# Control engineering
# Final exam

*November 7, 2022 - 1h30*

**Name & firstname:** ......................

**Diploma (IA2R/M3/Erasmus):** ...........

**Instructions:**

1. Do not forget to write your name above and include this sheet with your copy.
2. You can answer either in French or in English but do not mix both languages.
3. The only material you can consult is your personal A4 piece of paper.
4. You may use a hand calculator with no communication capabilities.
5. The exercises must be solved in the given order.
6. Good luck!

## Exercise 1 - Temperature control of the TCLab ($\approx$ 5 mn)

Three different feedback control strategies have been implemented on the TCLab kit that you have used in Lab 1.

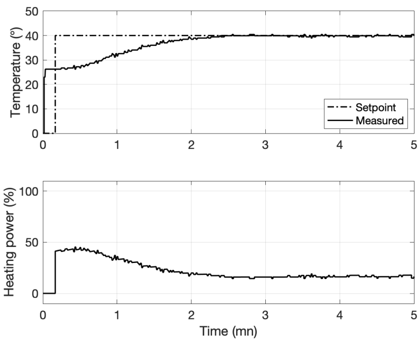


*Figure* 1: Control A

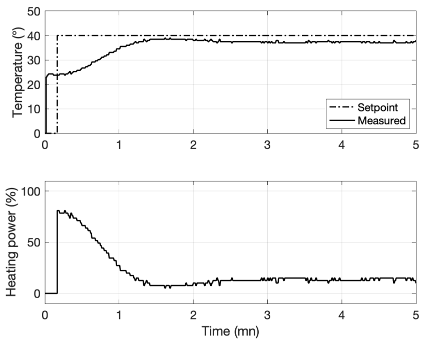


*Figure* 2: Control B

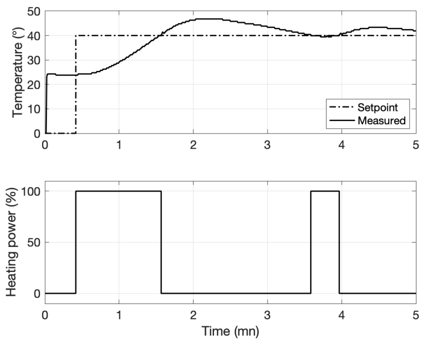


*Figure* 3: Control C

Match each one of the feedback control strategies below to one of the step responses (Control + letter). Justify your answer in each case.

1. On-off control - ..........................................................................
2. P control - ..........................................................................
3. PI control - ..........................................................................

## Exercise 2 - The Gang of Six (≈ 15mn)

The general block diagram of a control system is displayed in Figure 4. For most control designs, as explains by Brian Douglas in his video that was watched during Lab 2, six transfer functions appear to be important leading to the so-called **Gang of Six**.
Complete the missing transfer functions in the 3 empty boxes in Figure 4. Do not give the details of your calculations but the final results only.

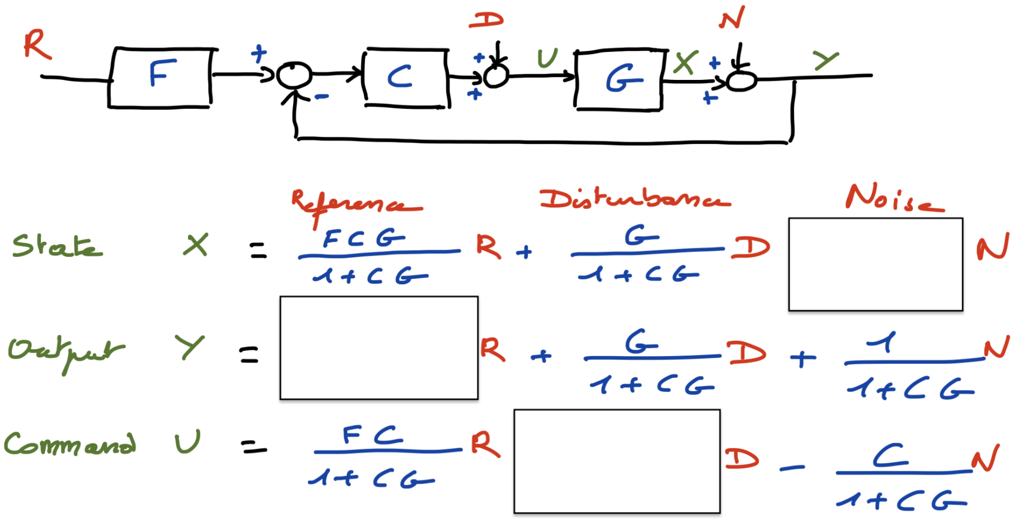


*Figure* 4: The **Gang of Six** important transfer functions in closed-loop control.

## Exercise 3 - Closed-loop block-diagram for the 3pi+ line follower robot (≈ 5mn)

The block diagram of the closed-loop control for the 3pi+ robot that was used in Lab 3 is displayed in Figure 5. Complete the block diagram to link the speed difference command and the nominal speed to the left and right wheel speeds.

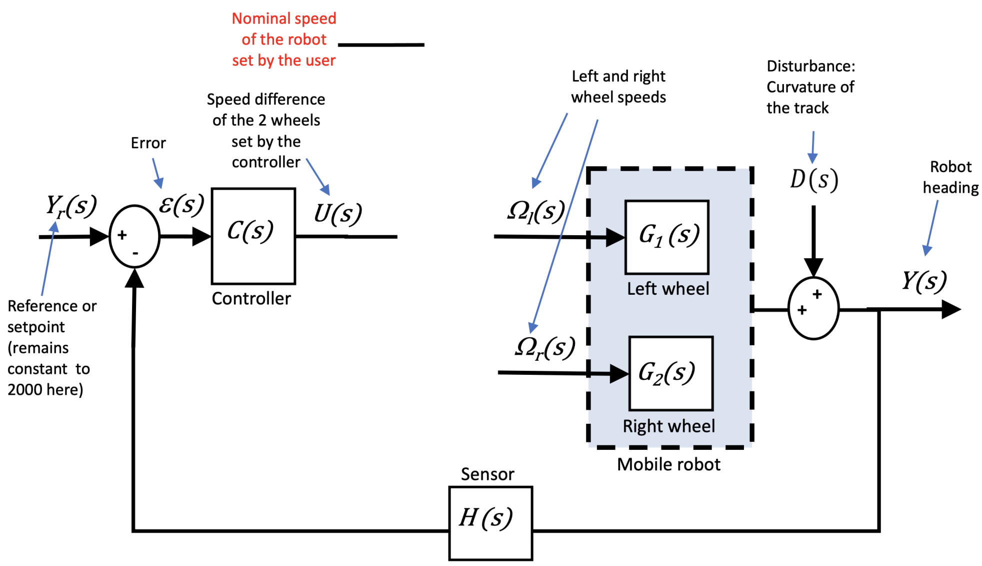


*Figure* 5: Block diagram of the closed-loop control for the 3pi+ robot.

## Exercise 4 - Automatic cruise control (≈ 50mn)

Automatic cruise control is found nowadays in most vehicles. The purpose of the cruise control system is to maintain a constant vehicle speed despite external disturbances, such as changes in wind or road slope. This is usually accomplished by measuring the vehicle speed, comparing it to the desired or reference speed, and automatically adjusting the throttle (accélérateur) according to a control law.

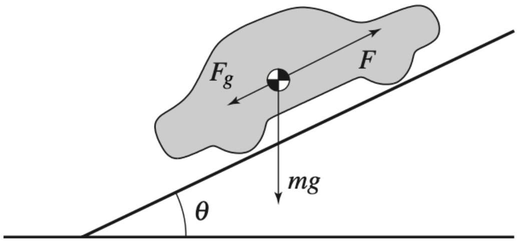


*Figure* 6: Simplified model of a car

The performance requirements for the cruise control are described in Table 1.

| Requirement | Assessment criteria | Level |
|---|---|---|
| Control the speed of the car | Step reference tracking<br>First overshoot<br>Settling time at 5 %<br>Disturbance rejection | No steady-state error<br>$D_{1\%} = 4.3\%$<br>$T_r^{5\%} = 40$ s<br>Rejection of slope effects |

Table 1: Performance requirements for cruise control

1. **Modelling**
   We consider here a simple model of the vehicle represented in Figure 6. The vehicle, of mass $m$, is acted on by a control force, $F(t)$ (in N). The force $F(t)$ represents the force generated at the road/tire interface and $\theta(t)$ (in rad) denotes the angle of the road with the horizontal axis. For this simplified model we will assume that we can control the force $F(t)$ directly and will neglect the dynamics of the powertrain, tires, etc., that go into generating the force. The resistive forces $F_g(t)$ due to rolling resistance and wind drag, are assumed to vary linearly with the vehicle velocity, $v(t)$ (in m/s) through the damping coefficient $b$, and act in the direction opposite the car motion.
   Assuming that the road angle with respect to the horizontal $\theta(t)$ remains small, summing forces in the horizontal direction and applying Newton's second law, we arrive at the following differential equation:
   $$m\dot{v}(t) + bv(t) = F(t) - mg\theta(t) \quad (1)$$
   where $g$ is the acceleration due to gravity.
   It is assumed that the physical parameters of the system are:
   - $m = 1000$ kg
   - $b = 100$ Ns/m
   - $g \approx 10$ m/s$^2$.

   **1.a.** Define the output $y(t)$, the input $u(t)$ and the disturbance variable $d(t)$ and their unit.

**1.b.** Let $Y(s)$, $U(s)$ and $D(s)$ denote the Laplace transforms of $y(t)$, $u(t)$ and $d(t)$ respectively. Show that equation (1) can be written in the Laplace domain as:

$$Y(s) = G(s)U(s) + G_D(s)D(s)$$

where $G(s) = \dfrac{K}{1+Ts}$ and $G_D(s) = \dfrac{K_D}{1+Ts}$.
Express the value of the two steady-state gains $K$ and $K_D$ along with the time-constant $T$ in terms of $m$, $b$ and $g$.

**1.c** Compute the poles of each model and conclude about the stability.

**1.d** Represent the system in the form of a block diagram having $U(s)$ and $D(s)$ as external inputs and $Y(s)$ as output.

2. **Proportional and integral (PI) feedback control**
The road is assumed to be flat $\theta(t) = d(t) = 0$.
The proposed strategy is to use a variation of the classical PI control as shown in Figure 7, where unlike the standard PI where the proportional term is usually applied to the error, it is applied to the output.

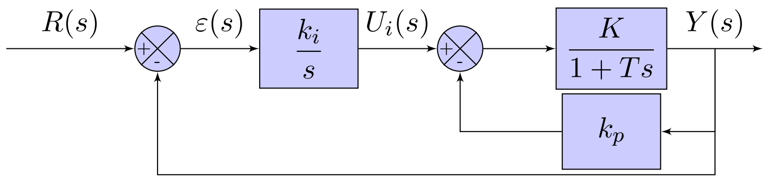


*Figure* 7: Block-diagram of the PI feedback configuration for the automatic cruise control with proportional effect on the output.

**2.a.** Determine the internal closed-loop transfer function $F_i(s) = \dfrac{Ys)}{U_i(s)}$ in terms of $k_p$, $K$ and $T$.

**2.b.** Represent the simplified closed-loop block-diagram.

**2.c.** Determine the closed-loop transfer function $F_{CL}(s) = \dfrac{Ys)}{R(s)}$ in terms of $k_p$, $k_i$, $K$ and $T$.

**2.d.** Determine the range of values for $k_p$ and $k_i$ that ensure the stability of the closed loop.

**2.e.** Calculate the steady-state tracking error in response to a step $r(t) = 25\Gamma(t)$ (25 m/s = 90 km/h) by using the final value theorem:

$$\lim_{t\to+\infty} \varepsilon(t) = \lim_{s\to 0} s\varepsilon(s)$$

**2.f.** Determine the value for $k_p$ and $k_i$ that make the closed-loop transfer function step response to have $D_{1\%} = 4.3$ % and $T_r^{5\%} = 40$ s. The following formula could be useful:

$$z = \sqrt{\frac{(\ln(D_1))^2}{\pi^2 + (\ln(D_1))^2}}; \qquad \omega_0 \approx \frac{3}{zT_r^{5\%}}$$

**2.g.** Calculate the peak time $T_{D_1} = \dfrac{\pi}{\omega_0\sqrt{1-z^2}}$ along with the car speed at the peak time $y(T_{D_1})$.

**2.h.** Plot the shape of the closed-loop PI-based control response to the step setpoint.

**2.i.** Are the performance requirements in servo mode satisfied with this PI structure and design?

3. **Slope effects**
The PI controller settings obtained above are retained.

**3.a.** The road at $t = t_0$ suddenly starts to climb with a low constant slope $\theta_0$, such as $\theta(t) = \theta_0 \Gamma(t - t_0)$. After the transient phase, is the steady-state speed of the car maintained at its initial value? Justify.

**3.b.** Now the slope of the road increases constantly such as $\theta(t) = \theta_0 \, r(t - t_0)$. Is the steady-state speed of the car maintained at its initial value? Justify.

**3.c.** Are the performance requirements in regulation mode satisfied with this PI structure and design?

## Exercise 5 - Feedback control of an inverted pendulum on a cart ($\approx$ 15mn)

Consider an inverted pendulum on a cart which is a very crude model of a Segway people mover as shown in Figure 8.

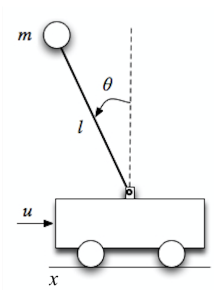


*Figure* 8: Simplified diagram of an inverted pendulum on a cart (left) - Segway people mover (right)

The simplest form of the inverted pendulum is depicted in Figure 8 where the following variables are defined:

- $x(t)$: back-and-forth position of the cart (in meter)
- $\theta(t)$: angular position (angle) of the pendulum made with the vertical (in rad/s)
- $m = 0.25$ kg: mass of pendulum
- $l = 0.5$ m: length of the pendulum
- $g \approx 10$ m/s$^2$: acceleration due to the gravity

Let us assume that the pendulum has a frictionless hinge at its base and that the variations of the angle made with the vertical are small. According to newtonian mechanics, we have the following relationship between the signals:

$$\ddot{\theta}(t) - \frac{g}{l}\theta(t) = -\frac{1}{l}\ddot{x}(t)$$

The goal is to move the cart position $x(t)$ back and forth in an attempt to keep the pendulum from tipping over, like balancing a broomstick on your finger by moving in response to the perceived angle the broomstick makes with the vertical.

Propose a controller form (P, PD and PI) to stabilize the system using feedback. Explain clearly why your proposed controller should be able to stabilize the pendulum system. Design your controller so that the closed-loop system has good tracking of a step reference.

## Appendices

### Some useful properties of the Laplace transform

$$\begin{aligned} L\left(\alpha x(t)+\beta y(t)\right) &= \alpha X(s)+\beta Y(s) \\ L\left(x(t-t_0)\right) &= e^{-t_0 s}X(s) \\ L\left(\dot{x}(t)\right) &= sX(s)-x(0) \\ L\left(\ddot{x}(t)\right) &= s^2X(s)-sx(0)-\dot{x}(0) \\ \lim_{t\to+\infty} x(t) &= \lim_{s\to 0} sX(s) \quad \text{if the limit exists} \end{aligned}$$

### Useful Laplace transform pairs

| Signal | Laplace transform |
|---|---|
| $\Gamma(t)$ | $\frac{1}{s}$ |
| $r(t) = t\Gamma(t)$ | $\frac{1}{s^2}$ |
| $e^{-at}\Gamma(t)$ | $\frac{1}{s+a}$ |

## Transfer function of a first-order systems

$$G(s) = \frac{K}{1+Ts}$$

## Transfer function of a second-order systems

$$G(s) = \frac{K}{\frac{s^2}{\omega_0^2}+2\frac{z}{\omega_0}s+1} = \frac{K\omega_0^2}{s^2+2z\omega_0 s+\omega_0^2}$$

### Characteristic values of a second-order system step response

| | |
|---|---|
| Value of the first overshoot in % | $D_{1\%} = e^{\frac{-\pi z}{\sqrt{1-z^2}}} \times 100$ |
| Time-instant of the first overshoot | $T_{D_1} = \frac{\pi}{\omega_0\sqrt{1-z^2}}$ |
| Value of the $n^{\text{th}}$ overshoot in % | $D_{n\%} = -(-D_1)^n \times 100$ |
| Time-instant of the $n^{\text{th}}$ overshoot | $T_{D_n} = n\,T_{D_1}$ |
| Settling-time at 5 % | $T_r^{5\%} \approx \frac{3}{z\omega_0}$ |