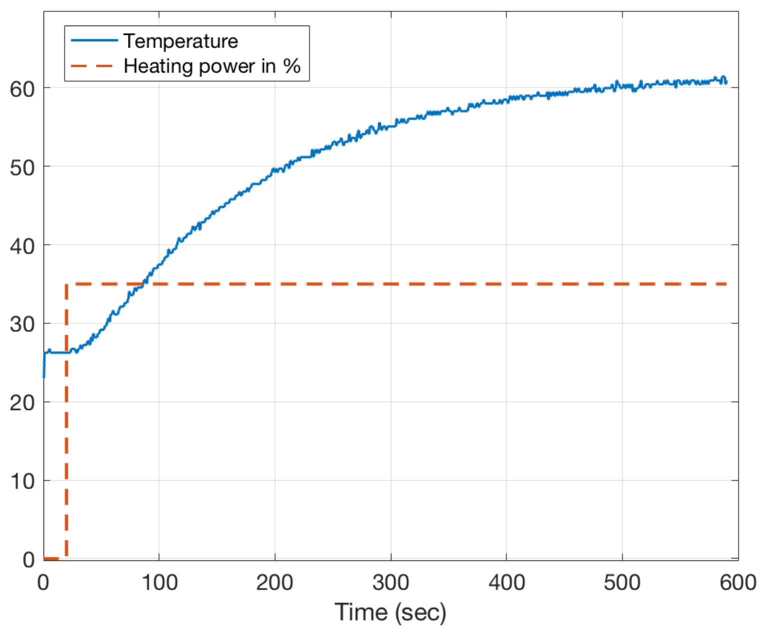


Figure 2.1: Step and temperature response of the thermal system

1. Propose a transfer function model form $G(s)$ for the system. Explain your choice.
2. Determine the numerical values of the different parameters of your chosen $G(s)$.
3. Deduce, from your identified transfer function model $G(s)$, the differential equation of the system.

Exercise 2.5 - Identification of a mechanical suspension system from step response test
Consider the position response $y(t)$ of a mechanical suspension system to a unit step input ($u(t) = \Gamma(t)$) plotted in Figure 2.2.

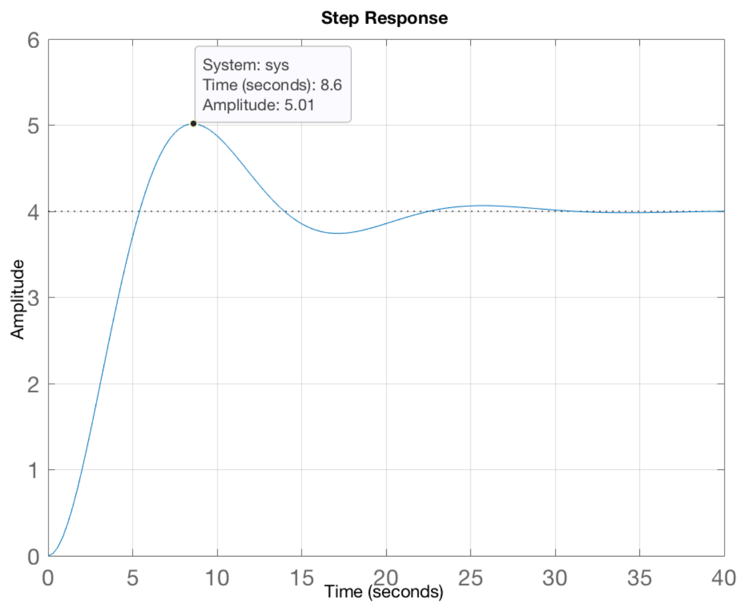


Figure 2.2: Response of a mechanical system to a unit step input

1. Propose a transfer function model form $G(s)$ for the system. Explain your choice.
2. Determine the numerical values of the different parameters of your chosen $G(s)$.
3. Deduce, from your identified transfer function model $G(s)$, the differential equation of the system.

Exercise 2.6 - Step response analysis
Figure 2.3 shows the unit step responses of five different linear time-invariant (LTI) systems. Pair each of the 5 step responses to one of the 7 transfer functions below. Explain your answers.

$$G_1(s) = \frac{0.1}{s+0.1}; \qquad G_2(s) = \frac{4}{s^2+2s+4}; \qquad G_3(s) = \frac{0.5}{s^2-0.1s+2}$$
$$G_4(s) = \frac{-0.5}{s^2+0.1s+2}; \qquad G_5(s) = \frac{1}{s+1}; \qquad G_6(s) = \frac{4}{s^2+0.8s+4}$$
$$G_7(s) = \frac{2}{s^2+s+3}$$

You can use Matlab to verify your solutions.

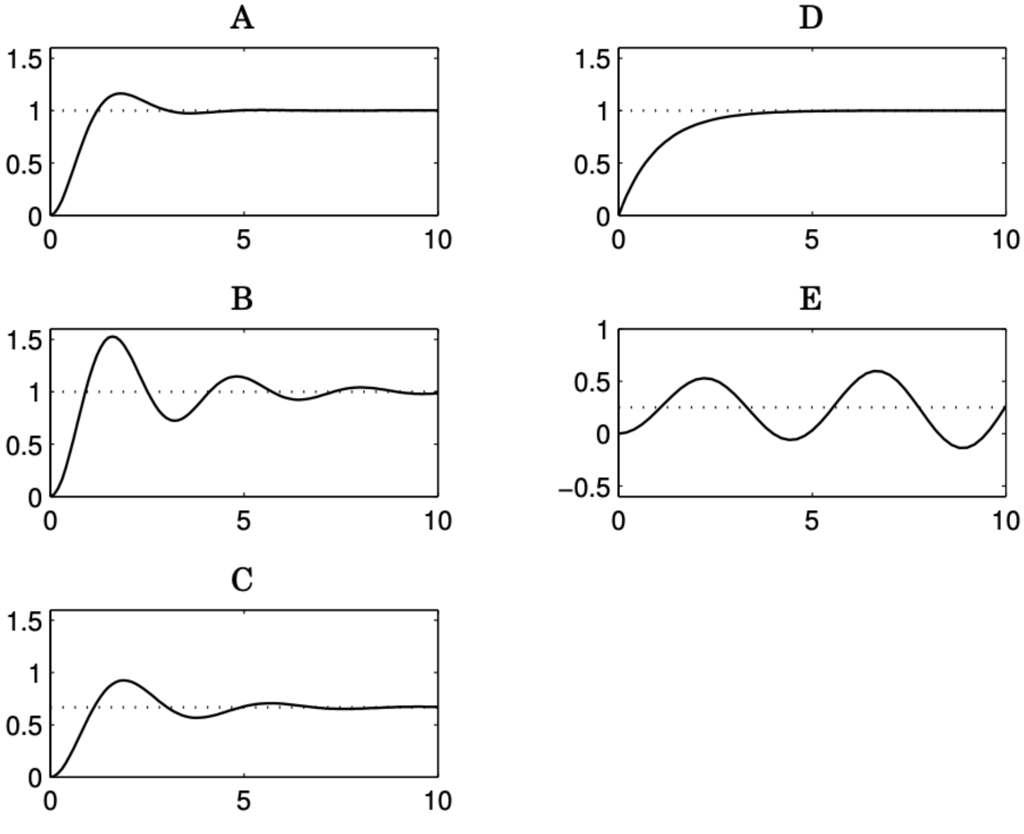


Figure 2.3: Step responses of different LTI systems

Stability and steady-state error analysis

Exercise 3.1 - Stability from the system poles
Determine the poles and zeros of the LTI systems described by the transfer functions below. Conclude about their stability:

$$G_1(s) = \frac{2}{s+2}; \qquad G_2(s) = \frac{2}{s^2+3s+2}; \qquad G_3(s) = \frac{1}{s^2+2s+2}$$
$$G_4(s) = \frac{2}{s^2+4}; \qquad G_5(s) = \frac{2}{s(s+2)}; \qquad G_6(s) = \frac{2}{s^2(s+2)}$$
$$G_7(s) = \frac{2(s^2-2s+2)}{(s+2)(s^2+2s+2)}; \qquad G_8(s) = \frac{200}{(s+2)(s^2-2s+2)}$$

Exercise 3.2 - Links between system poles and step responses
Pair the step responses and pole-zero diagrams in Figure 3.1. Give your solutions so that the pairs of plots that belong to the same system is written in the form Pole-zero-letter–Step-response-letter.

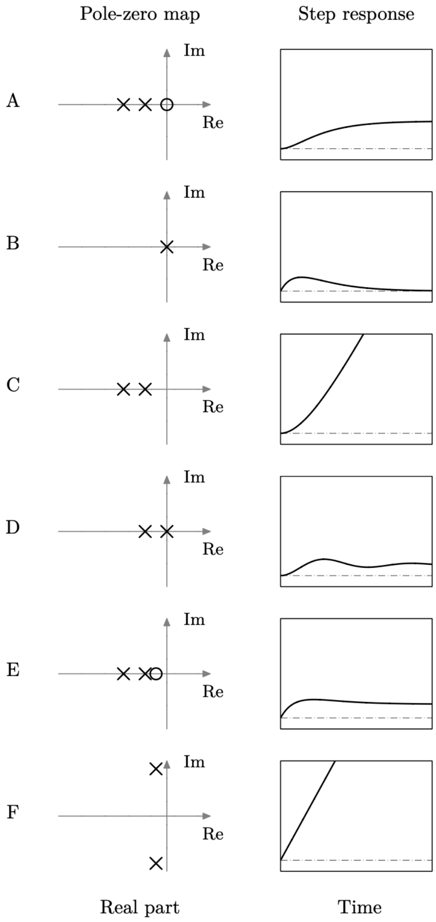


Figure 3.1: Pole-zero diagrams and step responses of different LTI systems. In the pole-zero diagram, imaginary and real parts have equal scaling, x marks poles, and o marks zeros.

Exercise 3.3 - Stability analysis by using the Routh-Hurwitz criterion
Study the stability of the LTI systems having the characteristic equations below. Specify, in case of instability, the number of unstable poles:

a) $s^3 - 25s^2 + 10s + 450 = 0$

b) $s^3 + 25s^2 + 450 = 0$

c) $s^3 + 25s^2 + 10s + 450 = 0$

d) $s^3 + 25s^2 + 10s + 50 = 0$

Exercise 3.4 - Elevator control system for supertall building
Yokohama's 70-floor Landmark Tower was completed in 1993 and it is still the tallest building in the Greater Tokyo Area with a height of 293m. Its elevators operate at a peak speed of 45 km/h! To reach such a speed without inducing discomfort in passengers, the elevator accelerates for longer periods, rather than more precipitously. Going up, it reaches full speed only at the 27th floor; it begins decelerating 15 floors later. The result is a peak acceleration similar to that of other skyscraper elevators–a bit less than a tenth of the force of gravity. Admirable ingenuity has gone into making this safe and comfortable. Special ceramic brakes had to be developed; iron ones would melt. Computer-controlled systems damp out vibrations. The lift has been streamlined to reduce the wind noise as it speeds up and down.

The closed-loop block-diagram for the control of the elevator's vertical position is shown in Figure 3.2.

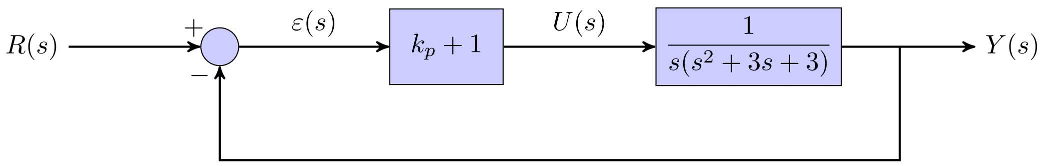


Figure 3.2: Vertical position control system for an elevator

1. Determine the range of values for k_p that ensures the stability of the control vertical position system.
2. By using the final value theorem, calculate the expected steady-state error for a step reference of amplitude A as a function of k_p.

Exercise 3.5 - Mobile robot steering control
The block diagram of the steering control system for a mobile robot is shown in Figure 3.3, where $R(s)$ and $Y(s)$ represent the Laplace transform of the desired and measured heading angle respectively.

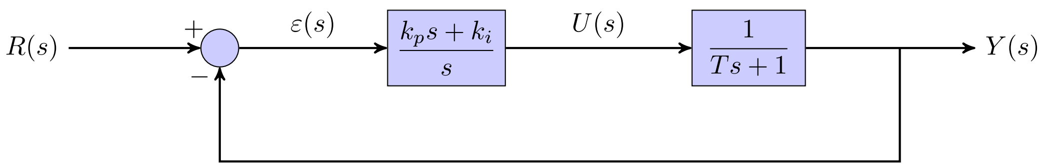


Figure 3.3: Block diagram of the steering control system for a mobile robot

1. Express $\varepsilon(s)$ in terms of $R(s)$, k_p, k_i and T.
2. When $k_p > 0$ and $k_i = 0$, determine the steady-state error for a step reference input $r(t) = A\Gamma(t)$.
3. When $k_p > 0$ and $k_i > 0$, determine the steady-state error for a step reference input $r(t) = A\Gamma(t)$.
4. When $k_p > 0$ and $k_i > 0$, determine the steady-state error for a ramp reference input $r(t) = At\Gamma(t)$.

Temperature control for battery systems

Exercise 4.1 - Empirical PID tuning rules

Several tuning rules are available for determining the PID controller parameter values when the system can be approximated as a first-order plus time-delay transfer function model determined from an open-loop step response (See Appendix). The proposed rules address different design specifications such as disturbance rejection or setpoint tracking mode.

1. The temperature response $\theta(t)$ of a continuous-time thermal process to a step input $(u(t) = 35 \times \Gamma(t-20))$ when there is no disturbance has been recorded and is plotted in Figure 4.1.

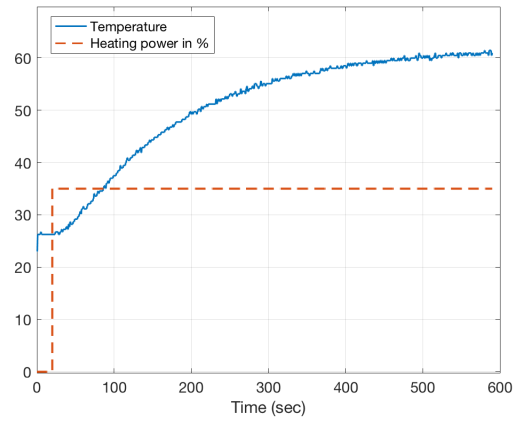


Figure 4.1: Step and temperature response of a thermal system

From the step response, determine the numerical parameter values of a first-order transfer function plus time-delay model (Exercise 2.4 revisited!).
We assume in the following that the thermal system can be reasonably well modeled by the first-order plus delay transfer function model

$$G(s) = \frac{1}{1+140s}e^{-10s}$$

2. Find the PI controller that results by applying the Ziegler-Nichols tuning rules to the process when the design specification is disturbance rejection.

3. Find the PI controller that results by applying the Chien-Hrones-Reswick tuning rules to the process when the design specification is setpoint tracking.

Exercise 4.2 - Temperature control

Extreme temperature changes result in many failures of electronic circuits or battery systems. Temperature control feedback systems can reduce the change of temperature by using a heater to overcome outdoor low temperatures. The analysis aims at illustrating some of the control tuning and performance issues associated with temperature control.

1. System modelling

The differential equation model of the thermal system takes the following form

$$T\frac{d\theta(t)}{dt} + \theta(t) = Ku(t-\tau) + K_D d(t) \tag{1}$$

where

- $\theta(t)$ is the temperature in $^\circ$Celsius
- $u(t)$ is the heating power in % such that $u(t) \in [0 - 100\%]$
- $d(t)$ represents the effects of a cold airstream

1.a. Define the output, the input and the disturbance variable.

1.b. Let $\Theta(s)$, $U(s)$ and $D(s)$ denote the Laplace transforms of $\theta(t)$, $u(t)$ and $d(t)$ respectively. Show that equation (1) can be written in the Laplace domain as:

$$\Theta(s) = G(s)U(s) + G_D(s)D(s)$$

where $G(s) = \dfrac{Ke^{-\tau s}}{1+Ts}$ and $G_D(s) = \dfrac{K_D}{1+Ts}$.

1.c. Represent the system in the form of a block diagram. We assume in the following that the numerical values of the model parameters are:

- $K = 1$
- $T = 140$ s
- $\tau = 10$ s
- $K_D = 0.5$

2. Performance analysis of a PI feedback control in servo mode

We suppose first that there is no disturbance ($d(t) = 0$). The block diagram of the temperature servo-control is displayed in Figure 4.2.

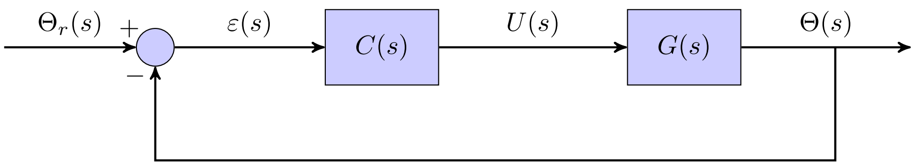


Figure 4.2: Block-diagram of the temperature control in servo mode

The design specifications for the temperature control in servo mode (or setpoint tracking mode) are the following:

- the steady-state error in response to a step on the temperature setpoint should be zero.
- the closed-loop system response to a step setpoint should be critically overdamped. No overshoot is tolerated.

The controller is chosen to be a PI controller of the following ideal form:

$$C(s) = K_p \frac{1+T_i s}{T_i s}$$

As the time-delay τ is much smaller than the time-constant T, the following approximation can be made

$$e^{-\tau s} = \frac{1}{e^{\tau s}} \approx \frac{1}{1+\tau s}$$

The transfer function model of the system then becomes

$$G(s) = \frac{K}{(1+\tau s)(1+Ts)}$$

When the system can be well approximated as an overdamped second-order transfer function model with two different time-constants, one approach to design the PI controller consists in setting the integral time-constant value T_i to compensate the dominant (larger) time-contant of the system (T here). This PI tuning rule is known as the compensation method of the dominant time-constant of the system.

2.a. The design of the PI controller is chosen such that the dominant time-constant of the system is compensated which results in setting the integral action $T_i = T$. With this setting for T_i, determine the open-loop transfer function $F_{OL}(s)$ in terms of K, K_p, τ, T.

2.b. Express the closed-loop transfer function $F_{CL}(s)$ in terms of K, K_p, τ, T.

2.c. Determine the range of values for K_p that ensures the stability of the feedback control system.

2.d. Calculate the steady-state tracking error in response to a step $\theta_r(t) = A\Gamma(t)$, *i.e.* :

$$\lim_{t\to+\infty} \varepsilon(t) = \lim_{t\to+\infty} (\theta_r(t) - \theta(t))$$

2.e. Determine the value for K_p that makes the closed-loop transfer function to have a damping ratio of $z = 1$.

2.f. Compare the designed PI controller with the one obtained by applying the empirical Chien-Hrones-Reswick tuning rules obtained in Exercise 4.1.

3. Performance analysis of a PI feedback control in regulation mode

We now investigate the performance of the feedback control in presence of a disturbance $d(t)$.
The design specifications for the temperature control in regulation mode (setpoint set to a constant value) are the following:

- the closed-loop system should be able to reject a constant cold airstream.

The controller is chosen to be a PI controller of the following ideal form:

$$C(s) = K_p \frac{1 + T_i s}{T_i s}$$

Let us assume that the temperature setpoint now equals to zero ($\theta_r(t) = 0$). The block-diagram of the regulation control is displayed in Figure 4.3. The PI controller $C(s)$ is the one that results from the dominant time-constant compensation method.

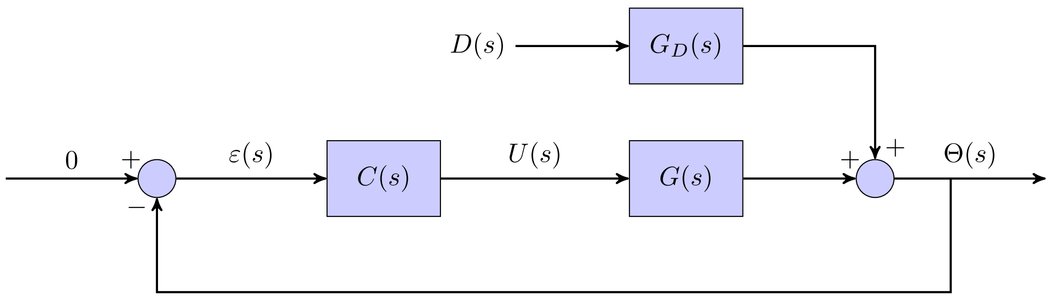


Figure 4.3: Block-diagram of the feedback control in regulation mode

3.a. Show that $\Theta(s)$ can be written as:

$$\Theta(s) = F_D(s)D(s)$$

where $F_D(s) = \dfrac{G_D(s)}{1 + C(s)G(s)}$

3.b. Express the transfer function $F_D(s)$ in terms of K, K_p, K_D, τ, T.

3.c. At time-instant t_0, a constant stream of cold air is acting on the system. Compute the steady-state value of the temperature $\theta(t)$ in response to a step $d(t) = \alpha\Gamma(t - t_0)$, *i.e.*

$$\lim_{t\to+\infty} \theta(t)$$

3.d. Conclude about the ability of the PI feedback control to reject a constant stream of cold air. Note that the PI parameters K_p and T_i can be chosen from the empirical Ziegler-Nichols tuning rules as solved in Exercise 4.1. Another option will also be investigated during the Lab.

Positional control of a DC servo-motor

One of the most common devices for actuating a control system is DC motors. A DC motor is any of a class of rotary electrical motors that converts direct current electrical energy into mechanical energy. Small DC motors are commonly used in tools, toys, appliances and in computer-related equipment such as disk drives or printers. Larger DC motors are currently used in propulsion of electric vehicles, robot systems, elevators, cranes and hoisting devices. Large DC motors are also widely used in the industry such as in drives for steel rolling mills. DC motors that are used in servo systems are called DC servo-motors. In DC servo-motors, the rotor inertias have been made very small. They are frequently used in robot control systems and other angular position or speed control system.
We study the angular position (positional) control of a DC servo-motor in this problem as they appear for each rotary joint of robotic arms such as the Canadarm2 as shown in Figure 5.1 for example.

Figure 5.1: Astronaut Stephen Robinson anchored to the end of Canadarm2 during STS-114, 2005. By NASA - http://spaceflight.nasa.gov/gallery/images/shuttle/sts-114/html/s114e6647.html

1. DC servo-motor modelling

The inductance in the armature circuit of a DC servo-motor is usually small and can therefore be neglected. In this case, the motor voltage-to-angular position transfer function for the DC servo-motor takes the following form

$$G(s) = \frac{\Theta(s)}{U(s)} = \frac{K}{s(1+Ts)} \tag{1}$$

- $\Theta(s) = \mathscr{L}\left[\theta(t)\right]$, where $\theta(t)$ is the angular position of the motor shaft in rad;
- $U(s) = \mathscr{L}\left[u(t)\right]$, where $u(t)$ is the applied motor voltage in V;
- K is the motor gain in rad/(V.s);
- T is the motor time-constant in second.

1.a. Define the input and the output of the DC servo-motor and their unit.

1.b. From equation (1), it can be seen that the transfer function involves a pure integrator term $\frac{1}{s}$. The transfer function model can also be seen as the cascade of a pure integrator and a simple first-order model.
It is indeed well-known that the angular velocity $\omega(t)$ is the time-derivative of the angular position $\theta(t)$

$$\omega(t) = \frac{d\theta(t)}{dt}$$

Both variables are linked in the Laplace domain by a pure integrator (or pure derivator) so that (1) can be expressed as:

$$\frac{\Theta(s)}{U(s)} = \frac{\Theta(s)}{\Omega(s)} \times \frac{\Omega(s)}{U(s)} = \frac{1}{s} \times \frac{K}{1+Ts} \tag{2}$$

$\Omega(s) = \mathscr{L}\left[\omega(t)\right]$, where $\omega(t)$ is the motor angular velocity (or speed) in rad/s.
From the analysis above, complete the block-diagram in Figure 5.2.

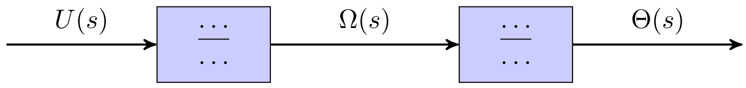
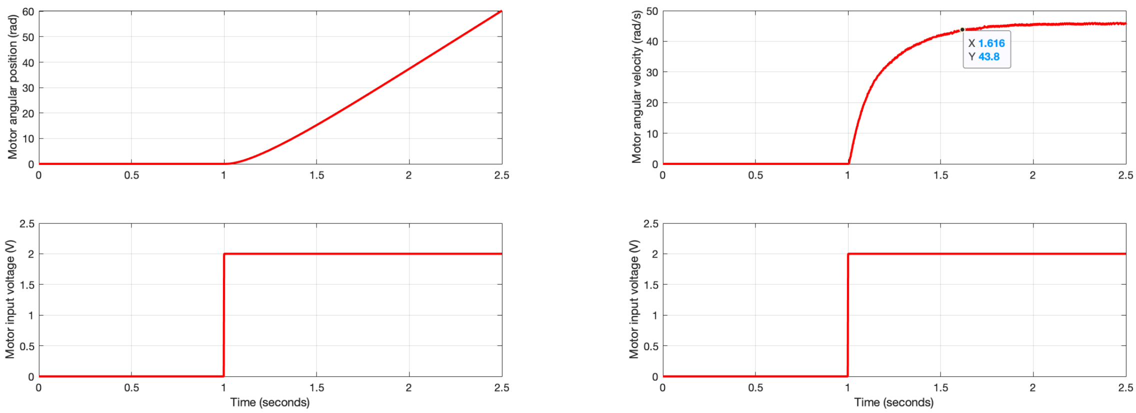


Figure 5.2: Block-diagram of the DC motor

1.c. Identifying a system having a pure integrator from a step response is tricky since the response is diverging. The angular position response (in rad) to a step input of amplitude 2V sent to the motor voltage is plotted in Figure 3(a). The response of the motor starts out slowly due to the time constant, but once that is out of the way the motor position ramps at a constant velocity. It is easier when the motor speed is also measured to identify the response between the motor speed and the input voltage since the voltage-to-angular velocity transfer function has the well-known first-order model form whose parameters can be easily estimated from the step response (the a priori knowledge about the pure integrator is then added in the final model):

$$\frac{\Omega(s)}{U(s)} = \frac{K}{1+Ts}, \tag{3}$$

The angular velocity response (in rad/s) to a step input of amplitude 2V sent to the motor voltage is plotted in Figure 3(b). From the step response plot, determine the parameters of the first-order model.



(a) Angular position response

(b) Angular velocity response

Figure 5.3: Angular position and velocity responses to a step motor input voltage

The performance requirements for the angular position control design are described in Table 3.

Requirement	Assessment criteria	Level
Control the position	position reference tracking	No steady-state error
	motor input voltage	limited to the range [-5V ; +5 V]
	Settling time at 5 %	As short as possible
	Overshoot	less than or equal to 5%
	Disturbance rejection	Rejection of load effects

Table 1: Performance requirements for angular position control

We assume in the following that the numerical values of the model parameters are:

- $K = 23$ rad/(V.s);
- $T = 0.2$ s

2. Servo-motor control using simple proportional feedback

Figure 5.4 shows a simple proportional feedback configuration of the positional servo system. This basic configuration has been used in industry for many years.

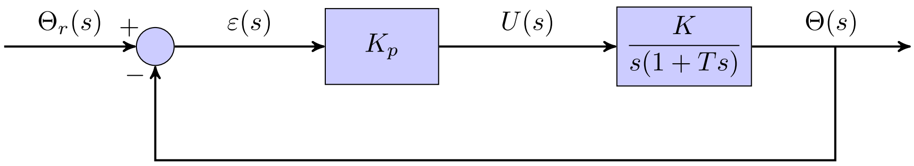


Figure 5.4: Block-diagram of the simple proportional feedback configuration of the positional servo system

2.a. Determine the open-loop transfer function $F_{OL}(s)$ in terms of K, K_p and T.

2.b. Express the closed-loop transfer function $F_{CL}(s)$ in terms of K, K_p and T.

2.c. Determine the range of values for K_p that ensures the stability of the feedback control system.

2.d. Calculate the steady-state tracking error in response to a step $\theta_r(t) = A\Gamma(t)$, *i.e.* :

$$\lim_{t\to+\infty} \varepsilon(t) = \lim_{t\to+\infty} (\theta_r(t) - \theta(t))$$

2.e. Determine the value for K_p that makes the closed-loop transfer function to have a damping ratio of $\frac{\sqrt{2}}{2}$.

2.f. Calculate the percent overshoot, the peak time T_{D_1}, the angular position value at the peak time $y(T_{D_1})$ and the settling time at 5 %.

2.g. Plot the shape of the closed-loop P control response to a unit step setpoint.

2.h. Are the performance requirements satisfied?

3. Servo-motor control using proportional and derivative feedback

We now consider the performance of a proportional and derivative (PD) control which involves a velocity feedback loop as shown in Figure 5.5. This control system represents a high-speed, high precision positional servo system. The positional servomotor systems of this type are used frequently in today's angular position control systems.

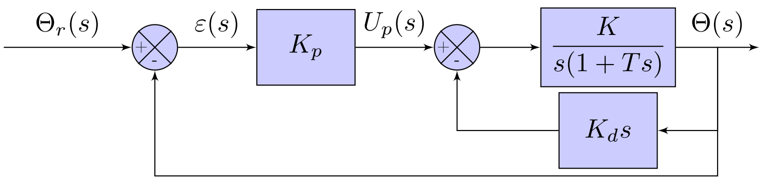


Figure 5.5: Block-diagram of the PD feedback configuration of the positional servo system

3.a. What is the advantage of implementing the derivative term on the output rather than on the error signal $\varepsilon(s)$?

3.b. Determine the internal closed-loop transfer function $F_i(s) = \dfrac{\Theta(s)}{U_p(s)}$ in terms of K, K_d and T.

3.c. Plot the simplified closed-loop block-diagram.

3.d. Determine the closed-loop transfer function $F_{CL}(s)$ in terms of K, K_p, K_d and T.

3.e. Determine the range of values for K_p and K_d that ensure the stability of the feedback control system.

3.f. Calculate the steady-state tracking error in response to a step $\theta_r(t) = A\Gamma(t)$, *i.e.* :

$$\lim_{t \to +\infty} \varepsilon(t) = \lim_{t \to +\infty} (\theta_r(t) - \theta(t))$$

3.g. Determine the value for K_p and K_d that make the closed-loop transfer function step response to have a percent overshoot of 4.3 % and a settling time at 5% of 0.05s.

3.h. Calculate the peak time and plot the shape of the closed-loop PD control response to a unit step setpoint. Are the performance requirements satisfied? Compare your plot with the closed-loop PD control response to a unit step setpoint displayed in Figure 5.6.

3.i. In practice, a pure derivative cannot be implemented because it will give very large amplification of the measurement noise. The gain of the derivative term must thus be limited at high frequencies. This is usually done by approximating the pure derivative term as

$$T_d s \approx \frac{T_d s}{1 + \dfrac{T_d s}{N}} \tag{4}$$

The low-pass filter transfer function on the right approximates the pure derivative well at low frequencies but the gain is limited to N at high frequencies. N is typically chosen in the range 3 to 20.

Modify the block-diagram of the closed-loop control to make appear the low pass-filter.

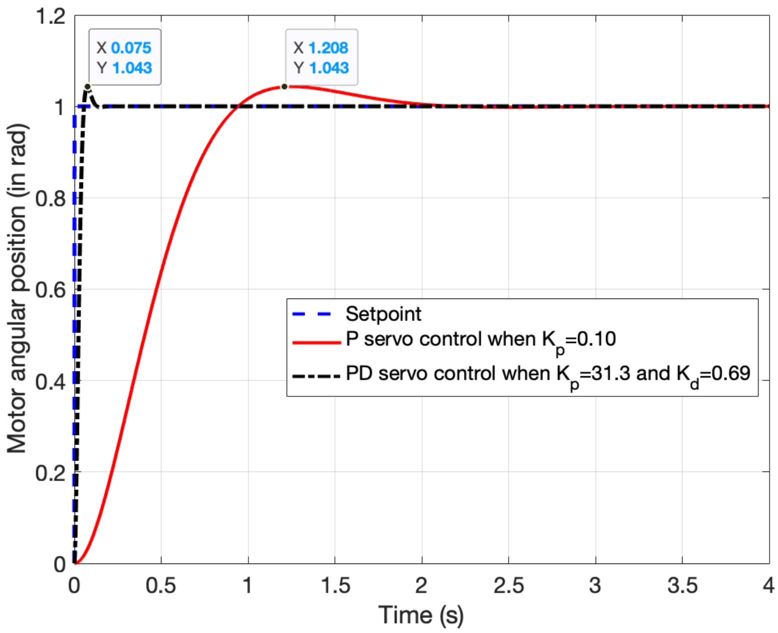


Figure 5.6: Closed-loop P and PD control responses to a unit step setpoint. Both controls satisfy the overshoot requirement but the PD control response is much faster.

More advanced PID-based control schemes - Cascade control & feedforward control

Exercise 6.1 - PI control of the rotary speed for a DC servo-motor

Control of speed is a common problem encountered by many control engineers, perhaps the most common well-known situation being the cruise control fitted to many automobiles. Here it will be assumed that the speed to be controlled is rotary. We investigate the angular velocity PI-based control for the Qube platform equipped with the disk that you have used and will use during the second lab.

The input of the system to be controlled is the voltage of the motor $u(t)$ in V while the output is the angular velocity or rotary speed $\omega(t)$ in rad/s.

The servomotor voltage-to-angular velocity transfer function takes the well-known first-order model form whose key parameters have been estimated from a step response:

$$\frac{\Omega(s)}{U(s)} = \frac{K}{1+Ts} = \frac{23}{1+0.2s} \tag{1}$$

The performance requirements for the rotary speed control are described in Table 3.

Requirement	Assessment criteria	Level
Control the rotary speed of the Qube disk	Step reference tracking First overshoot Settling time at 5 %	No steady-state error $D_{1\%} = 4.3\%$ $T_r^{5\%} = 50$ ms

Table 2: Performance requirements for rotary speed control

The proposed strategy is to use a variation of the classical PI control as shown in Figure 6.1, where unlike the standard PI where the proportional term is usually applied to the error, it is applied to the output.

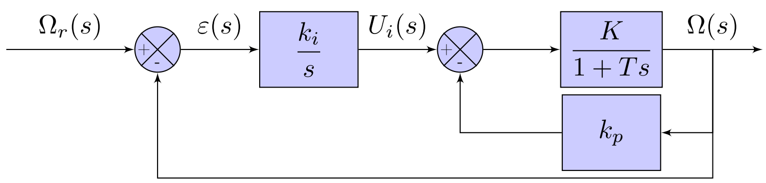


Figure 6.1: Block-diagram of the PI feedback configuration for the rotary speed servo system

1. Determine the internal closed-loop transfer function $F_i(s) = \dfrac{Y(s)}{U_i(s)}$ in terms of k_p, K and T.
2. Represent the simplified closed-loop block-diagram of the initial closed-loop block diagram shown in Figure 6.1.
3. Determine the closed-loop transfer function $F_{CL}(s)$ in terms of k_p, k_i, K and T.
4. Determine the range of values for k_p and k_i that ensure the stability of the closed loop.
5. Calculate the steady-state tracking error in response to a step $\omega_r(t) = 10\Gamma(t)$ by using the final value theorem:

$$\lim_{t\to+\infty} \varepsilon(t) = \lim_{s\to 0} s\varepsilon(s)$$

6. Determine the value for k_p and k_i that make the closed-loop transfer function step response to have $D_{1\%} = 4.3$ % and $T_r^{5\%} = 50$ ms. The following formula could be useful:

$$z = \sqrt{\frac{(\ln(D_1))^2}{\pi^2 + (\ln(D_1))^2}} \qquad \omega_0 = \frac{3}{T_r^{5\%}} \text{ when } z \approx 0.707$$

7. Calculate the peak time $T_{D_1} = \dfrac{\pi}{\omega_0\sqrt{1-z^2}}$ along with the amplitude of the response at the peak time $\omega(T_{D_1})$.
8. Plot precisely the shape of the closed-loop PI control response to the step setpoint.
9. Are the performance requirements satisfied with this PI structure and design?

Exercise 6.2 - Advanced PID-based control for an exoskeleton

An exoskeleton is a device that provides a human being with abilities that he or she does not have or has lost due to an accident. This type of device represented in Figure 6.2 can allow a person to lift heavy loads and considerably reduce the efforts to be made without the least fatigue. After having put on an exoskeleton adapted to his morphology and size, the user can make his movements while benefiting from a great fluidity.

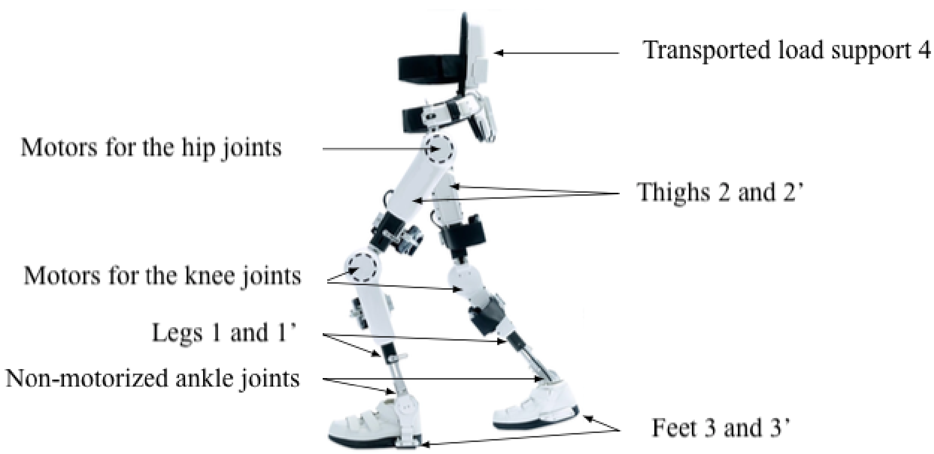


Figure 6.2: The different elements of an exoskeleton

An exoskeleton consists of:

- two legs **1** et **1'**;
- two thighs **2** and **2'**;
- two feet **3** et **3'**;
- a transported load support **4**;
- two motors for the hip joints;
- two motors for the knee joints;
- two non-motorized ankle joints.

The actuators equipping each joint (knees and hips) of the exoskeleton are synchronous motors of the "brushless" type coupled to speed reducers. The study focuses on the control of the knee motors only.

The request for movement of the exoskeleton user's is translated into a trapezoidal type speed command for the vertical movement. This request is finally translated into a position setpoint for the left and right knee motor axes. This position setpoint of the motor axis represented in Figure 6.3 shows zones that can be approximated by constants, ramps and parabolas.

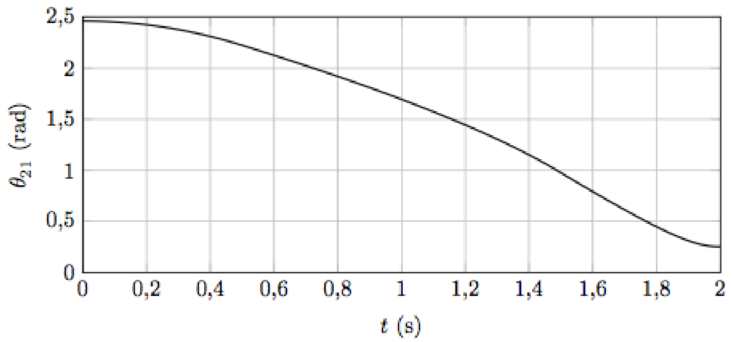


Figure 6.3: General form of the knee motor axis position setpoint

An extract of the performance requirements for the vertical movement control of the exoskeleton is described in Table 3.

Requirement	Assessment criteria	Level
Control the vertical movement	Angular position servo control accuracy - position error - velocity error - acceleration error	 $< 1\%$ $< 1\%$ $< 1\%$

Table 3: Performance requirements for the vertical movement control of the exoskeleton

From the performance requirements, to ensure a good synchronisation of the left and right motor axes, the steady-state accuracy in response to a step, a ramp or a parabola must be lower than 1%. We study in the following two different control strategies to satisfy the performance requirements for the angular position servo control accuracy.

6.2.1 - Cascade control of the knee motor angular position

A first strategy to control the position of the motor axes is adopted for each knee joint. We note :

- $\theta_r(t)$, the reference or setpoint of the motor angular position (in rad);
- $\theta(t)$, the angular position of the motor shaft (in rad);
- $c_r(t)$, the reference or setpoint of the motor torque (in N.m);
- $c(t)$, the motor torque (in N.m);
- $c_r(t)$, the resistive torque (in N.m);
- $\omega_r(t)$, the reference or setpoint of the motor angular velocity (in rad/s);
- $\omega(t)$, the angular velocity of the motor (in rad/s);

and the parameters or transfer function:

- J, inertia of the moving assembly, related to the level of the motor axis (in kg.m^2);
- f, the equivalent viscous friction coefficient for the moving assembly (in kg.m^2/s);
- K_1, the proportional controller gain of the position control loop (in s^{-1});
- $C_\Omega(s)$, the PI controller of the angular velocity control loop;
- $M(s)$, the equivalent closed-loop transfer function of the motor torque control loop.

The closed-loop block-diagram of the first control strategy is given in Figure 6.4.

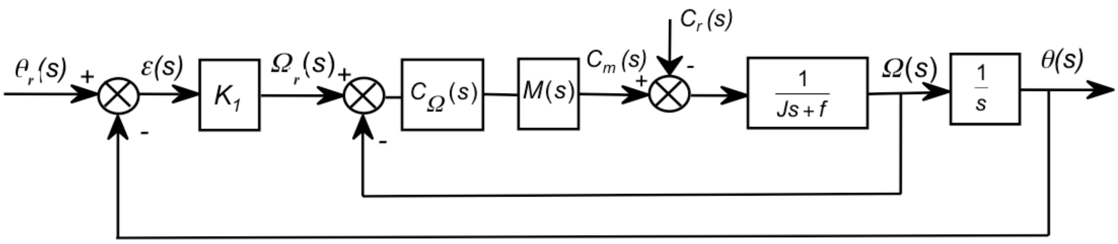


Figure 6.4: Block-diagram of the cascaded feedback control strategy

We assume in the following that the resistive torque varies very slowly in comparison with the dynamic of the servo-control, which leads to assume $C_r(s) = 0$. We also assume that the dynamic of the motor torque servo control loop is negligible which leads to consider that:

$$M(s) = 1$$

1. Determine the steady-state gain K and time-constant T of the voltage-to-angular velocity transfer function,
$$G(s) = \frac{K}{1+Ts} = \frac{1}{Js+f}$$

We will use K and T in the following.

The controller $C_\Omega(s)$ for the angular velocity servo-control is chosen to be a PI controller of the following form:
$$C_\Omega(s) = K_2\left(\frac{1+T_is}{T_is}\right)$$
The integral time-constant T_i is chosen in order to compensate the time-constant of the voltage-to-angular velocity transfer function, $T_i = T$. The controller $C_\Omega(s)$ takes the form:
$$C_\Omega(s) = K_2\left(\frac{1+Ts}{Ts}\right)$$

2. Show that the transfer function of the internal loop is
$$F_\Omega(s) = \frac{\Omega(s)}{\Omega_r(s)} = \frac{KK_2}{KK_2+Ts}$$
3. Represent the simplified closed-loop block-diagram of the initial closed-loop block diagram shown in Figure 6.4.
4. Determine the transfer function of the closed-loop $F_{CL}(s) = \dfrac{\theta(s)}{\theta_r(s)}$ in terms of K, K_1, K_2 and T.
5. Determine the conditions for K_1 and K_2 that ensure the stability of the closed loop.

The accuracy of the complete servo-control can be evaluated from the following errors:

- the position error: $\varepsilon_p = \lim\limits_{t\to+\infty} \varepsilon(t)$ in response to a unit step setpoint: $\theta_r(t) = \Gamma(t)$,
$$\theta_r(s) = \frac{1}{s};$$
- the velocity error: $\varepsilon_v = \lim\limits_{t\to+\infty} \varepsilon(t)$ in response to a unit ramp setpoint: $\theta_r(t) = t\Gamma(t)$,
$$\theta_r(s) = \frac{1}{s^2};$$
- the acceleration error: $\varepsilon_a = \lim\limits_{t\to+\infty} \varepsilon(t)$ in response to a parabola setpoint: $\theta_r(t) = \frac{1}{2}t^2\Gamma(t)$,
$$\theta_r(s) = \frac{1}{s^3}.$$

6. Express $\varepsilon(s)$ in terms of $\theta_r(s)$, K, K_1, K_2 and T.

   By using the final value theorem:
$$\lim_{t\to+\infty} \varepsilon(t) = \lim_{s\to 0} s\varepsilon(s)$$

7. Determine the position error ε_p. Are the performance requirements satisfied for the position error?
8. Determine the velocity error ε_v. Find the value of K_1 to satisfy the performance requirements for the velocity error.
9. Determine the acceleration error ε_a. Are the performance requirements satisfied for the acceleration error?

6.2.2 - Cascade plus feedforward control of the knee motor angular position

To satisfy the performance requirements for the three errors, a second control feedback strategy with a feedforward controller is tested as shown in Figure 6.5, where:

$$F_\Omega(s) = \frac{1}{1 + T_\Omega s} \quad \text{et} \quad T_\Omega = 0.02\,\mathrm{s}$$

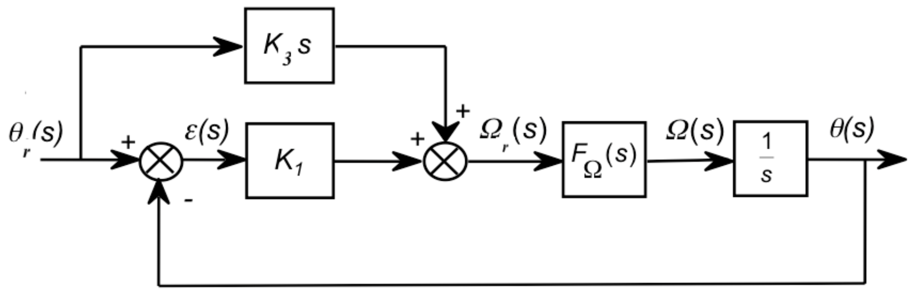


Figure 6.5: Block-diagram of the feedback plus feedforward control strategy

10. Express $\varepsilon(s)$ in terms of $\theta_r(s)$, K_1, K_3 and T_Ω.
11. Determine the position error ε_p. Are the performance requirements satisfied for the position error?
12. Determine the velocity error ε_v. Find the value of K_3 to get a zero velocity error.
13. Determine the acceleration error ε_a when we set $K_3 = 1$. Find the value of K_1 to satisfy the performance requirements for the acceleration error.

Appendices

Useful properties of the Laplace transform

Some useful properties which have found practical use in Control Engineering are summarized below.

Property	signal	Laplace transform
linearity	$ax(t) + by(t)$	$aX(s) + bY(s)$
time-delays	$x(t - \tau)$	$e^{-\tau s}X(s)$
convolution	$y(t) = h(t) * u(t)$	$Y(s) = H(s)U(s)$
differentiation	$\dot{x}(t)$	$sX(s) - x(0)$
	$\ddot{x}(t)$	$s^2X(s) - sx(0) - \dot{x}(0)$
integration	$\int_0^t x(\tau)d\tau$	$\frac{X(s)}{s}$
initial value theorem	$\lim_{t \to 0+} x(t) = \lim_{s \to +\infty} sX(s)$	
final value theorem	$\lim_{t \to +\infty} x(t) = \lim_{s \to 0} sX(s)$	if the limit exists

Some Laplace transform pairs

Signal	Laplace transform
$\delta(t)$	1
$\Gamma(t)$	$\frac{1}{s}$
$r(t) = t\Gamma(t)$	$\frac{1}{s^2}$
$t^2\Gamma(t)$	$\frac{2}{s^3}$
$e^{-at}\Gamma(t)$	$\frac{1}{s+a}$
$t^n e^{-at}\Gamma(t)$	$\frac{n!}{(s+a)^{n+1}}$
$\cos(\omega_0 t)\Gamma(t)$	$\frac{s}{s^2+\omega_0^2}$
$\sin(\omega_0 t)\Gamma(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
$e^{-at}\cos(\omega_0 t)\Gamma(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
$e^{-at}\sin(\omega_0 t)\Gamma(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

A few important transfer functions

First-order systems

$$G(s) = \frac{K}{1+Ts}$$

The 2 characteristic parameters of a first order system are:

- K: steady-state gain
$$K = \lim_{s\to 0} G(s)$$
- T: time-constant

Characteristic values of a first-order system step response

Rise-time at 63%	$T_m^{63\%} = T$
Rise-time at 95%	$T_m^{95\%} \approx 3T$
Settling-time at 5 %	$T_r^{5\%} \approx 3T$

Second-order systems

$$G(s) = \frac{K}{\frac{s^2}{\omega_0^2} + 2\frac{z}{\omega_0}s + 1} = \frac{K\omega_0^2}{s^2 + 2z\omega_0 s + \omega_0^2}$$

The 3 characteristic parameters of a second-order system are:

- K: steady-state gain
- z : damping ratio ($z > 0$)
- ω_0 : undamped natural frequency

Characteristic values of a underdamped second-order system step response ($z < 1$)

Value of the first overshoot in %	$D_{1\%} = e^{\frac{-\pi z}{\sqrt{1-z^2}}} \times 100$
Time-instant of the first overshoot	$T_{D_1} = \frac{\pi}{\omega_0\sqrt{1-z^2}}$
Value of the n^{th} overshoot in %	$D_{n\%} = -(-D_1)^n \times 100$
Time-instant of the n^{th} overshoot	$T_{D_n} = n\,T_{D_1}$
Pseudo-period	$T_p = \frac{2\pi}{\omega_0\sqrt{1-z^2}}$
Settling-time at 5 %	$T_r^{5\%} \approx \frac{3}{\omega_0 z}$ if $z < 1$; $\quad T_r^{5\%} \approx \frac{3}{\omega_0}$ if $z \approx 0.707$
Settling-time at x % for a given z	$T_r^{x\%} = \frac{\ln\left(\frac{100}{x\sqrt{1-z^2}}\right)}{\omega_0 z}$
Rise-time (100%)	$T_m^{100\%} = \frac{\pi - \text{acos}(z)}{\omega_0\sqrt{1-z^2}}$

Model identification from step responses

Identification of a first-order model

$$G(s) = \frac{K}{1 + Ts}$$

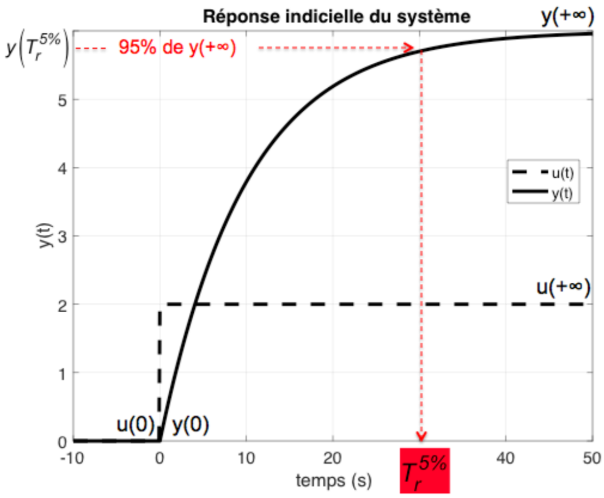


Figure 6.6: Step response of a first-order system

From the step response displayed in Figure 6.6, it is necessary to determine the steady-state gain K and the time constant T. The procedure is as follows:

1. Find the final and initial values of the response and of the step. Deduce K from:

$$K = \frac{y(+\infty) - y(0)}{u(+\infty) - u(0)}$$

2. Find $y(T_r^{5\%})$, deduce from it $T_r^{5\%}$ then T:

$$T = \frac{T_r^{5\%}}{3}$$

Identification of a second-order underdamped model

$$G(s) = \frac{K}{\frac{s^2}{\omega_0^2} + 2\frac{z}{\omega_0}s + 1} = \frac{K\omega_0^2}{s^2 + 2z\omega_0 s + \omega_0^2}$$

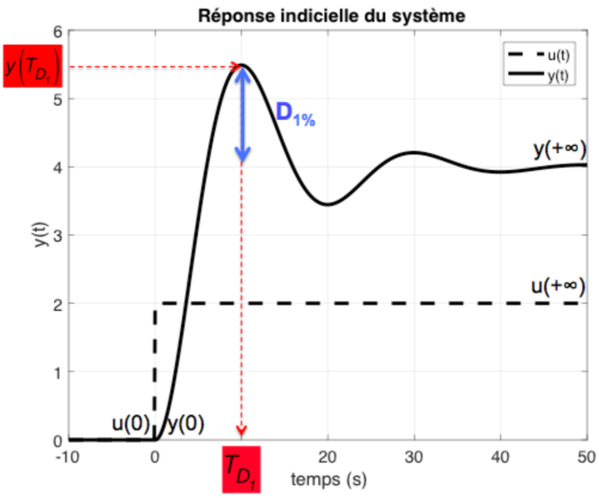


Figure 6.7: Step response of a second-order system

From the step response displayed in Figure 6.7, it is necessary to determine the steady-state gain K, the damping ratio z and the undamped natural frequency ω_0. The procedure is as follows:

1. Find the final and initial values of the response and of the step. Deduce K from:
$$K = \frac{y(+\infty) - y(0)}{u(+\infty) - u(0)}$$

2. Find the final and initial values of the response and that of the first overshoot $y(t_{D_1})$. Deduce from it D_1, then z:
$$D_1 = \frac{y(T_{D_1}) - y(+\infty)}{y(+\infty) - y(0)}$$
$$z = \sqrt{\frac{(\ln(D_1))^2}{(\ln(D_1))^2 + \pi^2}}$$

3. Find the time-instant of the first overshoot T_{D_1}. Deduce from it ω_0:
$$\omega_0 = \frac{\pi}{T_{D_1}\sqrt{1 - z^2}}$$

Identification of a first-order plus time-delay model by the Broïda method

Broïda has suggested to approximate the underdamped step response of any n-th order system by a first-order plus time-delay model

$$G(s) = \frac{Ke^{-\tau s}}{1 + Ts}$$

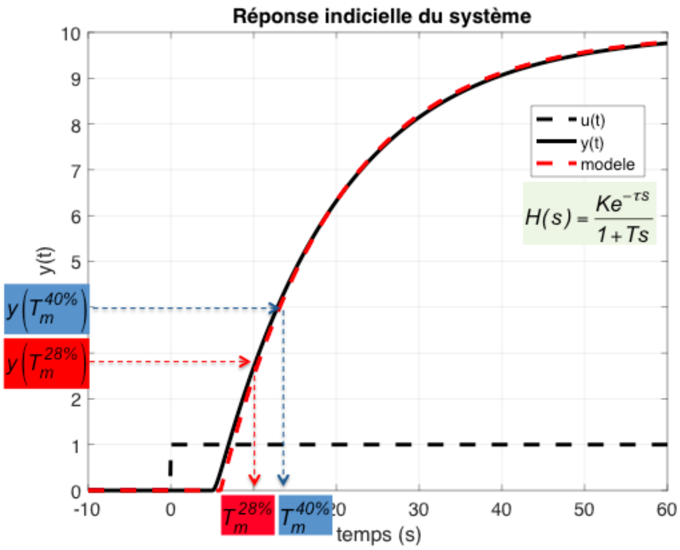


Figure 6.8: Underdamped step response of any n-th order system approximated by a first-order plus time-delay model

From the step response displayed in Figure 6.8, it is necessary to determine the steady-state gain K, the time-constant T and the pure time-delay τ. The procedure is as follows:

1. Find the final and initial values of the response and of the step. Deduce K from:

$$K = \frac{y(+\infty) - y(0)}{u(+\infty) - u(0)}$$

2. Find $y(T_m^{28\%})$ and $y(T_m^{40\%})$, deduce $T_m^{28\%}$ and $T_m^{40\%}$ then calculate :

$$\tau = 2,8T_m^{28\%} - 1,8T_m^{40\%}$$
$$T = 5,5\left(T_m^{40\%} - T_m^{28\%}\right)$$

PID tuning by using empirical rules

A PID controller defined in its so-called *ideal* form is defined as:

$$C(s) = K_p\left(1 + \frac{1}{T_i s} + \frac{T_d s}{1 + \frac{T_d}{N}s}\right) = K_p + K_i\frac{1}{s} + K_d\frac{s}{1 + \frac{T_d}{N}s} \tag{1}$$

The individual effects of these three K_p, K_i and K_d parameters appearing in (1) on the closed-loop performance of stable plants are summarized in Table 1 below. A nice animation that illustrates well the individual effects is available at: `https://en.wikipedia.org/wiki/File:PID_Compensation_Animated.gif`

TABLE 1 **Effects of independent P, I, and D tuning on closed-loop response. For example, while K_I and K_D are fixed, increasing K_P alone can decrease rise time, increase overshoot, slightly increase settling time, decrease the steady-state error, and decrease stability margins.**

	Rise Time	Overshoot	Settling Time	Steady-State Error	Stability
Increasing K_P	Decrease	Increase	Small Increase	Decrease	Degrade
Increasing K_I	Small Decrease	Increase	Increase	Large Decrease	Degrade
Increasing K_D	Small Decrease	Decrease	Decrease	Minor Change	Improve

Tuning of PID controllers - Take away message

Several tuning rules are available for determining the PID controller parameter values. The proposed rules address different design specifications such as load disturbance rejection or setpoint tracking.

There is no single best method and no one correct answer. You need to approach each problem individually and learn to make your judgment. Generally, all the methods require iterative tuning.

Tuning of the PID controller by using the Ziegler-Nichols rules

Assume a continuous-time process is reasonably well modeled by a **first-order plus time-delay transfer function model**:

$$G(s) = \frac{Ke^{-\tau s}}{1 + Ts}$$

The most popular tuning rules are those attributed to Ziegler-Nichols. Their aim is to provide ***satisfactory load disturbance rejection***. Ziegler-Nichols rules have been set to have a one quarter-amplitude decay ratio. A one quarter-amplitude decay ratio is a response where the amount of undershoot decays to one-fourth of the previous amplitude of the overshoot every whole cycle after being upset by a disturbance, as shown in Figure A.1. It may be too oscillatory (underdamped) for some systems.

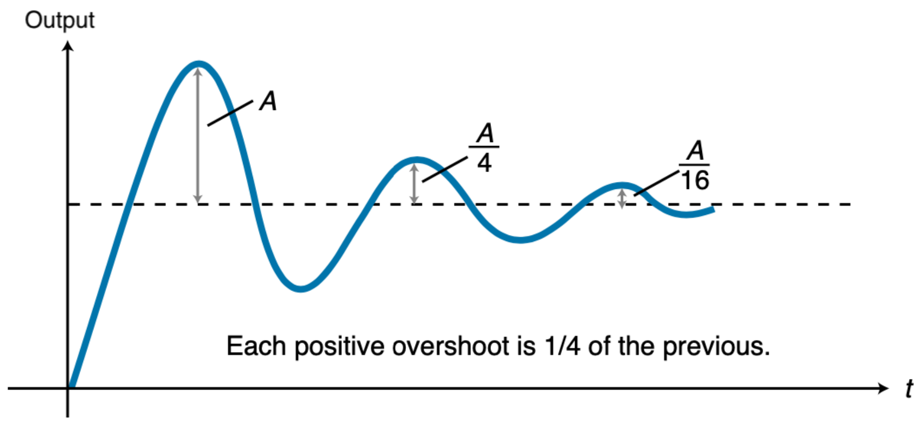


Figure A.1: This response is the desired tuned system response. The closed-loop system reduces the magnitude of each overshoot by one-quarter of the previous one.

Table 4 shows the Ziegler-Nichols rules for P, PI, or PID controllers defined in its so-called *ideal* form (see (1)).

Controller type	K_p	T_i	T_d
P	$\frac{T}{K\tau}$		
PI	$0.9\frac{T}{K\tau}$	3τ	
PID	$1.2\frac{T}{K\tau}$	2τ	0.5τ

Table 4: Ziegler-Nichols disturbance rejection tuning rules for a first-order-plus delay model determined from an open-loop step response

Tuning of the PID controller by using the Chien-Hrones-Reswick rules

As previously said, the Ziegler-Nichols rule aim is to provide satisfactory disturbance rejection. For ***setpoint tracking or servo control***, it can be advantageous to use the empirical rules suggested by Chien-Hrones-Reswick which are given in Table 5 for P, PI, and PID controllers in their *ideal* form (see equation (1)).

Controller type	K_p	T_i	T_d
P	$0.3\frac{T}{\tau}$		
PI	$0.35\frac{T}{\tau}$	1.2T	
PID	$0.6\frac{T}{\tau}$	T	0.5T

Table 5: Chien-Hrones-Reswick setpoint tracking tuning rules for a first-order-plus delay model determined from an open-loop step response

English to French glossary

bandwidth	:	bande passante
crane	:	grue
closed-loop system	:	système bouclé
cut-off frequency	:	fréquence (ou pulsation) de coupure
damped frequency	:	pulsation amortie
damping ratio	:	coefficient d'amortissement
drag	:	traînée
feedback	:	contre-réaction
feedback system	:	système à contre-réaction
hoisting device	:	dispositif de levage
impulse response	:	réponse impulsionnelle
integral wind-up	:	emballement (de l'action) intégral
input	:	entrée
joint	:	articulation
gain	:	gain
heading angle	:	angle de cap
linear time-invariant (LTI)	:	linéaire invariant dans le temps
motor shaft	:	arbre moteur
output	:	sortie
overdamped	:	sur-amorti
overshoot	:	dépassement
rise time	:	temps de montée
road grade	:	inclinaison de la route
robot arm joint	:	articulation d'un bras de robot
root locus	:	lieu des racines
setpoint	:	consigne
settling time	:	temps de réponse
steady-state gain	:	gain statique
steady-state response	:	réponse en régime permanent
steering	:	direction
step response	:	réponse indicielle
stream	:	courant
time-delay	:	retard pur
time-invariant	:	invariant dans le temps
transient response	:	réponse transitoire
throttle	:	accélérateur
undamped	:	non amorti
undamped natural frequency	:	pulsation propre non amortie
underdamped	:	sous-amorti