Control engineering
Solutions to
problems n°1
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Laplace transform, transfer function and block-diagram analysis of linear time-invariant (LTI) dynamic systems

Version of November 10, 2024

Exercise 1.1 - Laplace transform of an exponential signal

1.1.a. $\Gamma(t)$ is the step function defined as

$$\Gamma(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

1.1.b. $x(t) = e^{-at}\Gamma(t)$ for $a = 1$ is plotted in Figure 1.1.

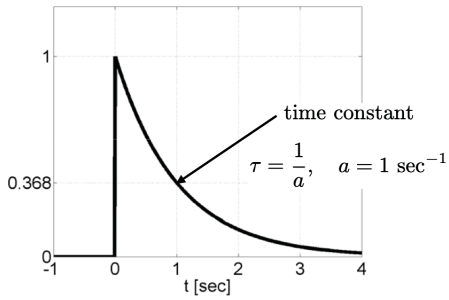

Figure 1.1: Exponential signal $x(t) = e^{-at}\Gamma(t)$ for $a = 1$.

1.1.c. $x(t)$ is causal since $x(t) = 0$ for $t < 0$.

1.1.d. $\Gamma(t)$ makes the signal $x(t)$ to be causal since $\Gamma(t) = 0$ for $t < 0$.

1.1.e. According to the Laplace transform definition

$$\begin{aligned} X(s) = \mathscr{L}[x(t)] &= \int_0^{+\infty} x(t)e^{-st}dt \\ &= \int_0^{+\infty} e^{-t}e^{-st}dt \\ &= \int_0^{+\infty} e^{-(s+1)t}dt \\ &= \left[\frac{-1}{s+1}e^{-(s+1)t}\right]_0^{+\infty} \\ &= \frac{-1}{s+1}(0-1) \\ &= \frac{1}{s+1} \end{aligned}$$

Note that there exist conditions for the improper integral to converge. The Laplace transform of $x(t) = e^{-at}\Gamma(t)$ for $a < 0$ is unbounded if $s < a$; therefore, the real part of s must be restricted to be larger than $-a$ for the integral to be finite. This requirement almost always holds for functions that are useful in process modeling and control and therefore will not be specified.

Take-home message

The Laplace transform of an exponential function is important because exponential functions appear in the solution to all linear differential equations (see Exercise 1.3 below).

Exercise 1.2 - Inverse Laplace transform

We seek the inverse Laplace transform $y(t) = \mathscr{L}^{-1}[Y(s)]$: i.e., a function $y(t)$ such that $\mathscr{L}[y(t)] = Y(s)$. Applying a partial fraction expansion of $Y(s)$

$$Y(s) = \frac{2}{(s+3)(s+5)} = \frac{A_1}{s+3} + \frac{A_2}{s+5}$$

where

$$\begin{aligned}
A_1 &= \lim_{s\to-3}(s+3)Y(s) = \lim_{s\to-3}\frac{2(s+3)}{(s+3)(s+5)} = \lim_{s\to-3}\frac{2}{(s+5)} = 1 \\
A_2 &= \lim_{s\to-5}(s+5)Y(s) = \lim_{s\to-5}\frac{2(s+5)}{(s+3)(s+5)} = \lim_{s\to-5}\frac{2}{(s+3)} = -1
\end{aligned}$$

So it comes,

$$Y(s) = \frac{2}{(s+3)(s+5)} = \frac{1}{s+3} - \frac{1}{s+5}$$

From the table of Laplace transforms we know that

$$\begin{aligned}
\mathscr{L}^{-1}\left[\frac{1}{s+3}\right] &= e^{-3t}\Gamma(t) \\
\mathscr{L}^{-1}\left[\frac{1}{s+5}\right] &= e^{-5t}\Gamma(t)
\end{aligned}$$

Using these results and the linearity theorem of the inverse Laplace transform, we obtain

$$y(t) = \mathscr{L}^{-1}[Y(s)] = e^{-3t}\Gamma(t) - e^{-5t}\Gamma(t) = \left(e^{-3t} - e^{-5t}\right)\Gamma(t)$$

Take-home message

Partial fraction expansion and the Laplace transform are key mathematical tools in Control engineering.

Exercise 1.3 - Solution of differential equations The general procedure to solve ordinary differential equations using the Laplace transform consists of four steps, as shown in Figure 1.2[1]. The overall idea is to find the solution of the differential equation in the Laplace domain, use partial fraction expansion to look for a Laplace transform pair given in the table of Laplace transform.

[1] From Seborg et al., *Process dynamics and control*, Wiley 2011.

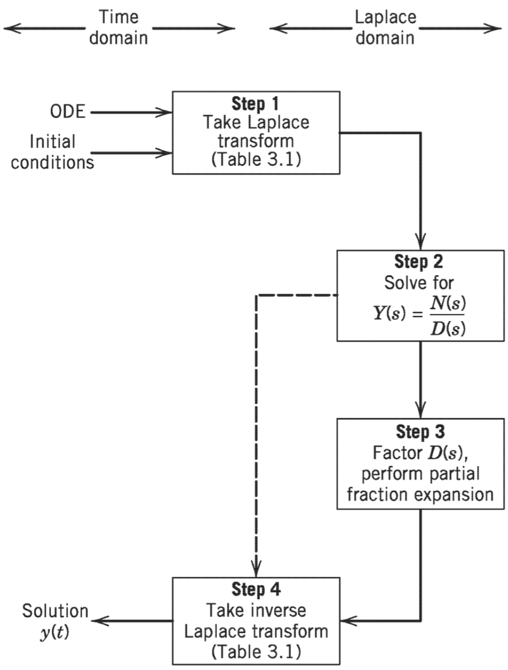

Figure 1.2: The general procedure for solving an ordinary differential equation using the Laplace transform (*Reference to Table 3.1 above is the table of Laplace transform*).

1.3.a. To calculate $y_1(t)$ we start by taking the Laplace transform of both sides of the differential equation:

$$\mathscr{L}[\dot{y}_1(t)] = \mathscr{L}[-2y_1(t)]$$

Here, we apply the time derivative property at the left side (with $y_1(0) = 1$ here), and the linear combination property to the right side, to get

$$sY_1(s) - 1 = -2Y_1(s)$$

Solving for $Y_1(s)$ gives

$$Y_1(s) = \frac{1}{s+2}$$

To get the corresponding $y_1(t)$ from this $Y_1(s)$ we look for a proper Laplace transform pair in the table of Laplace transform.

$$y_1(t) = \mathscr{L}^{-1}[Y_1(s)] = e^{-2t}\Gamma(t)$$

1.3.b. To calculate $y_2(t)$ we start by taking the Laplace transform of both sides of the differential equation:

$$\mathscr{L}[\dot{y}_2(t) + 2y_2(t)] = \mathscr{L}[\Gamma(t)]$$

We then apply the time derivative property at the left side (with $y_1(0) = 1$ here) along with the linear combination property and by using the Laplace transform of the step signal ($\mathscr{L}[\Gamma(t)] = \frac{1}{s}$) in the right side of the equation, to get

$$sY_2(s) - 1 + 2Y_2(s) = \frac{1}{s}$$
$$(s+2)Y_2(s) = 1 + \frac{1}{s}$$
$$(s+2)Y_2(s) = \frac{s+1}{s}$$

Solving for $Y_2(s)$ gives

$$Y_2(s) = \frac{s+1}{s(s+2)}$$

Applying a partial fraction expansion of $Y_2(s)$

$$Y_2(s) = \frac{s+1}{s(s+2)} = \frac{A_1}{s} + \frac{A_2}{s+2}$$

where

$$A_1 = \lim_{s\to 0} sY_2(s) = \lim_{s\to 0} \frac{s(s+1)}{s(s+2)} = \lim_{s\to 0} \frac{s+1}{(s+2)} = \frac{1}{2}$$
$$A_2 = \lim_{s\to -2} (s+2)Y_2(s) = \lim_{s\to -2} \frac{(s+2)(s+1)}{s(s+2)} = \lim_{s\to -2} \frac{s+1}{s} = \frac{1}{2}$$

So it comes,

$$Y_2(s) = \frac{s+1}{s(s+2)} = \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s+2}\right)$$

To get the corresponding $y_2(t)$ from this $Y_2(s)$ we look for proper Laplace transform pairs in the table of Laplace transform.

$$y_2(t) = \mathscr{L}^{-1}[Y_2(s)] = \frac{1}{2}\left(\mathscr{L}^{-1}\left[\frac{1}{s}\right] + \mathscr{L}^{-1}\left[\frac{1}{s+2}\right]\right) = \frac{1}{2}\left(1 + e^{-2t}\right)\Gamma(t)$$

1.3.c. By applying the same method and using the Laplace transform of the Dirac impulse ($\mathscr{L}[\delta(t)] = 1$), it comes for $y_3(t)$

$$Y_3(s) = \frac{10}{s^2 + 10s + 16}$$

Applying a partial fraction expansion of $Y_3(s)$

$$Y_3(s) = \frac{10}{(s+2)(s+8)} = \frac{5}{3}\left(\frac{1}{s+2} - \frac{1}{s+8}\right)$$

To get the corresponding $y_3(t)$ from this $Y_3(s)$ we look for proper Laplace transform pairs in the table of Laplace transform.

$$y_3(t) = \mathscr{L}^{-1}[Y_3(s)] = \frac{5}{3}\left(e^{-2t} - e^{-8t}\right)\Gamma(t)$$

Take-home message

In order to facilitate the solution of a differential equation describing a control system, the equation is transformed into an algebraic form. This transformation is done with the help of the Laplace transform, that is the time-domain differential equation is converted into a Laplace algebraic equation, which is easier to manipulate.

Exercise 1.4 - Second-order model step responses with numerator dynamics

Let us consider the response of the following second-order differential equation model

$$3\ddot{x}(t) + 18\dot{x}(t) + 24x(t) = a\dot{u}(t) + 6u(t), \qquad \dot{x}(0) = x(0) = 0; \quad u(0) = 0$$

when the input $u(t) = \Gamma(t)$ is a unit-step function.

Applying the Laplace transform on both sides of the differential equation model and using the Laplace transform of the unit-step function ($\mathscr{L}[\Gamma(t)] = \frac{1}{s}$)

$$X(s) = \frac{as+6}{s(3s^2 + 18s + 24)} = \frac{as+6}{3s(s^2 + 6s + 8)}$$

Note that the model has numerator dynamics if $a \neq 0$.

Applying a partial fraction expansion of $X(s)$

$$X(s) = \frac{\frac{a}{3}s + 2}{s(s+2)(s+4)} = \frac{1}{4}\frac{1}{s} + \frac{a-3}{6}\frac{1}{s+2} + \frac{3-2a}{12}\frac{1}{s+4}$$

To get the corresponding $x(t)$ from this $X(s)$ we look for proper Laplace transform pairs in the table of Laplace transform.

$$x(t) = \left(\frac{1}{4} + \frac{a-3}{6}e^{-2t} + \frac{3-2a}{12}e^{-4t}\right)\Gamma(t)$$

The plot of the response is given in Figure 1.3 for several values of a. Notice that the value of a has an impact on the transient response. In particular, when $a < 0$, the response becomes negative before changing direction and converging to its positive steady-state value while an overshoot in the response occurs when $a > 3$ and the height of the overshoot increases as a increases.The slope of the responses at the origin is also impacted by a non-zero value of a. However, the value of a does not affect the steady-state response.

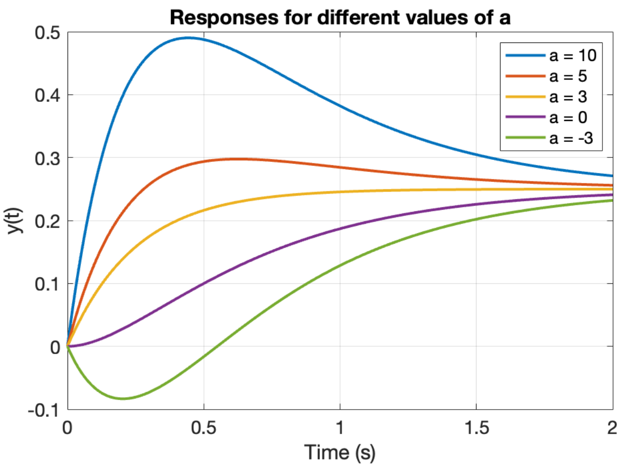

Figure 1.3: Second-order model step responses for different numerator dynamics.

Take-home message
The transient step response of the second-order model can be impacted by the presence of a numerator dynamic (when the transfer function model has one or more zeros=roots of the numerator). The numerator dynamic has no effect on the steady-state response.

Exercise 1.5 - Transfer function of a mechanical system

1.5.a. The transfer function of a system is the ratio of the Laplace transformed output to the Laplace transformed input when all initial conditions are zero.
To calculate the transfer function, we start by applying the Laplace transform of both sides of the differential equation:

$$\mathscr{L}[m\ddot{y}(t) + b\dot{y}(t) + ky(t)] = \mathscr{L}[u(t)]$$

We then apply the time derivative property at the left side (assuming the initial conditions $\dot{y}(0) = y(0) = 0$) along with the linear combination property and by using the Laplace transform of the signal ($\mathscr{L}[u(t)] = U(s)$) in the right side of the equation, to get

$$\left(ms^2 + bs + k\right)Y(s) = U(s)$$

The transfer function of the mechanical system is then defined as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + bs + k}$$

The mechanical system can be represented in the form of the following block-diagram where the transfer function $G(s)$ appears inside the block.

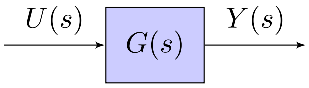

Figure 1.4: Block-diagram of the mechanical system

Note that, from the block-diagram above, the following relationship can always be deduced

$$Y(s) = G(s) \times U(s)$$

Take-home message

The control for a dynamic linear time-invariant system whether electrical, mechanical, thermal, hydraulic, etc. can be represented by a differential equation. This differential equation is usually derived according to physical laws governing a system. Applying the Laplace transform to both sides of the differential equation will lead to the transfer function model. Transfer function is the key mathematical model description of a dynamical system in classical control design. Control systems will use block-diagram representation where transfer function appears inside the block.

1.5.b. The system order is $n = 2$ (highest power of the denominator of $G(s)$).
The steady-state gain is defined as

$$K = \lim_{s \to 0} G(s) = \frac{1}{k}$$

The poles are the roots of the denominator while the zeros are the roots of the numerator.
$G(s)$ has no zero but has two poles

$$\begin{aligned}
\text{if } b^2 > 4mk, \quad & p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \\
\text{if } b^2 = 4mk, \quad & p_{1,2} = \frac{-b}{2m} \\
\text{if } b^2 < 4mk, \quad & p_{1,2} = \frac{-b \pm j\sqrt{4mk - b^2}}{2m}
\end{aligned}$$

Take-home message

From the transfer function model, it is usual to deduce a few characteristics of the system such as: order, steady-state gain, poles and zeros.

Exercise 1.6 - Equivalent transfer function of simple closed-loop block-diagram

Any system in which the output is measured and compared with the reference, the difference being used to actuate the system until the output equals the input is called a closed-loop or feedback control system.
The elements of a closed-loop control system are represented in block-diagram form using the transfer function approach. The general form of such a system is shown in Figure 1.5.
The transfer function $T(s)$ relating $R(s)$ and $Y(s)$ is termed the closed-loop transfer function.

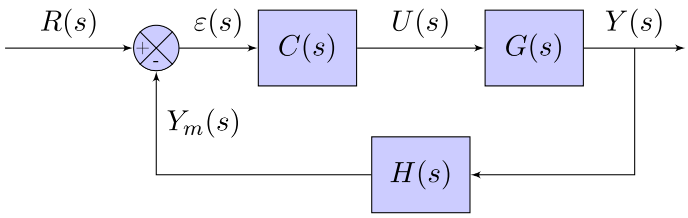

Figure 1.5: Classical block-diagram of a simple closed-loop feedback system

From Figure 1.5, starting from the right side of the closed-loop block diagram, we can write

$$Y(s) = G(s)U(s) \tag{1}$$
$$U(s) = C(s)\varepsilon(s) \tag{2}$$
$$\varepsilon(s) = R(s) - Y_m(s) \tag{3}$$
$$Y_m(s) = H(s)Y(s) \tag{4}$$

Substituting (4) into (3) and (3) into (2) and then (2) into (1) yields

$$Y(s) = G(s)C(s)\big(R(s) - H(s)Y(s)\big) \tag{5}$$
$$Y(s) + C(s)G(s)H(s)Y(s) = C(s)G(s)R(s) \tag{6}$$
$$\big(1 + C(s)G(s)H(s)\big)Y(s) = C(s)G(s)R(s) \tag{7}$$

The closed-loop transfer function is then defined as

$$T(s) = \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)} \tag{8}$$

Take-home message
Block-diagram is the conceptual description of a feedback control system. Many graphical simulation tools such as SIMULINK use block-diagrams to build simulators of closed-loop feedback systems.

Exercise 1.7 - Transfer function of simple closed-loop block-diagram. A case study

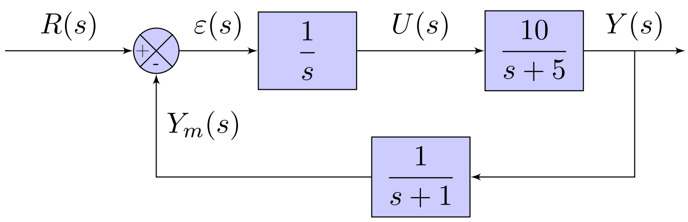

Figure 1.6: Case study of a simple closed-loop feedback system

In Figure 1.6, we have

$$C(s) = \frac{1}{s}$$
$$G(s) = \frac{10}{s+5}$$
$$H(s) = \frac{1}{s+1}$$

Then the closed-loop transfer function from (8), after derivation, is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{10s + 10}{s^3 + 6s^2 + 5s + 10}$$

Step responses of important systems
& System identification from step response test

Version of November 10, 2024

Exercise 2.1 - Step response of a first-order system

1. The given transfer function can be identified to the standard first-order transfer function model

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{1+Ts} = \frac{2}{1+10s} \tag{1}$$

from which it can be deduced by direct identification:

- the steady-state gain $K = 2$
- the time-constant $T = 10$ seconds

The poles and zeros of a system are the roots of the denominator and numerator of a transfer function $G(s)$.
The system has no zero and it has one pole:

$$p_1 = -\frac{1}{T} = -\frac{1}{10}$$

2. The unit step response of a standard first-order model takes the form

$$y(t) = K\left(1 - e^{-t/T}\right)\Gamma(t) \tag{2}$$

It is not asked here to demonstrate this result which is well-known. However, for those who need a reminder we briefly recall below how to proceed by calculating first the step response in the Laplace domain

$$Y(s) = \frac{K}{1+Ts} \times U(s) = \frac{\frac{K}{T}}{s+\frac{1}{T}} \times \frac{1}{s}$$

Then by using partial fraction expansion, it comes

$$Y(s) = K\left(\frac{1}{s} - \frac{1}{s+\frac{1}{T}}\right)$$

from which by using the table of Laplace transform, equation (2) is obtained.

To calculate the slope at the origin, we need to determine the time-derivative of the step response and then evaluate it at $t = 0$. Note that the function $\Gamma(t)$ appearing at the right end of (2) is there to indicate that the step-response is causal ($y(t) = 0$ when $t < 0$). It should not be considered in the derivation of the time-derivative of $y(t)$. The calculation is then straightforward and yields

$$\dot{y}(t) = \frac{K}{T}e^{-t/T}$$

At the origin, *i.e.* for $t = 0$, we have

$$\dot{y}(0) = \frac{K}{T}e^0 = \frac{K}{T} = \frac{1}{5} \neq 0$$

The slope at the origin of the step response of a standard first-order model is different than 0. This is a distinct feature with the step response of a standard second-order model where the slope at the origin is null.

3. It can easily be seen from (2) that for $t = T$, the output reaches 63% of its final value and the response has reached 95% of its final value after a time $t = 3T$.
The rise-time at 63% and 95% and the settling-time at 5% are obtained for a standard first-order model as (see Appendices)

Rise-time at 63%	$T_m^{63\%} = T = 10$ seconds
Rise-time at 95%	$T_m^{95\%} \approx 3T = 30$ seconds
Settling-time at 5 %	$T_r^{5\%} \approx 3T = 30$ seconds

4. The unit step response of the standard first-order system is plotted in Figure 2.1.

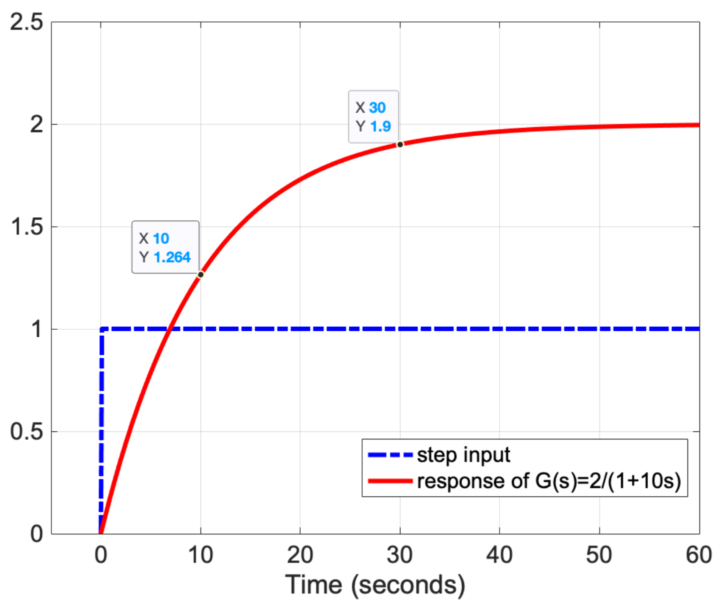

Figure 2.1: Step response of the first-order system described by (1)

In practice, very few experimental step responses exhibit exactly first-order dynamic behavior. This is because the true system is usually neither first-order nor linear. Only the simplest processes exhibit such ideal dynamics. In order to account for higher-order dynamics that are neglected in a standard first-order model, a time-delay term is usually included. This modification can improve the agreement between model and experimental overdamped responses.

Take-home message
The step response of a standard first-order is overdamped (no overshoot). The slope at the origin is different to zero.

Exercise 2.2 - Step response of a first-order plus time-delay system

1. A time-delay, as its name suggests, is an element that produces an output which is a time-delayed version of its input. A time-delayed version of τ seconds of a signal $x(t)$ is denoted as $x(t-\tau)$. By using the delay property of the Laplace transform,

$$\mathscr{L}[x(t-\tau)] = e^{-\tau s}X(s)$$

the time-delay makes appear an $e^{-\tau s}$ term in the Laplace domain.

The transfer function of a first-order plus time-delay model takes the following form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{Ke^{-\tau s}}{1+Ts}$$

where

- K is the steady-state gain
- T is the time-constant
- τ is the time-delay

From the numerical values given, we have

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2e^{-20s}}{1+10s}$$

2. The step response of the first-order time-delayed system is plotted in Figure 2.2. It is simply a delayed version by $\tau = 20$ seconds of the step response of the standard first-order system represented in Figure 2.1.

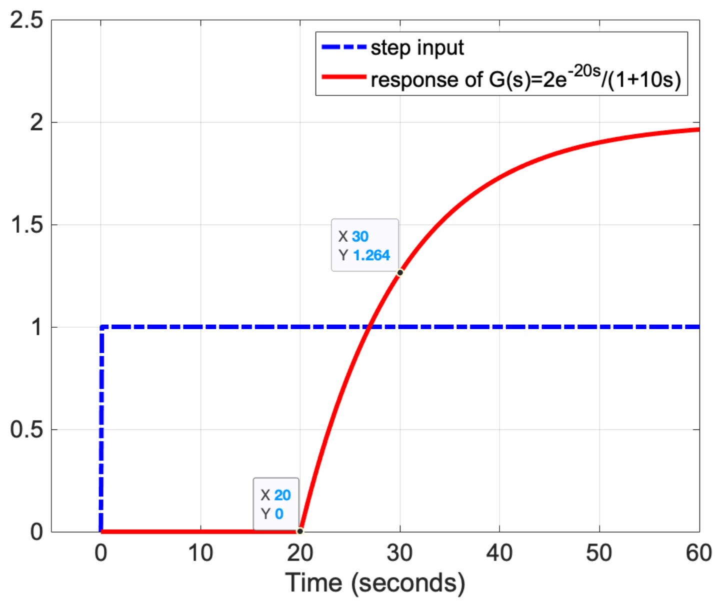

Figure 2.2: Step response of the first-order plus-delay system

Take-home message

The step response of a time-delayed system is a delayed version of the step response of the system (with no delay).

Exercise 2.3 - Step response of a second-order system

1. By taking the Laplace transform of both sides of the differential equation, it comes

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s^2 + 2s + 10} \tag{3}$$

2. The transfer function is of order $n = 2$ (highest power of the denominator). A standard second-order system can be represented by the following transfer function

$$G(s) = \frac{K}{\frac{s^2}{\omega_0^2} + 2\frac{z}{\omega_0}s + 1} = \frac{K\omega_0^2}{s^2 + 2z\omega_0 s + \omega_0^2} \tag{4}$$

where the 3 characteristic parameters of a second-order system are:

- K: steady-state gain
- z : damping ratio ($z > 0$)
- ω_0 : undamped natural frequency

By direct identification of $G(s)$ in (3) with the standard form given in the right end of equation (4), we have:

- $K\omega_0^2 = 10$
- $2z\omega_0 = 2$ rad/s
- $\omega_0^2 = 10$

which results in

- $K = 1$
- $\omega_0 \approx 3.16$ rad/s
- $z \approx 0.316$

The system has no zero and it has two complex conjugate poles:

$$p_{1,2} = -1 \pm 3j$$

3. $0 < z < 1$, the response is therefore underdamped.
4. Knowing the numerical values of z and ω_0, by using formula given in the appendices,

- Value of the first overshoot in % : $D_{1\%} = e^{\frac{-\pi z}{\sqrt{1-z^2}}} \times 100 \approx 35.1\%$
- Value of the 2nd overshoot in % : $D_{2\%} = -(-D_1)^2 \times 100 = -(-0.351)^2 \times 100 \approx -12.3\%$
- Time-instant of the first overshoot : $T_{D_1} = \frac{\pi}{\omega_0\sqrt{1-z^2}} \approx 1.05$s
- Time-instant of the 2nd overshoot : $T_{D_2} = 2\,T_{D_1} \approx 2.1$s

5. The underdamped step response of the second-order system described by (3) is plotted in Figure 2.3.

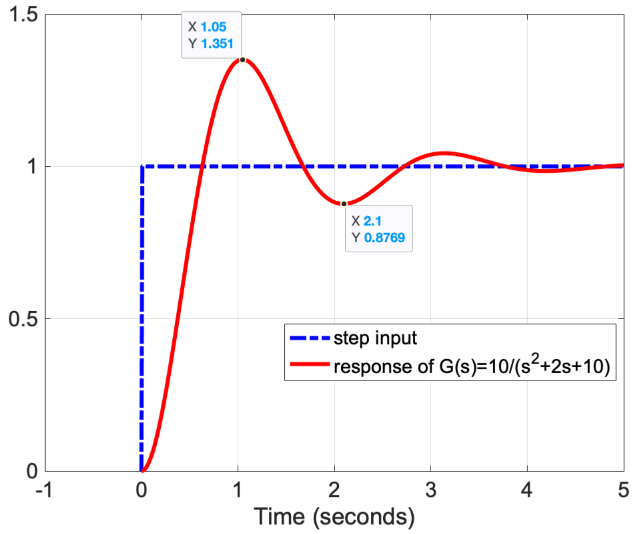

Figure 2.3: Underdamped step response of the standard second-order system described by (3)

Take-home message

The step response of a standard second-order system depends on the value of the damping ratio z *as illustrated in Figure 2.4 :*

- *diverging response when* $z < 0$
- *undamped response when* $z = 0$
- *underdamped response when* $0 < z < 1$
- *critically damped response when* $z = 1$
- *overdamped response when* $z > 1$

The slope at the origin is always null.

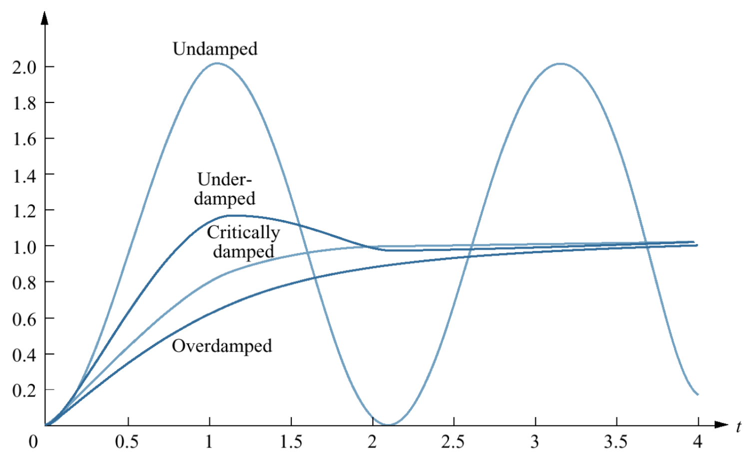

Figure 2.4: Possible step responses of a standard second-order system described by (4)

Exercise 2.4 - Identification of a mechanical suspension system from step response test

1. The step response is clearly under-damped. There is no apparent time-delay in the response which starts immediately after the step input. The mechanical system can be modelled by a standard second-order transfer function:

$$G(s) = \frac{K\omega_0^2}{s^2 + 2z\omega_0 s + \omega_0^2}$$

2. From the step response, it is necessary to determine the steady-state gain K, the damping ratio z and the undamped natural frequency ω_0. The procedure (given in the Appendices) is as follows:

1. From a reading of the plot, find the final ($y(+\infty)$, $u(+\infty)$) and initial ($y(0)$, $u(0)$) values of the response and of the step input. Deduce K from:

$$K = \frac{y(+\infty) - y(0)}{u(+\infty) - u(0)} = \frac{4-0}{1-0} = 4$$

2. From a reading of the plot, find the final ($y(+\infty) = 4$) and initial values ($y(0) = 0$) of the response and that of the first overshoot $y(t_{D_1}) = 5$. Deduce D_1 from:

$$D_1 = \frac{y(T_{D_1}) - y(+\infty)}{y(+\infty) - y(0)} = \frac{5-4}{4-0} = 0.25$$

   then z is calculated from:

$$z = \sqrt{\frac{(\ln(D_1))^2}{(\ln(D_1))^2 + \pi^2}} \approx 0.40$$

3. From the plot, read the corresponding time-instant of the first overshoot $T_{D_1} = 8.6$ seconds. Deduce from it ω_0:

$$\omega_0 = \frac{\pi}{T_{D_1}\sqrt{1-z^2}} \approx 0.40 \text{ rad/s}$$

The second-order transfer function is therefore:

$$G(s) = \frac{K\omega_0^2}{s^2 + 2z\omega_0 s + \omega_0^2} = \frac{0.64}{s^2 + 0.32s + 0.16}$$

3. The differential equation can be retrieved from the definition of the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{0.64}{s^2 + 0.32s + 0.16}$$

$$(s^2 + 0.32s + 0.16)Y(s) = 0.64U(s)$$

By applying the inverse Laplace transform on both sides above, it comes

$$\ddot{y}(t) + 0.32\dot{y}(t) + 0.16y(t) = 0.64u(t), \quad y(0) = \dot{y}(0) = 0$$

Take-home message

Underdamped step response can be approximated by standard second-order (plus time-delay) transfer function model which parameters can be easily estimated from the first overshoot (value and time-instant).

Exercise 2.5 - Step response analysis

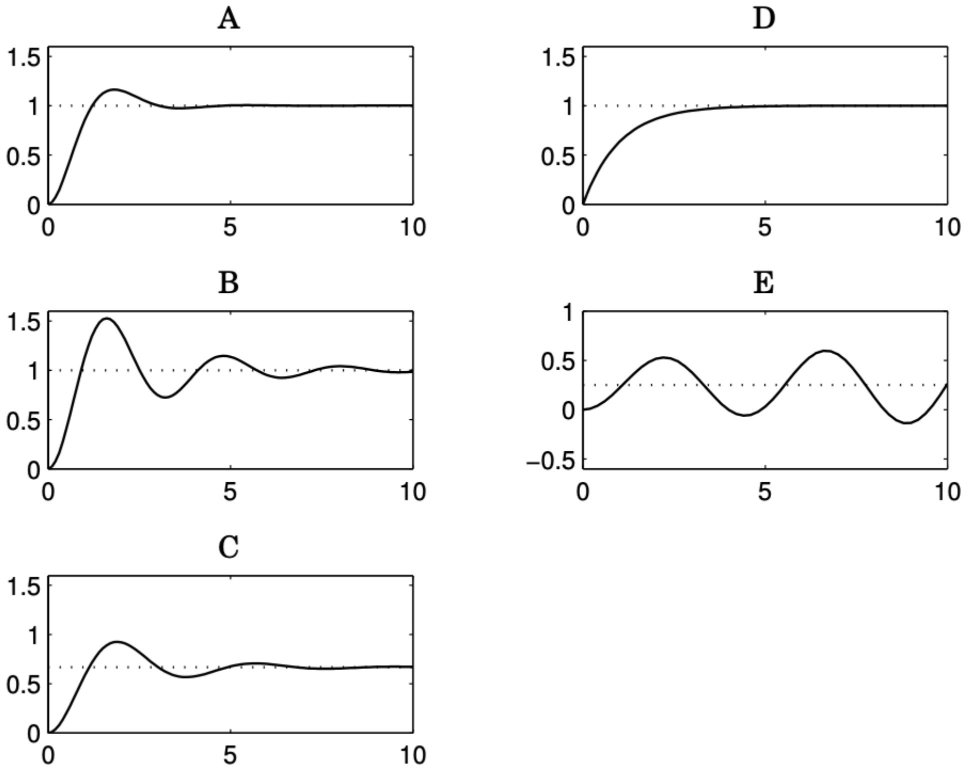

Figure 2.5: Step responses of different LTI systems

Here the best to find the pairs is to determine the characteristic parameters (order, steady-state gain, time-constant for a first-order system or damping ratio and undamped natural frequency for a second order system) of the 7 transfer functions and then find the corresponding step response amongst the 5 responses plotted in Figure 2.5.
The standard first and second-order transfer function forms to be used for an easier and quicker determination of the main characteristic parameters are:

$$G(s) = \frac{K/T}{s + 1/T} \quad \text{or} \quad G(s) = \frac{K\omega_0^2}{s^2 + 2z\omega_0 s + \omega_0^2}$$

After some calculations, we have

$$G_1(s) = \frac{0.1}{s + 0.1}; \qquad n = 1; K = 1; T = 10;$$

$$G_2(s) = \frac{4}{s^2 + 2s + 4}; \qquad n = 2; K = 1; z = 0.5 (D_{1\%} \approx 16\%); \omega_0 = 2;$$

$$G_3(s) = \frac{0.5}{s^2 - 0.1s + 2}; \qquad n = 2; K = 0.25; z = -0.035 (G_3(s) \text{ is unstable}); \omega_0 = \sqrt{2};$$

$$G_4(s) = \frac{-0.5}{s^2 + 0.1s + 2}; \qquad n = 2; K = -0.25; z = 0.035; \omega_0 = \sqrt{2};$$

$$G_5(s) = \frac{1}{s + 1}; \qquad n = 1; K = 1; T = 1;$$

$$G_6(s) = \frac{4}{s^2 + 0.8s + 4}; \qquad n = 2; K = 1; z = 0.2 (D_{1\%} \approx 40\%); \omega_0 = 2;$$

$$G_7(s) = \frac{2}{s^2 + s + 3}; \qquad n = 2; K = \frac{2}{3}; z = \frac{\sqrt{3}}{6}; \omega_0 = \sqrt{3};$$

From the step responses plotted in Figure 2.5, we can observe the following distinguishing features

- A : underdamped second-order type of response, $K = 1$, $D_{1\%} \approx 16\%$
- B : underdamped second-order type of response, $K = 1$, $D_{1\%} \approx 40\%$
- C : underdamped second-order type of response, $K \approx \frac{2}{3}$
- D : overdamped first-order type of response, $K = 1$, $T = 10$
- E : underdamped second-order type of response, $K = 0.25$, oscillatory diverging response (unstable system $z < 0$)

We can then deduce the following pairing:

$$A \rightarrow G_2(s)$$
$$B \rightarrow G_6(s)$$
$$C \rightarrow G_7(s)$$
$$D \rightarrow G_5(s)$$
$$E \rightarrow G_3(s)$$

As suggested, you can use Matlab[2] to verify the solutions above. Matlab is indeed a useful tool to analyse linear time-invariant (LTI) systems. We recall below several useful commands to define a transfer function, analyse its main features and plot its step response.

Definition of a transfer function in Matlab
Let us consider two transfer functions:

$$H(s) = \frac{2}{0.1s + 1}$$
$$G(s) = \frac{0.2}{(s + 0.1)(s + 1)}$$

To define a transfer function in Matlab, several methods are possible depending on the form in which it is expressed. For $H(s)$, the easiest way is to directly define the transfer function as

```
s = tf('s'); % indicates that the variable s will be considered as the Laplace
variable
H=2/(0.1*s+1)
H =
2
---------
0.1 s + 1
Continuous-time transfer function.
```

$H(s)$ could have been also defined by entering the coefficients of the numerator and denominator (in decreasing power of s)) using the command `tf` :

```
H=tf([2],[0.1 1])
H =
2
---------
0.1 s + 1
Continuous-time transfer function.
```

[2]For those who are new in Matlab, follow `Matlab onramp` where you will learn the essentials of Matlab through a free two-hour introduction tutorial on commonly used commands and workflows

For $G(s)$, the form is factorized. The easiest way is to directly enter it in this form to avoid having to develop:

```
G=0.2/((s+0.1)*(s+1))
G =
0.2
---------------
s^2 + 1.1 s + 0.1
Continuous-time transfer function.
```

Don not forget to add the symbol `*` in the denominator definition of `G` or you will get an error message !

The command `zpk` is used to enter a transfer function given the poles and zeros. $G(s)$ could thus have been defined by the following command:

```
G=zpk([],[-0.1,-1],2)
2
------------
(s+0.1) (s+1)
Continuous-time zero/pole/gain model.
```

To obtain the poles of a transfer function, use the `pole` command:

```
pole(G)
-0.1
-1
```

For the zeros of a transfer function, use the `zero` command:

```
zero(G)
```

To plot the pole and zero diagram, use the `pzplot` command:

```
pzplot(G);
grid on
```

Step responses of a system in Matlab

To plot the step response of a system, we can use the `step` command:

```
figure
step(G);
grid on
```

Note the role of the command `figure` so that the step response is plotted in a new figure window (in Figure(2) here), otherwise the pole-zero plot will be overwritten.

To get the characteristic values of the step response, you can use the `stepinfo` command:

```
stepinfo(G)
struct with fields:
RiseTime:  22.1501
SettlingTime:  40.1739
SettlingMin:  18.0103
SettlingMax:  19.9858
Overshoot:  0
Undershoot:  0
Peak:  19.9858
PeakTime:  73.5906
```

You can change the default setting of the settling time and the rise time of the `stepinfo` function to match the definitions of your choice.

For example, definitions of $t_r^{5\%}$ and $t_m^{63\%}$ given in the lecture can be specified by the following command:

```
stepinfo(G,'SettlingTimeThreshold',0.05,'RiseTimeLimits',[0 0.63])
```

Stability and steady-state error analysis
Version of November 10, 2024

Exercise 3.1 - Stability from the system poles

The main characteristics of second-order systems are summarized in Figure 3.1. They will be useful for the pairing.

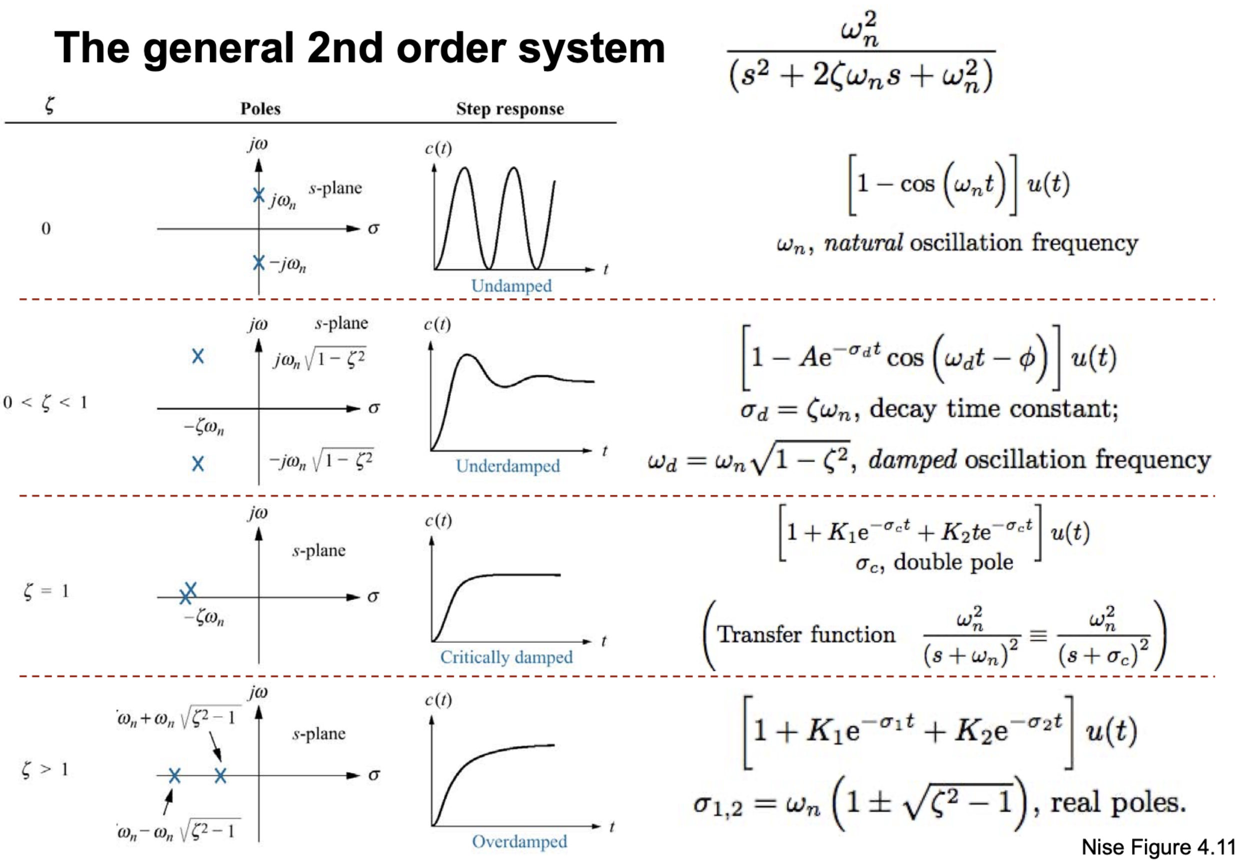

Figure 3.1: Main characteristics of second-order systems (*from N. Nise, Control systems engineering, Wiley*. The damping ratio z and undamped natural frequency ω_0 are denoted above ζ and ω_n respectively.

The poles and zeros of the transfer functions are first calculated. Zeros do not affect the stability of a system; they only influence the transient response of the system to a given input. The stability is therefore determined from the transfer function poles which should all lie in the left half of the complex plane.

$$G_1(s) = \frac{2}{s+2};$$
$$z_1 = \varnothing; \quad p_1 = -2 < 0; G_1(s) \text{ is stable}$$
$$G_2(s) = \frac{2}{s^2+3s+2}$$
$$z_1 = \varnothing; \quad p_1 = -1 < 0; p_2 = -2 < 0; G_2(s) \text{ is stable}$$
$$G_3(s) = \frac{1}{s^2+2s+2};$$
$$z_1 = \varnothing; \quad p_{1,2} = -1 \pm j; \mathscr{R}(p_{1,2}) < 0, G_3(s) \text{ is stable}$$
$$G_4(s) = \frac{2}{s^2+4};$$
$$z_1 = \varnothing; \quad p_{1,2} = \pm 2j; 2 \text{ conjugate poles on the imaginary axis}, G_4(s) \text{ is marginally stable}$$
$$G_5(s) = \frac{2}{s(s+2)};$$
$$z_1 = \varnothing; \quad p_1 = 0; p_2 = -2 < 0; 1 \text{ unique pole at the origin}, G_5(s) \text{ is marginally stable}$$
$$G_6(s) = \frac{2}{s^2(s+2)};$$
$$z_1 = \varnothing; \quad p_{1,2} = 0; p_3 = -2 < 0; 1 \text{ double pole at the origin}, G_6(s) \text{ is unstable}$$
$$G_7(s) = \frac{2(s^2-2s+2)}{(s+2)(s^2+2s+2)};$$
$$z_{1,2} = 1 \pm j; \quad p_1 = -2; p_{2,3} = -1 \pm j; \mathscr{R}(p_{1,2,3}) < 0, G_7(s) \text{ is stable}$$
$$G_8(s) = \frac{200}{(s+2)(s^2-2s+2)};$$
$$z_1 = \varnothing; p_1 = -2; p_{2,3} = 1 \pm j; \mathscr{R}(p_{2,3}) > 0, G_8(s) \text{ is unstable}$$

Take-home message
The stability of a system can be determined from its poles:

- *Stable systems have poles only in the left-hand plane (LHP);*
- *Unstable systems have at least one pole in the right-hand plane and/or poles of multiplicity greater than 1 on the imaginary axis;*
- *Marginally stable systems have a pole in the origin or two pure conjugate imaginary poles and the other poles in the left-hand plane.*

Exercise 3.2 - BIBO stability. Links between system poles and step responses
We first recall the definition of *Bounded-Input Bounded-Output* (BIBO) stability which states that:

- A system is stable if every bounded input yields a bounded output.
- A system is unstable if any bounded input yields a unbounded output.

A step is a bounded input, the system response should also be bounded for the system to be stable.

The pairs of plots that belong to the same system given in the form pole-zero-letter–step-response-letter are the following:
A–B; B–F; C–A; D–C; E–E; F–D.

- Pole-zero diagram B has a single pole in the origin which gives a ramp as step response, that is, B–F.

- Pole-zero diagram D also has a pole in the origin which gives an infinitely growing step response, D–C.
- Pole-zero diagram F has complex poles which gives an oscillatory step response, F–D
- Pole-zero diagram A has a zero in the origin which gives final value zero, A–B.
- Pole-zero diagram C cannot be step response E, since two real poles and no zeros give no overshoot. Hence C–A,
- Step response E is the only alternative left for pole-zero diagram E, E–E.

Take-home message
BIBO stability is an important and generally desirable system characteristic. A system is BIBO stable if every bounded input results in a bounded output. In terms of time-domain features, a linear system is BIBO stable if and only if its impulse response is absolutely integrable or its step response converges to a finite value.

Exercise 3.3 - Stability analysis by using the Routh-Hurwitz criterion
Equating the transfer function denominator polynomial to zero defines the characteristic equation of a system. The Routh-Hurwirtz stability criterion is a simple criterion that enables us to determine the number of closed-loop poles that lie in the right-half s plane without having to factor the denominator polynomial.
The Routh-Hurwirtz stability criterion is based on a characteristic equation that has the form

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$$

where it is arbitrarily assume that $a_n > 0$. If $a_n < 0$, simply multiply the characteristic equation by -1 to generate a new equation that satisfies this condition.
A necessary (but not sufficient) condition for stability is that all of the coefficients a_i in the characteristic equation be positive. If any coefficient is negative or zero, then at least one root of the characteristic equation lies to the right of, or on, the imaginary axis, and the system is unstable. If all of the coefficients are positive, we can then construct the Routh array.
The necessary and sufficient condition for a system to be stable is that all terms in the first column of the Routh array are positive.
Routh stability criterion also states that the number of roots of the characteristic equation with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array.

a) $s^3 - 25s^2 + 10s + 450 = 0$
Since the s^2 term is negative, the system is therefore unstable. *Recall that a necessary condition for stability is that all of the coefficients in the characteristic equation must be positive.*

b) $s^3 + 25s^2 + 450 = 0$
Because the s term is missing, its coefficient is zero. Thus, the system is unstable. *Recall that a necessary condition for stability is that all of the coefficients in the characteristic equation must be positive.*

c) $s^3 + 25s^2 + 10s + 450 = 0$
All of the coefficients in the characteristic equation are positive. Let us construct the array of coefficients (the first two rows are obtained directly from the given polynomial. The remaining terms are obtained from these by using the appropriate formula (see lecture notes). If any coefficients are missing, they may be replaced by zeros in the array)

$$\begin{array}{c|cc} s^3 & 1 & 10 \\ s^2 & 25 & 450 \\ s^1 & \frac{250-450}{25} = -8 & 0 \\ s^0 & 450 & \end{array}$$

In the first column of the Routh array above, there is one negative value, the system is therefore unstable. Furthermore there are two sign changes in the first column ($25 \to -8$ and $-8 \to 450$), hence there are two roots with positive real parts.

d) $s^3 + 25s^2 + 10s + 50 = 0$ All of the coefficients in the characteristic equation are positive. Let us construct the array of coefficients

$$\begin{array}{c|cc} s^3 & 1 & 10 \\ s^2 & 25 & 50 \\ s^1 & \frac{250-50}{25} = 8 & 0 \\ s^0 & 50 & \end{array}$$

All terms of the first column of the Routh array above are positive, the system is therefore stable.
Take-home message
The necessary and sufficient condition for a system to be stable is that all terms in the first column of the Routh array are positive.

Exercise 3.4 - Elevator control system for supertall building
1. The closed-loop transfer function is

$$F_{CL}(s) = \frac{k_p + 1}{s^3 + 3s^2 + 3s + k_p + 1}$$

A necessary condition for stability is that all of the coefficients in the characteristic equation must be positive, hence $k_p > -1$. The Routh array is

$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 3 & k_p + 1 \\ s^1 & \dfrac{8 - k_p}{3} & 0 \\ s^0 & k + 1 & \end{array}$$

So, for stability it is required that $-1 < k_p < 8$.

2. The steady-state error is the error after the transient response has decayed leaving only the steady-state response.
To calculate the steady-state error, we use the final-value theorem

$$\lim_{t \to +\infty} \varepsilon(t) = \lim_{s \to 0} s\varepsilon(s)$$

From the closed-loop block diagram, we have

$$\begin{aligned} \varepsilon(s) = R(s) - Y(s) &= R(s)\Big(1 - \frac{Y(s)}{R(s)}\Big) \\ &= R(s)\big(1 - F_{CL}(s)\big) \end{aligned}$$

Therefore, for a unit step input as reference $R(s) = \dfrac{1}{s}$, the steady-state error becomes

$$\begin{aligned} \lim_{t \to +\infty} \varepsilon(t) &= \lim_{s \to 0} s\varepsilon(s) \\ &= \lim_{s \to 0} sR(s)\Big(1 - F_{CL}(s)\Big) \\ &= \lim_{s \to 0} s \times \frac{1}{s}\Big(1 - F_{CL}(s)\Big) \\ &= 1 - F_{CL}(0) = 0 \end{aligned}$$

When the closed-loop system is stable ($-1 < k_p < 8$), there is no steady-state error in the vertical position to a step reference. The elevator will therefore reach the desired floor without any position error.

Take-home message

1. *The Routh-Hurwirtz criterion can be used to determine the range of the controller gain that ensures the stability of the closed-loop system;*

2. *The final-value theorem is useful for evaluating the steady-state error of the closed-loop in response to a given reference input.*

Exercise 3.5 - Mobile robot steering control

1. From the closed-loop block diagram, we have

$$\begin{aligned}\varepsilon(s) = R(s) - Y(s) &= R(s)\Big(1 - \frac{Y(s)}{R(s)}\Big)\\ &= R(s)\big(1 - F_{CL}(s)\big)\end{aligned}$$

where $F_{CL}(s)$ is given by

$$F_{CL}(s) = \frac{k_p s + k_i}{Ts^2 + (1 + k_p)s + k_i}$$

By using the final-value theorem, the steady-state error, in the general case, is expressed as

$$\begin{aligned}\lim_{t\to+\infty} \varepsilon(t) &= \lim_{s\to 0} s\varepsilon(s)\\ &= \lim_{s\to 0} sR(s)\Big(1 - F_{CL}(s)\Big) = \lim_{s\to 0} sR(s)\Big(1 - \frac{k_p s + k_i}{Ts^2 + (1 + k_p)s + k_i}\Big)\\ &= \lim_{s\to 0} sR(s)\Big(\frac{Ts^2 + s + k_p s + k_i - k_p s - k_i}{Ts^2 + (1 + k_p)s + k_i}\Big)\\ &= \lim_{s\to 0} s^2 R(s)\Big(\frac{Ts + 1}{Ts^2 + (1 + k_p)s + k_i}\Big)\end{aligned}$$

2. When $k_i = 0$ and $k_p > 0$, for a step input as reference $R(s) = \dfrac{A}{s}$, the steady-state error becomes

$$\begin{aligned}\lim_{t\to+\infty} \varepsilon(t) &= \lim_{s\to 0} s\varepsilon(s)\\ &= \lim_{s\to 0} A\Big(\frac{Ts + 1}{Ts + 1 + k_p}\Big)\\ &= \frac{A}{1 + k_p}\end{aligned}$$

If $k_i = 0$, there is a steady-state error in the robot position for a step reference. The mobile robot will not be able to track the desired steering reference without any steady-state error, the latter could however be reduced by choosing a large value for k_p.

3. When $k_i > 0$ and $k_p > 0$, for a step input as reference $R(s) = \dfrac{A}{s}$, the steady-state error becomes

$$\begin{aligned}\lim_{t\to+\infty} \varepsilon(t) &= \lim_{s\to 0} s\varepsilon(s)\\ &= \lim_{s\to 0} As\Big(\frac{Ts + 1}{Ts^2 + (1 + k_p)s + k_i}\Big)\\ &= 0\end{aligned}$$

If $k_i > 0$ and $k_p > 0$, there is no steady-state error in the robot position for a step reference. The mobile robot will track the desired steering reference without any steady-state error.
4. When $k_i > 0$ and $k_p > 0$, for a ramp input as reference $R(s) = \frac{A}{s^2}$, the steady-state error becomes

$$\begin{aligned}\lim_{t\to+\infty} \varepsilon(t) &= \lim_{s\to 0} s\varepsilon(s) \\ &= \lim_{s\to 0} A\Big(\frac{Ts+1}{Ts^2+(1+k_p)s+k_i}\Big) \\ &= \frac{A}{k_i}\end{aligned}$$

If $k_i > 0$ and $k_p > 0$, there is a steady-state error in the robot position for a ramp steering reference. The mobile robot will track the desired position with some steady-state error as illustrated below.

The transient response of the mobile robot to a triangular wave input when $k_i > 0$ and $k_p > 0$ is shown in Figure 3.2. The response clearly shows the effect of the steady-state error, which may be considered as negligible if k_i is sufficiently large. Note that the output attains the desired velocity as required by the reference, but it exhibits a steady-state error e_{ss}.

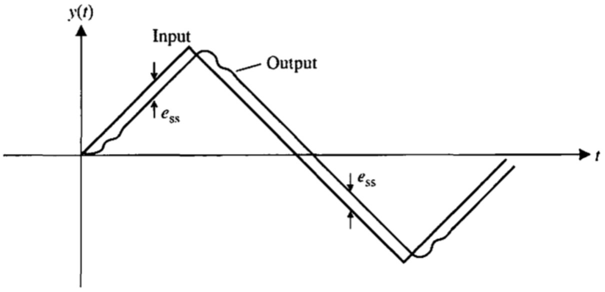

Figure 3.2: Triangular wave input reference and feedback control output response

Take-home message
The steady-state error of a closed-loop system depends not only upon the chosen controller $C(s)$ for a given system $G(s)$, but also on the reference signal which usually takes the form of a step, a ramp or a parabola.

Cruise control of a vehicle

Version of November 10, 2024

1. **Modelling**

1.a. The goal is to control the speed of the car by adjusting the driving force despite external disturbances, such as changes in the road grade. Therefore

- the output is the speed of the car in m/s, $y(t) = v(t)$;
- the input is the driving force of the car in N, $u(t) = f(t)$;
- the disturbance variable is the slope grade in rad, $d(t) = \phi(t)$.

1.b. The differential equation describing the vehicle dynamics has the following form

$$m\dot{v}(t) + bv(t) = f(t) - mg\sin(\phi(t)) \tag{1}$$

The model is not linear because of the presence of the sinus function that applies to the road angle.

1.c. Since the analysis and control design techniques studied in this course apply only to linear systems, the differential equation model needs to be linearized. When the road angle remains small, we can use the following small angle approximation of the nonlinear sinus function:

$$\sin\big(\phi(t)\big) \approx \phi(t)$$

After substituting the above approximation into the differential equation (1), we arrive at the linearized equation of motion:

$$m\dot{v}(t) + bv(t) = f(t) - mg\phi(t) \tag{2}$$

or by using the general input, output and disturbance variables

$$m\dot{y}(t) + by(t) = u(t) - mgd(t) \tag{3}$$

1.d. To obtain the transfer function model, we must first take the Laplace transform of the linearized system equation (3) assuming zero initial conditions.

$$msY(s) + bY(s) = U(s) - mgD(s) \tag{4}$$

where $Y(s)$, $U(s)$ and $D(s)$ denote the Laplace transforms of $y(t)$, $u(t)$ and $d(t)$ respectively.

$$Y(s) = \frac{1}{ms+b}U(s) + \frac{-mg}{ms+b}D(s) \tag{5}$$

or by factorising the b term in the denominator to write the transfer functions in the form of standard (canonical) first-order transfer function

$$Y(s) = \frac{\frac{1}{b}}{1+\frac{m}{b}s}U(s) + \frac{\frac{-mg}{b}}{1+\frac{m}{b}s}D(s) \tag{6}$$

which can be then directly identified to the standard first-order transfer function form as

$$Y(s) = \frac{K}{1+Ts}U(s) + \frac{K_D}{1+Ts}D(s) \tag{7}$$

with $G(s) = \dfrac{K}{1+Ts}$ and $G_D(s) = \dfrac{K_D}{1+Ts}$.
From (6) and (7), we have

$$K = \frac{1}{b}; \quad K_D = \frac{-mg}{b}; \quad T = \frac{m}{b}$$

If the physical parameters of the system are:

$$m = 1000 \text{ kg}; \qquad b = 100 \text{ Ns/m}; \qquad g = 10 \text{ m/s}^2$$

then

$$K = \frac{1}{100} = 0.01; \quad K_D = \frac{-1000 \times 10}{100} = -100; \quad T = \frac{1000}{100} = 10\text{s}$$

1.e. The denominators of both first-order transfer functions $G(s)$ and $G_D(s)$ are the same. They have therefore the same unique pole

$$p_1 = -\frac{1}{T} = -\frac{b}{m} = -0.1$$

As the time-constant $T > 0$, the pole is real negative. The transfer functions $G(s)$ and $G_D(s)$ are therefore stable.

1.f. The block-diagram of the car model for cruise control is represented in Figure 4.1.

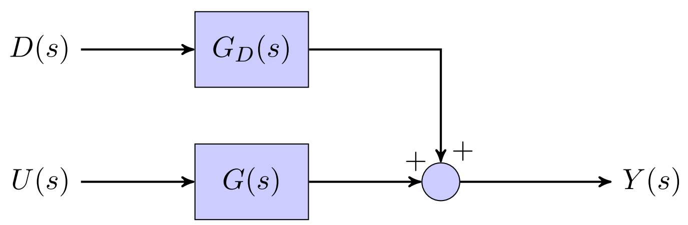

Figure 4.1: Block-diagram of the car model for cruise control

In the following study, we will assume that the road is flat $\phi(t) = 0$ and therefore that there is no disturbance acting on the control loop $d(t) = 0$. The block-diagram of the car model can be simplified as

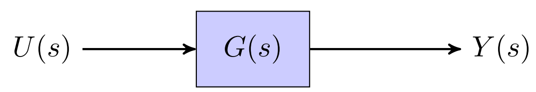

Figure 4.2: Block-diagram of the car model for cruise control when the road is flat

The performance specification for the cruise control in response to a step on the speed setpoint are the following:

- Settling-time at 5 % $\leq$ 10 s
- Percent overshoot $\leq$ 10%
- Steady-state error $\leq$ 2%

2. **Proportional feedback (P) control**
Let $R(s)$ denote the Laplace transform of the speed reference (or setpoint) $r(t)$. We want to drive at a constant speed of $r(t) = 25\Gamma(t)$ (25 m/s = 90 km/h).

2.a. A simple feedback proportional (P) controller is first implemented according to the following control law:

$$u(t) = k_p\varepsilon(t) \qquad \text{where } k_p > 0$$
$$\varepsilon(t) = r(t) - y(t)$$

Taking the Laplace transform of the equations above, we have:

$$U(s) = k_p\varepsilon(s)$$
$$\varepsilon(s) = R(s) - Y(s)$$

from which the controller transfer function can be deduced

$$C(s) = \frac{U(s)}{\varepsilon(s)} = k_p$$

The controller is a simple proportional gain, hence the name of proportional feedback control.

2.b. The closed-loop block diagram of the P feedback cruise control is represented below

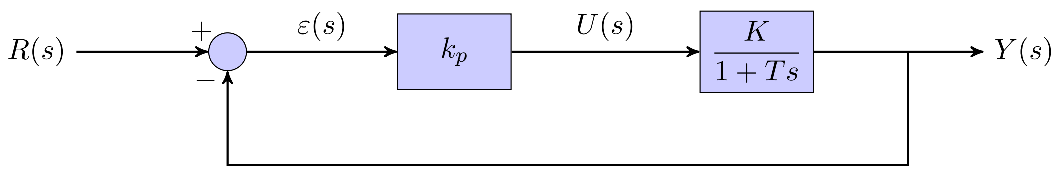

Figure 4.3: Block-diagram of the simple P feedback cruise control

2.c. The open-loop transfer function is

$$F_{OL}(s) = C(s)G(s) = \frac{k_pK}{1+Ts}$$

The closed-loop transfer function is

$$F_{CL}(s) = \frac{C(s)G(s)}{1+C(s)G(s)} = \frac{k_pK}{Ts+1+k_pK}$$

2.d. The range of values for k_p that ensures the stability of the closed-loop control can be here evaluated from the pole of the closed-loop transfer function. The necessary and sufficient condition for a system to be stable is that the real part of the poles are all negative.
$F_{CL}(s)$ is of order 1. Its unique pole can be easily determined

$$p_1 = -\frac{1+k_pK}{T}$$

from which it is required for the closed-loop to be stable that

$$k_p > -\frac{1}{K}$$

2.e. Let us first calculate the error

$$\begin{aligned}\varepsilon(s) &= R(s) - Y(s)\\ &= R(s)\left(1 - \frac{Y(s)}{R(s)}\right)\\ &= R(s)\left(1 - F_{CL}(s)\right)\\ &= R(s)\left(1 - \frac{k_pK}{Ts+1+k_pK}\right)\\ &= R(s)\left(\frac{Ts+1}{Ts+1+k_pK}\right)\end{aligned}$$

By using the final value theorem, the steady-state error can be expressed as

$$\begin{aligned}\lim_{t\to+\infty}\varepsilon(t) &= \lim_{s\to 0} s\varepsilon(s)\\ &= \lim_{s\to 0} sR(s)\left(\frac{Ts+1}{Ts+1+k_pK}\right)\end{aligned}$$

For a setpoint $r(t) = 25\Gamma(t)$, the steady-state error becomes

$$\begin{aligned}\lim_{t\to+\infty}\varepsilon(t) &= \lim_{s\to 0} s\frac{25}{s}\left(\frac{Ts+1}{Ts+1+k_pK}\right)\\ &= \frac{25}{1+k_pK}\end{aligned}$$

2.f. When k_p=900, the steady-state error is

$$\lim_{t\to+\infty}\varepsilon(t) = \frac{25}{1+\frac{900}{100}} = 2.5\text{m/s (9 km/h)}$$

As shown in Figure 4.4, there is indeed a steady-state error of 9 km/h (2.5 m/s) which represents an error of 10% of the desired speed. Therefore the requirement for the steady-state error, which should be $\leq 2\%$, is not satisfied (2% of 25 m/s is 0.5 m/s or 1.8 km/h).

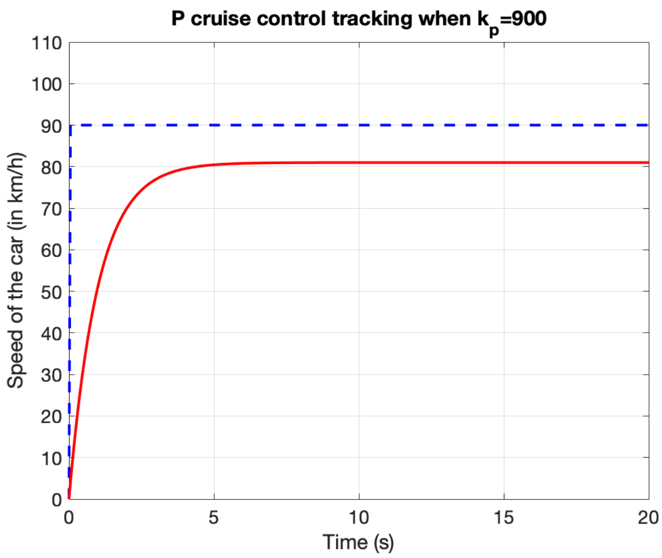

Figure 4.4: P cruise control when $k_p = 900$

2.g. Let us determine the value of k_p to satisfy the steady-state error specification.

$$\lim_{t \to +\infty} \varepsilon(t) = \frac{25}{1 + k_p K} \leq \frac{2}{100} \times 25$$
$$1 + k_p K \geq 50$$
$$k_p \geq \frac{49}{0.01}$$
$$k_p \geq 4900$$

Let us, for example, consider the case when $k_p = 5000$. With this large proportional gain value, the steady-state error is now smaller than 2% for a speed setpoint of 90 km/h (25 m/s). The settling-time has also been reduced substantially as shown in Figure 4.5.

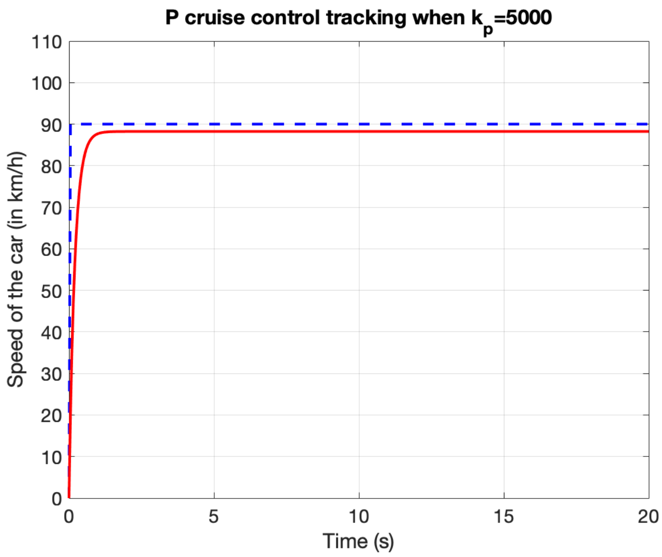

Figure 4.5: P cruise control when $k_p = 5000$

However, this response is unrealistic in practice because a real cruise control system generally cannot change the speed of the vehicle from 0 to 90km/h (25 m/s) in less than 0.5 seconds due to power limitations of the engine and drivetrain.

Actuator limitations are very frequently encountered in practice in control engineering, and consequently, the required control amplitude must always be considered when designing any controller.

The solution to this problem in this case is to choose a lower proportional gain, k_p, that will give a reasonable settling-time, and add an integral term in the controller to eliminate the steady-state error.

3. **Proportional and integral (PI) feedback control**

3.a. We now consider a proportional integral (PI) controller, given by the following transfer function:

$$C(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s}$$

3.b. The new open-loop transfer function becomes

$$F_{OL}(s) = \frac{k_p s + k_i}{s} \times \frac{K}{1 + Ts}$$

The closed-loop transfer function now becomes

$$F_{CL}(s) = \frac{K(k_p s + k_i)}{Ts^2 + (1 + k_p K)s + k_i K} \tag{8}$$

$F_{CL}(s)$ is now of second order and has a zero.

3.c. The Routh-Hurwirtz criterion can be used to determine the range of the PI controller gains that ensures the stability of the closed-loop system.
A necessary condition for stability is that all of the coefficients in the denominator must be positive (since $T > 0$), hence $k_p > -\frac{1}{K}$ and $k_i > 0$. The Routh array can then be built

$$\begin{array}{c|cc} s^2 & \mathrm{T} & k_i K \\ s^1 & 1 + k_p K & 0 \\ s^0 & k_i K & \end{array}$$

So, the range of the PI controller gains k_p and k_i that ensure the stability of the closed-loop system is

$$k_i > 0; \quad k_p > -\frac{1}{K}$$

3.d. Let us again calculate the error for the PI controller

$$\begin{aligned} \varepsilon(s) &= R(s) - Y(s) \\ &= R(s)\left(1 - F_{CL}(s)\right) \\ &= R(s)\left(1 - \frac{K(k_p s + k_i)}{Ts^2 + (1 + k_p K)s + k_i K}\right) \\ &= R(s)\left(\frac{s(Ts + 1)}{Ts^2 + (1 + k_p K)s + k_i K}\right) \end{aligned}$$

By using the final value theorem, the steady-state error can be calculated as

$$\begin{aligned} \lim_{t \to +\infty} \varepsilon(t) &= \lim_{s \to 0} s\varepsilon(s) \\ &= \lim_{s \to 0} sR(s)\left(\frac{s(Ts + 1)}{Ts^2 + (1 + k_p K)s + k_i K}\right) \end{aligned}$$

For a setpoint $r(t) = 25\Gamma(t)$, the steady-state error becomes

$$\begin{aligned} \lim_{t \to +\infty} \varepsilon(t) &= \lim_{s \to 0} s\frac{25}{s}\left(\frac{s(Ts + 1)}{Ts^2 + (1 + k_p K)s + k_i K}\right) \\ &= 0 \end{aligned}$$

If $k_i > 0$ and $k_p > -\frac{1}{K}$, there is no steady-state error in the car speed for a step reference. The car will track the desired speed setpoint without any steady-state error.
The integral control has reduced the steady-state error to zero that was the goal for adding the integral term.

3.e. The question now is to find a way to set the PI controller gains so that the performance requirements (settling-time and overshoot) are satisfied. Different methods exist and here we will design the PI controller algebraically. That is, we will choose the PI controller gains k_p and k_i to satisfy the settling time and percent overshoot requirements; this is equivalent to placing the closed-loop transfer function poles in some desired locations and this is why this design method is sometimes named controller design via algebraic pole placement.

By inspection, it can be noticed that the closed-loop transfer function given in (8) is second-order with a zero. Let us rewrite the transfer function so that the lowest power term of the denominator is 1

$$F_{CL}(s) = \frac{\frac{k_p}{k_i}s + 1}{\frac{T}{k_iK}s^2 + \frac{1+k_pK}{k_iK}s + 1} \tag{9}$$

The system does not therefore match the canonical form recalled below because of the presence of the zero.

$$G_{canon}(s) = \frac{1}{\frac{1}{\omega_0^2}s^2 + \frac{2z}{\omega_0}s + 1} \tag{10}$$

Despite the presence of the zero, we will initially treat our system as if it did have a canonical form. Therefore, the initial design will not reflect the true behavior of our closed-loop system with zero (not to mention the uncertainty in the plant model). Despite these limitations, this design will provide a good starting point and will prove qualitatively helpful in the tuning of our controller.

Matching the denominator of our closed-loop transfer function $F_{CL}(s)$ in (8) to the canonical form given in (10), we get the following relationships between the PI controller gains and damping ratio and undamped natural frequency (which are linked to the desired closed-loop pole locations).

$$\frac{T}{k_iK} = \frac{1}{\omega_0^2} \tag{11}$$

$$\frac{1+k_pK}{k_iK} = 2\frac{z}{\omega_0} \tag{12}$$

from which the PI controller gains can be expressed as

$$k_i = \frac{T\omega_0^2}{K} \tag{13}$$

$$k_p = \frac{1}{K}\left(\frac{2zKk_i}{\omega_0} - 1\right) \tag{14}$$

We can then choose the PI controller gains k_p and k_i in an attempt to satisfy the original system requirements on percent overshoot and settling time. In this process we will use the formula given, which again assume a canonical second-order underdamped system (which we don't have) since it makes it possible to use the following formula (valid for a canonical second order system)

$$z = \sqrt{\frac{(\ln(D_1))^2}{\pi^2 + (\ln(D_1))^2}}$$

$$\omega_0 \approx \frac{3}{T_r^{5\%}z}$$

The percent overshoot requirement (not in %), $D_1 = 0.1$, gives the following value for z

$$z = 0.5912$$

The settling-time requirement, $T_r^{5\%} = 10$s, gives the following value for ω_0

$$\omega_0 = 0.5077$$

A more accurate value could have been calculated by using the abacus given in the Appendix of the Control Engineering problems but here we use the approximation for $T_r^{5\%}$ given in the questions. By using these numerical values in (13) and (14), it comes

$$k_i = 257 \tag{15}$$

$$k_p = 500 \tag{16}$$

If we choose to set the PI controller gains to meet the above constraints, the closed-loop system will not be guaranteed to meet the specification requirements because the closed-loop transfer function is not canonical (it has a zero). That being said, the above relationships are very useful in qualitatively guiding the tuning of the controller gains.

From Figure 4.6, we can see indeed that for $k_p = 500$ and $k_i = 257$, the percent overshoot is about 20% of the setpoint amplitude (which is above the 10% requirement) while the settling time is 8.7 seconds is less than the 10 s requirement. Usually choosing appropriate PID controller gains requires a trial and error process. The best way to attack this tedious process is to adjust one variable (k_p or k_i here) at a time and observe how changing one variable influences the system output.

For example, by doubling the proportional gain ($k_p = 1000$), the closed-loop system response now meets the specification requirements as illustrated in Figure 4.7.

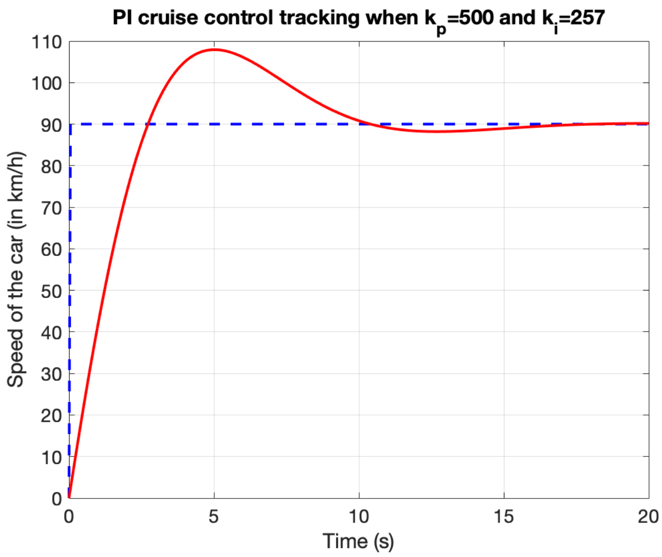

Figure 4.6: PI cruise control for $k_p = 500$ and $k_i = 257$

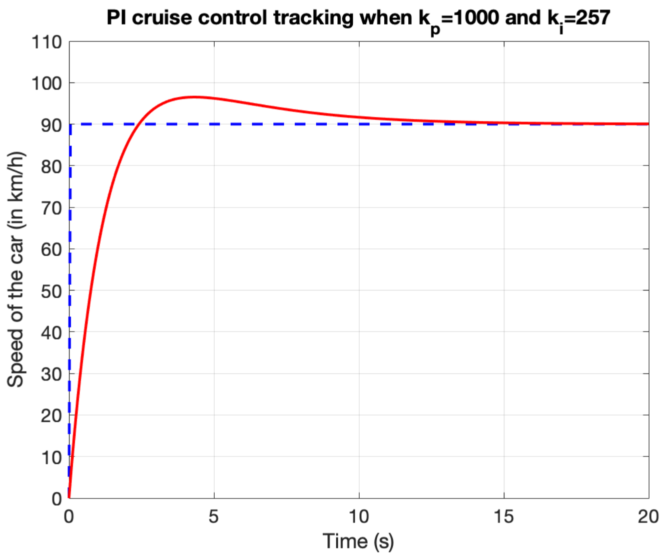

Figure 4.7: PI cruise control for $k_p = 1000$ and $k_i = 257$